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ΜΗΧΑΝΙΚΗ. — **Spectral decomposition of the compliance tensor for anisotropic plates,** by Academician *Pericles S. Theocaris* and *Dimitrios Sokolis**.

Keywords: Spectral decomposition, compliance tensor, eigenvector, plane eigenangle.

ABSTRACT

The spectral decomposition of compliance **S** is extended to the principal stress planes offering a possibility of characterization of the elastic properties of anisotropic media under plane-stress conditions. It is shown that the three eigenvalues of **S**, together with a «new» dimensionless parameter ω_p, called the plane eigenangle, constitute the essential parameters for an invariant description of the elastic behaviour of anisotropic plates. Both the variational limits of the eigenangle ω_p and the restrictive bounds for the values of the Poisson's ratios imposed by thermodynamics are considered. Finally, it is shown that the plane eigenangle ω_p may be employed as a monoparametric indication of the anisotropy of the material.

1. INTRODUCTION

After the ineffectual attempts (Olszak and Urbanowski, 1956; Olszak and

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* Π. Σ. ΘΕΟΧΑΡΗ - Δ. ΣΟΚΟΛΗ, **Φασματικὴ ἀνάλυσις τῶν τανυστῶν ἐνδόσεως S εἰς ἐπιπέδους πλάκας ἀνισοτρόπων ὑλικῶν.**

Maciejewska, 1985) to extend the theory of separation of elastic energies from the isotropic to anisotropic bodies into a term of dilatational and another one of distortional type of energy, it was succeeded (Rychlewski, 1984a, b) to prove the possibility of always decomposing the compliance, stiffness or failure fourth-rank tensors into their eigenvectors. This decomposition was shown to be the simplest one, rendering orthonormal components of stresses and strains, thus yielding a simple means of separating the total energy of the deformed body into its terms.

Then, using the spectral decomposition principle (Theocaris and Philippidis, 1989a, b; 1990; 1991), it was succeeded to decompose the elastic stiffness, or compliance or failure fourth-rank tensors for the transversely isotropic body. In this way, the elastic energy densities in transversely isotropic bodies, such as the uniaxial fiber reinforced composites could be splitted readily by evaluating the eigenvalues in terms of the components of their compliance tensor in a Cartesian coordinate system, whose axes were collinear with the principal strength directions of the solid. As a result, the elastic strain energy density of the transversely isotropic materials was divided into discrete constituents, each associated to an elastic eigenstate, pointing out the absence of a pure distortional energy component.

From the spectral decomposition of the compliance tensor emerged four invariant parameters, that is four elastic constants, which are described in terms of the typical engineering elastic constants. These are the four eigenvalues of the compliance fourth-rank tensor, which together with a dimensionless quantity presented a full characterization of the elastic properties of a transversely isotropic medium. In addition, the dimensionless parameter, called eigenangle ω, determined the arrangement and orientation of the eigentensors in the principal stress space.

However, since all the experimental evidence today exists for plane stress problems, the three-dimensional spectral decomposition of the fourth-rank tensors has to be extended to encompass the equally important two-dimensional equivalent. Then, the evaluation of the spectral decomposition of the compliance tensor on the principal stress planes offers a possibility of characterization of the elastic properties of anisotropic media under plane-stress conditions. Furthermore, it suggests a way for the separation of the total elastic energy density of plane laminae into distinct elements (Theocaris, 1989).

According to this two-dimensional spectral decomposition, the elastic properties of a material can be described properly by means of three eigenvalues, corresponding to the three energy-orthogonal eigenstates, together with a «new» dimensionless parameter ω_p, called the plane eigenangle, which influences the alignment of the eigentensors on the principal stress plane.

Furthermore, the variation of the eigenangle ω_p is investigated within the limits set by the classical thermodynamics principles. However, it is indicated that the positiveness of the eigenvalues of the compliance tensor imposes new restrictive bounds for the values of the Poisson's ratios. Finally, it is shown that the plane eigenangle ω_p may be used as a momoparametric index of the anisotropy of the material. A few examples of representative uniaxial fiber composites and inorganic crystals of the hexagonal system illustrate the results of our theoretical analysis.

2. SPECTRAL DECOMPOSITION OF COMPLIANCE TENSOR ON THE (σ_1, σ_3) PLANE

It was previously proved that the fourth-rank compliance tensor may be spectrally decomposed into four energy-orthogonal states, each of which with the ability to decompose the stress and strain tensors into four energy-orthogonal stress, strain states (Theocaris and Philippidis, 1989a, b; 1990; 1991).

In this paper, the spectral decomposition of the compliance tensor will be considered on the principal stress plane (σ_1, σ_3). In this way, it will be shown that energy may be separated into two orthonormal components.

Consider now the decomposition of the compliance tensor **S** of a transversely isotropic linear elastic solid. We assume the Cartesian coordinate system, which the stress and strain tensors are referred to, being oriented along the principal material directions, with the 33-axis normal to the isotropic (transverse) plane. Following the classical analysis for transversely isotropic materials and defining the usual elastic moduli and Poisson's ratios E_L, E_T, ν_L, ν_T, where the subscript T denotes the elastic properties on the isotropic plane and the subscript L the corresponding ones on the normal (longitudinal) plane, we obtain the following basic stress-strain relationships:

$$\varepsilon_1 = \frac{1}{E_T}(\sigma_1 - \nu_T\,\sigma_2) - \frac{\nu_L}{E_L}\,\sigma_3 \tag{1a}$$

$$\varepsilon_2 = \frac{1}{E_T}(\sigma_2 - \nu_T\,\sigma_1) - \frac{\nu_L}{E_L}\,\sigma_3 \tag{1b}$$

$$\varepsilon_3 = \frac{1}{E_L}\,\sigma_3 - \frac{\nu_L}{E_L}\,(\sigma_1 + \sigma_2) \tag{1c}$$

$$2\varepsilon_{12} = \frac{1}{G_T}\,\sigma_{12}\ ,\quad 2\varepsilon_{13} = \frac{1}{G_L}\,\sigma_{13}\ ,\quad 2\varepsilon_{23} = \frac{1}{G_L}\,\sigma_{23} \tag{1d}$$

However, we restrict our attention to plane stress situations for which $\sigma_2 = 0$. In addition, if the deformation along the 22-direction remains the same for all (σ_1, σ_3) planes, then $(\varepsilon_{12}, \varepsilon_{23}) = 0$. Therefore, the stress-strain relationships become:

$$\varepsilon_1 = \frac{1}{E_T}\,\sigma_1 - \frac{\nu_L}{E_L}\,\sigma_3 \tag{2a}$$

$$\varepsilon_2 = -\frac{\nu_T}{E_T}\,\sigma_1 - \frac{\nu_L}{E_L}\,\sigma_3 \tag{2b}$$

$$\varepsilon_3 = \frac{1}{E_L}\,\sigma_3 - \frac{\nu_L}{E_L}\,\sigma_1 \tag{2c}$$

$$2\varepsilon_{13} = \frac{1}{G_L}\,\sigma_{13} \tag{2d}$$

However, the component of strain along the 22-direction does not produce an energy component, since its scalar product with the component of stress is always equal to zero. In this case, the compliance tensor **S** is associated to the following square matrix:

$$\mathbf{S} = \begin{bmatrix} \frac{1}{E_T} & -\frac{\nu_L}{E_L} & 0 \\ -\frac{\nu_L}{E_L} & \frac{1}{E_L} & 0 \\ 0 & 0 & \frac{1}{2G_L} \end{bmatrix} \tag{3}$$

The eigenvalues of the associated square matrix of rank three to tensor **S** defined above were evaluated to be:

$$\lambda_1 = \frac{1}{2E_L} + \frac{1}{2E_T} + \left[\left(\frac{1}{2E_L} - \frac{1}{2E_T}\right)^2 + \frac{\nu_L^2}{E_L^2}\right]^{1/2} \tag{4a}$$

$$\lambda_2 = \frac{1}{2E_L} + \frac{1}{2E_T} - \left[\left(\frac{1}{2E_L} - \frac{1}{2E_T}\right)^2 + \frac{\nu_L^2}{E_L^2}\right]^{1/2} \tag{4b}$$

$$\lambda_3 = \frac{1}{2G_L} \tag{4c}$$

The corresponding three idempotent tensors of the spectral decomposition of $\mathbf{S}$ were also evaluated to be:

$$\mathbf{E}_1 = E^1_{ijkl} = \mathbf{f} \otimes \mathbf{f} = f_{ij}\, f_{kl} \tag{5a}$$

$$\mathbf{E}_2 = E^2_{ijkl} = \mathbf{g} \otimes \mathbf{g} = g_{ij}\, g_{kl} \qquad \mathbf{a}, \mathbf{b}, \mathbf{f}, \mathbf{g} \in \mathbf{L} \tag{5b}$$

$$\mathbf{E}_3 = E^3_{ijkl} = \frac{1}{2}\,(a_{ik}\, b_{jl} + a_{il}\, b_{jk} + a_{jk}\, b_{il} + a_{jl}\, b_{ik}) \tag{5c}$$

where $\mathbf{L}$ is the second-rank symmetric tensor space over $\mathbf{R}^3$.

Tensors $\mathbf{f}$ and $\mathbf{g}$ are axisymmetric and depend on the components of the compliance tensor $\mathbf{S}$. They are given by:

$$\mathbf{f} = \cos\omega_p\, \mathbf{b} + \sin\omega_p\, \mathbf{a} \tag{6a}$$

$$\mathbf{g} = \sin\omega_p\, \mathbf{b} - \cos\omega_p\, \mathbf{a} \tag{6b}$$

with

$$\tan 2\omega_p = -\frac{2\nu_L}{E_L} \Big/ \left(\frac{1}{E_T} - \frac{1}{E_L}\right) \tag{7}$$

and the second-rank symmetric tensors $\mathbf{a}$ and $\mathbf{b}$ are defined as follows:

$$\mathbf{a} = \mathbf{k} \otimes \mathbf{k}, \quad \mathbf{b} = \mathbf{j} \otimes \mathbf{j}, \tag{8}$$

with $\mathbf{k}$ and $\mathbf{j}$ the unit vectors of $\mathbf{R}^3$, associated with the 33 and 11-directions of the Cartesian coordinate system. For the eigenvalues and the associated idempotent tensors defined in relations (5),...,(8), it is valid that:

$$\mathbf{S} = \lambda_1\mathbf{E}_1 + \lambda_2\mathbf{E}_2 + \lambda_3\mathbf{E}_3 \tag{9}$$

The idempotent tensors $\mathbf{E}_m$, m = 1,...,3 decompose the unit element $\mathbf{I}$ of the fourth-rank symmetric tensor space and satisfy the following set of equations:

$$\mathbf{I} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 \tag{10a}$$

$$\mathbf{E}_m \cdot \mathbf{E}_n = 0, \quad m \neq n \tag{10b}$$

$$\mathbf{E}_m \cdot \mathbf{E}_m = \mathbf{E}_m \tag{10c}$$

Tensors $\mathbf{E}_m$, m = 1,...,3 divide the second-rank symmetric tensor space, $\mathbf{L}$, into orthogonal subspaces $\mathbf{L}_{\lambda_m}$, consisting of eigentensors of the compliance tensor $\mathbf{S}$.

It should be noted that in the case of an isotropic elastic body, it is valid that: $E_L = E_T = E$, $G_L = G_T = G$ and $\nu_L = \nu_T = \nu$. Then, relations (8) and (9) may be written as:

$$\mathbf{I} = \mathbf{E}_P + \mathbf{E}_D \tag{11a}$$

$$\mathbf{S} = \lambda_p\mathbf{E}_P + \lambda_D\mathbf{E}_D \tag{11b}$$

in which

$$\lambda_D = \frac{1}{2G}, \quad \lambda_p = \frac{1}{3K}, \quad K = \frac{E}{3(1-\nu),}. \quad \mathbf{E}_P = \frac{1}{2}(\mathbf{1} \otimes \mathbf{1}) \cdot \tag{12}$$

If the stress states $\boldsymbol{\sigma}_m$ constitute the eigenstates of tensor $\mathbf{S}$ they should satisfy the eigenvalue equation:

$$\mathbf{S} \cdot \boldsymbol{\sigma}_m = \lambda_m \boldsymbol{\sigma}_m \tag{13}$$

in which the index m varies between 1 and 3, and the λ_m values are described in terms of relations (4). Therefore, the eigentensors of the transversely isotropic compliance tensor, **S**, are derived by the orthogonal projection of a second-rank symmetric tensor $\boldsymbol{\sigma}$ on subspaces $\mathbf{L}_{\lambda m}$, produced by the linear operators $\mathbf{E}_m$, as follows:

$$\boldsymbol{\sigma}_m = \mathbf{E}_m \cdot \boldsymbol{\sigma} \ , \quad m = 1,...,3 \tag{14}$$

Denoting by $\boldsymbol{\sigma}$ the contracted stress tensor in the form of a 3-D vector, this tensor is given by:

$$\boldsymbol{\sigma} = [\sigma_1, \sigma_3, \sigma_{13}]^T \tag{15}$$

Carrying out the calculations implied by relations (13), it was found that:

$$\boldsymbol{\sigma}_1 = (\cos\omega_p(\sigma_1) + \sin\omega_p(\sigma_3))\,[\cos\omega_p, \sin\omega_p, 0]^T \tag{16a}$$

$$\boldsymbol{\sigma}_2 = (\sin\omega_p(\sigma_1) - \cos\omega_p(\sigma_3))\,[\sin\omega_p, -\cos\omega_p, 0]^T \tag{16b}$$

$$\boldsymbol{\sigma}_3 = [0, 0, \sigma_{13}]^T \tag{16c}$$

Moreover, it should be noted that relations (16) state that the stress eigenstates corresponding to a spectral decomposition of the compliance tensor **S** for transversely isotropic plates, break down the generic stress tensor $\boldsymbol{\sigma}$ into three elements, that is:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3 \tag{17}$$

As can be observed, eigentensors $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are dependent on the value of the plane eigenangle ω_p, given by relation (7), and the engineering elastic constants of the material. On the contrary, the third eigentensor $\boldsymbol{\sigma}_3$ is independent of the eigenangle ω_p and the material properties, thus remaining the same for all transversely isotropic materials. Therefore, the three eigenvalues λ_1, λ_2, and λ_3, together with the eigenangle ω_p constitute the four invariant elastic constants required for the description of the elastic behaviour of transversely isotropic plates.

If we now consider the definition of the strain energy density we have that:

$$\begin{aligned}2T(\boldsymbol{\sigma}) &= \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon} = \boldsymbol{\sigma}\cdot \mathbf{S}\, \boldsymbol{\sigma} = \\ &= (\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3)\cdot(\lambda_1\mathbf{E}_1+\lambda_2\mathbf{E}_2+\lambda_3\mathbf{E}_3)\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2+\boldsymbol{\sigma}_3) \\ &= \lambda_1\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_1+\lambda_2\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_2+\lambda_3\boldsymbol{\sigma}_3\cdot\boldsymbol{\sigma}_3 \end{aligned} \tag{18}$$

Therefore, the strain energy density is given by:

$$2T(\boldsymbol{\sigma}) = T(\boldsymbol{\sigma}_1) + T(\boldsymbol{\sigma}_2) + T(\boldsymbol{\sigma}_3) \tag{19}$$

that is the elastic potential is decomposed in distinct energy components, each associated with the same eigenstress tensor. We denote by $T(\boldsymbol{\sigma}_m)$ the following quantity:

$$T(\boldsymbol{\sigma}_m) = \lambda_m(\boldsymbol{\sigma}_m\cdot\boldsymbol{\sigma}_m), \quad m = 1,...,3 \tag{20}$$

Thus, any eigenstate $\boldsymbol{\sigma}_m$ has its own potential $T(\boldsymbol{\sigma}_m)$, which does not depend on the action of the other $\boldsymbol{\sigma}_m$'s. From relations (18), (19) and (20) it can be seen that, in order for the strain energy density T to be positive definite, the eigenvalues have to be positive definite, that is:

$$\lambda_m(m = 1...3) > 0 \tag{21}$$

Substituting relations (16) into relation (20), the following expressions are obtained for the strain energy density parts of a transversely isotropic medium:

$$\begin{aligned}T(\boldsymbol{\sigma}_1) = &\left\{\frac{1}{2E_L}+\frac{1}{2E_T}+\left[\left(\frac{1}{2E_L}-\frac{1}{2E_T}\right)^2+\frac{\nu_L^2}{E_L^2}\right]^{1/2}\right\} \\ &\times\left[\cos\omega_p(\sigma_1)+\sin\omega_p(\sigma_3)\right]^2\end{aligned} \tag{22a}$$

$$\begin{aligned}T(\boldsymbol{\sigma}_2) = &\left\{\frac{1}{2E_L}+\frac{1}{2E_T}-\left[\left(\frac{1}{2E_L}-\frac{1}{2E_T}\right)^2+\frac{\nu_L^2}{E_L^2}\right]^{1/2}\right\} \\ &\times\left[\sin\omega_p(\sigma_1)-\cos\omega_p(\sigma_3)\right]^2\end{aligned} \tag{22b}$$

$$T(\boldsymbol{\sigma}_3) = \frac{1}{2G_L}(\sigma_{13})^2 \tag{22c}$$

It is observed from relations (22) that the two energy components $T(\boldsymbol{\sigma}_1)$ and $T(\boldsymbol{\sigma}_2)$ depend upon the value of the plane eigenangle ω_p given by relation (7), and are associated with both shape distortion and volume change of the medium. The third energy component $T(\boldsymbol{\sigma}_3)$ is independent of the value of the eigenangle ω_p and is solely associated with shape distortion of the medium.

Finally, the value of the plane eigenangle ω_p should be compared with the value of the eigenangle ω obtained by the general spectral decomposition of the compliance tensor, defined as:

$$\tan 2\omega = -\frac{2\sqrt{2}\nu_L}{E} \Big/ \left(\frac{1-\nu_T}{E_T} - \frac{1}{E_L}\right) \tag{23}$$

It should be noted that the value of the eigenangle ω is associated to that of the plane eigenangle ω_p when the value of the transverse Poisson's ratio ν_T vanishes. Bearing in mind that the stress eigenstates, obtained by the spectral decomposition of the compliance tensor on the principal stress space, are aligned in parallel and normally to the principal diagonal plane (σ_3, δ_{12}), then, the $\sqrt{2}$ factor corresponds to the projection of these eigenstates on the principal stress plane (σ_1, σ_3) over an angle $\pi/4$.

3. VARIATION INTERVAL OF THE EIGENANGLE ω_p

When, the eigentensors $\boldsymbol{\sigma}_m$, $m = 1...3$ are projected on the principal stress plane (σ_1, σ_3), the projection of eigentensor $\boldsymbol{\sigma}_3$ vanishes. The projections of $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are represented by two orthogonal vectors with associated unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ having as direction cosines (Fig. 1):

$$\mathbf{e}_1 = [\cos\omega_p, \sin\omega_p]^T \tag{24a}$$

$$\mathbf{e}_2 = [\sin\omega_p, -\cos\omega_p]^T \tag{24b}$$

The unit vector $\mathbf{e}_1$ subtends with axis $O\sigma_3$ an angle equal to $(\omega_p - \pi/2)$, whereas the unit vector $\mathbf{e}_2$ is inclined to the same axis by an angle $(\pi - \omega_p)$.

It should be noted that the elastic strain energy associated with both the eigendeformation tensor $\bar{\mathbf{e}}_1 = \lambda_1\boldsymbol{\sigma}_1$ and the eigendeformation tensor $\bar{\mathbf{e}}_2 = \lambda_2\boldsymbol{\sigma}_2$ is a mixed type of energy, that is both distortional and dilatational.

Moreover, since the eigentensors $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ depend on material elastic properties, their corresponding strain energies differ for the various kinds of transversely isotropic solids.

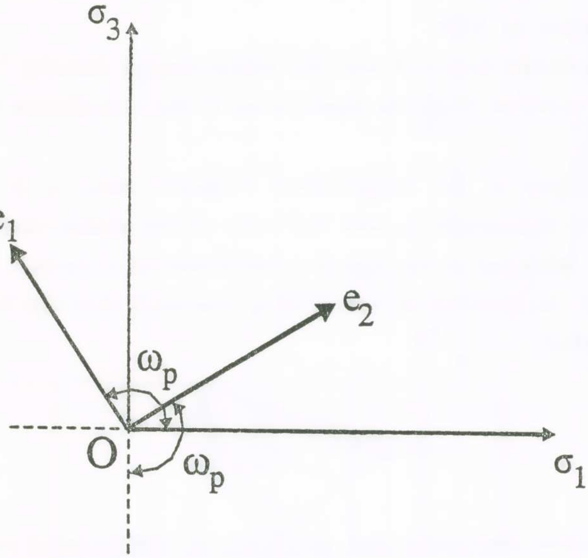

Fig. 1. Geometric representation of the eigentensors of the transversely isotropic compliance tensor on the principal stress plane (σ_1, σ_3).

For an isotropic material, for which $E_L = E_T = E$ and $\nu_L = \nu_T = \nu$, the eigenangle ω_p takes the value $\omega_p = 135^o$. As a result, the eigentensor $\boldsymbol{\sigma}_2$ becomes a spherical tensor, whereas $\boldsymbol{\sigma}_1$ becomes a deviatoric tensor, their corresponding strain energies being the dilatational and distortional elastic energy respectively.

$$\mathbf{e}_1 = [-1/\sqrt{2}, 1/\sqrt{2}]^T \tag{25a}$$

$$\mathbf{e}_2 = [1/\sqrt{2}, 1/\sqrt{2}]^T \tag{25b}$$

In addition, it can be shown that the eigenangle ω_p takes values in the interval (0^o, 180^o), whereas for an isotropic material it was shown that ω_p equals 135^o. However, there are two possible angles ω_p which correspond to the isotropic medium, namely 45° and 135°. The value of ω_p equal to 45° is associated with a negative value of Poisson's ratio, whereas the value of ω_p equal to 135° refers to a positive one.

However, although a negative value of Poisson's ratio is admissible in terms of thermodynamics, isotropic elastic behavior will be associated to a value of ω_p equal to 135°.

The compliance tensor **S** and the strain energy density T can be assured to be strictly positive, if all the eigenvalues of the compliance tensor are strictly positive.

The variation of the longitudinal Poisson's ratio ν_L is shown in Fig. 2 in terms of the eigenangle ω_p and the ratio of the elastic moduli E_L/E_T. Each of the graphs included in the figure corresponds to a distinct value of the ratio E_L/E_T and the interval of values of ν_L was such that the following inequality was satisfied:

$$|\nu_L| \leqslant \left(\frac{E_L}{E_T}\right)^{1/2} \tag{26}$$

An extensive discussion and definition of the limits of variation of Poisson's ratios in anisotropic materials based on a more stringent bound for these variables is undertaken in a forthcoming companion paper. See also Theocaris (1994; 1996), and Theocaris and Philippidis (1992; 1994).

As far as the variation of the eigenangle ω_p with respect to the engineering elastic constants is concerned, for some values of the ratio of longitudinal to transverse elastic moduli E_L/E_T, it is observed that diminishing the ratio E_L/E_T reduced significantly the interval of variation of the eigenangle ω_p. When $E_L/E_T = 0.4$, the interval of variation of ω_p is approximately [58°, 122°], whereas the interval of values ν_L is [-0.63, 0.63]. However, since no negative value of ν_L has ever been measured, the associated phenomenological interval of values of ω_p is the set [90°, 122°].

In the same figure, the ratio of the moduli was considered equal to unity ($E_L/E_T = 1$). This special case corresponds to an isotropic material with different values of Poisson's ratio ν_L. Then, it is noted that negative values of Poisson's ratio correspond to $\omega_p = 45°$, whereas positive Poisson's ratios yields a value of $\omega_p = 135°$.

When the ratio of the moduli E_L/E_T was set equal to two, the thermody-

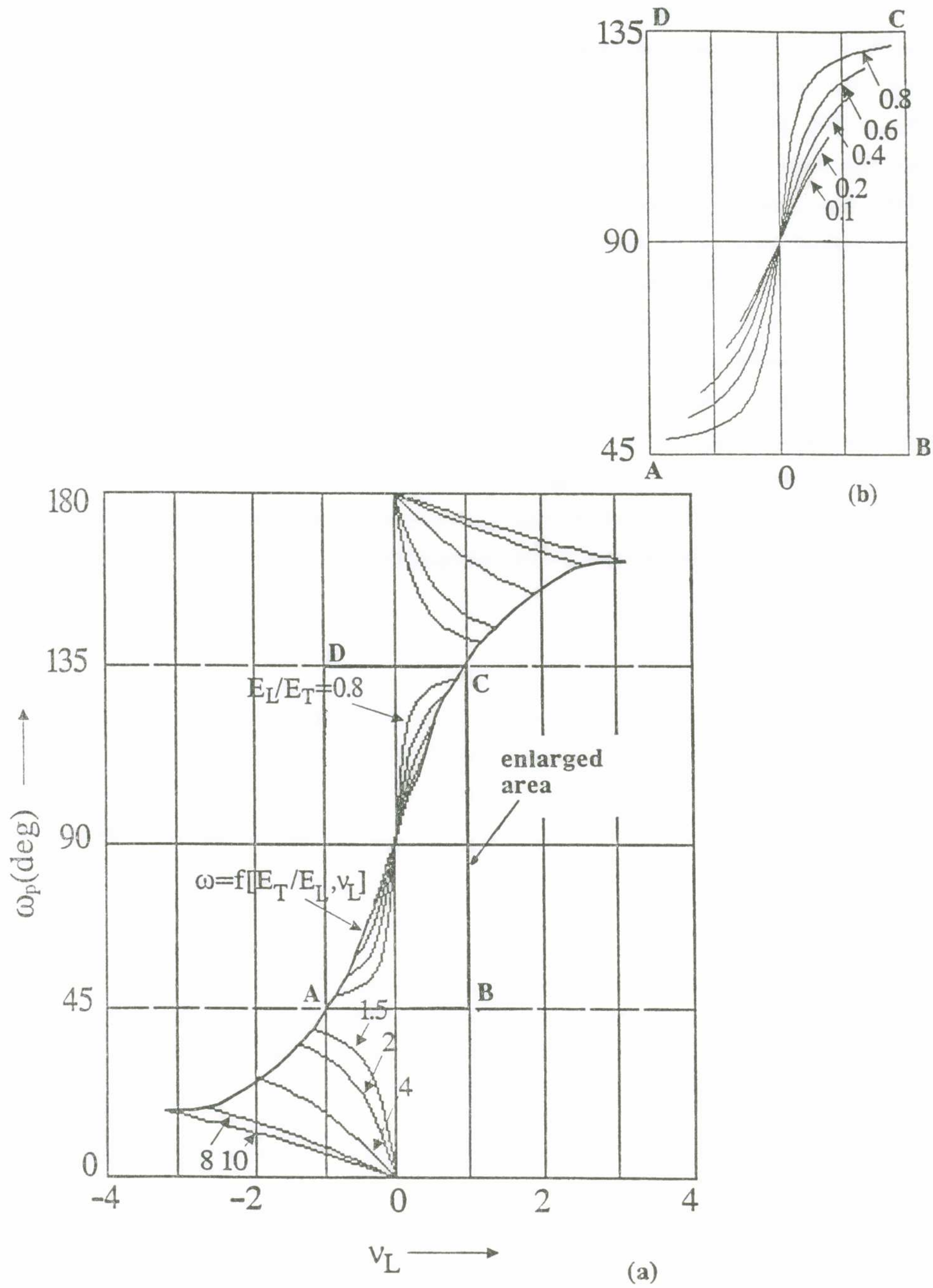

Fig. 2. Variation of the longitudinal Poisson's ratio ν_L with the eigenangle ω_p for the entire thermodynamically acceptable spectrum of values for some values of the ratio of longitudinal to transverse elastic moduli E_L/E_T.

namically admissible interval of variation of the eigenangle ω_p was expanded to [0^o, 175^o], whereas the interval of values of ν_L was [- 1.414, 1.414]. In addition, the corresponding phenomenological interval of eigenangle ω_p values is [90^o, 175^o]. Finally, when the ratio of the moduli E_L/E_T was taken equal to ten, the interval of values of the eigenangle ω_p was further expanded to [0^o, 180^o], the interval of values of ν_L was [-3.16, 3.16], and the associated phenomenological interval of variation of the eigenangle ω_p was [90^o, 180^o].

In Table 1, the value of the eigenangle ω_p, as well as the engineering elastic constants are tabulated for a large spectrum of transversely isotropic-composite media and for real natural materials possessing an axis of infinitc order of elastic symmetry, as for example crystals of the hexagonal system. The first section of Table 1 contains fiber composite solids with various fiber and matrix types. The second section is comprised of inorganic crystals of the hexagonal system. The third section contains a compression annealed pyrolytic graphite material and woven fabric comporites.

TABLE 1.

The values of the elastic properties and the plane eigenangle ω_p for a series of transversely isotropic media

Transversely isotropic medium	GPa			ν_L	ν_T	ω_p	Ref.
	E_L	E_T	G_L				
Thornel 75S Gr/Epoxy	305.19	6.464	6.3	0.366	0.539	179.55	
Thornel 50S Gr/Epoxy	221.73	7.450	6.1	0.278	0.491	179.45	Smith, 1972
Courtaulds HTS C/Epoxy	158.76	10.623	6.7	0.300	0.398	178.77	
T 300/5208 Gr/Epoxy	125.36	10.624	5.9	0.327	0.410	178.27	Knight, 1982
Thornel 50 Gr/Aluminum	160.00	29.800	18.9	0.440	0.419	174.31	Blessing and Elban, 1981
Borsic 1100 Aluminum	233.53	138.03	C0.5	0.240	0.410	162.62	Gieske and Allred, 1974

Boron/Aluminum alloy 6061	230.00	139.00	56.9	0.170	0.480	166.28	Read and Ledbetter, 1977
SiO_2 β-quartz	94.16	106.27	36.1	0.247	0.064	128.51	
$BaTiO_3$	147.92	122.25	54.6	0.238	0.364	146.90	
H_2O (-16°C)	11.765	9.620	3.184	0.282	0.413	145.79	
$BaTiO_3$ + 5% $CaTiO_3$	118.76	124.22	47.4	0.314	0.304	132.99	Huntington, 1958
Co	313.48	211.86	75.53	0.216	0.489	158.99	
Cd	28.17	81.3	18.52	0.262	0.122	109.36	
Zn	35.24	119.33	38.3	0.257	−0.063	108.05	
Compression Annealed, Pyrolytic Graphite	37.00	920	0.250	0.010	0.160	90.60	Blakslee et al., 1970
Graphite/Epoxy 1	7.36	148	3.82	0.52	0.31	113.88	
Graphite/Epoxy 2	9.31	132	4.61	0.48	0.28	112.18	Ishikawa and Chou, 1982
Graphite/Epoxy 3	8.82	113	4.46	0.42	0.30	111.18	
Kevlar/Epoxy	5.5	85.3	2.54	0.50	0.40	113.47	Zweben and Norman, 1976
Glass/Epoxy	15.9	47.5	6.23	0.40	0.27	115.14	Chevalier and Nouamani, 1990
Glass/Polyimide	15.7	41.2	5.59	0.46	0.30	118.05	Ishikawa and Chou, 1982

4. DISCUSSION

The spectral decomposition of the elastic compliance fourth-rank tensor **S** for transversely isotropic plates permits the separation of the stress and strain tensors in energy-orthogonal components.

The decomposition of the stress tensor $\boldsymbol{\sigma}$ obtained for transversely isotropic solids, yielded three energy-orthogonal stress states, which separate the elastic strain energy directly. The stress tensor may be effectively described by eigentensors $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$. Additionally, the normality of the eigentensors of stress and strain corresponding to a different that the former stress - eigentensor was shown.

Furthermore, the three eigenvalues of the compliance tensor **S** together with the value of the eigenangle ω_p may be used together for an invariant description of the elastic behavior. According to relation (7), the eigenangle ω_p for an isotropic body is equal to 135° and, in general, varies between 0° and 180°. For highly anisotropic fiber composites, the value of the eigenangle ω_p is very nearly equal to 180°, whereas for matrix composites, characterised by a moderate anisotropy, the values of the eigenangle ω_p approach the limiting case of the isotropic material, that is 135°.

Considering for example, Thornel 75S Gr/Epoxy (Smith, 1972), the value of the eigenangle ω_p was equal to 179.55°, and therefore, the resulting strain tensor from a pure hydrostatic loading differs much from the spherical one. As a result, this material does not possess the property of isotropic materials to exhibit extremely high strength under hydrostatic pressure. On the other hand, by taking as an example the crystal $BaTiO_3 + 5\%\ CaTiO_3$ (Huntington, 1958), the eigenangle ω_p was evaluated equal to 137.16°, so that the stress eigentensor $\boldsymbol{\sigma}_1$ is very nearly a hydrostatic loading, and the associated strain eigentensor is almost a spherical tensor. Consequently, these materials possess nearly infinite strength under hydrostatic loading.

According to equation (7), the value of the eigenangle ω_p is a function of the longitudinal and transverse elastic moduli E_L, E_T and the longitudinal Poisson's ratio ν_L. In addition, the eigenangle ω_p is independent of the value of the longitudinal shear modulus G_L. However, the value of the shear modulus, G_L, is very important for the characterisation of the fracture toughness of a material, since it is responsible for the distribution of strains. Moreover, according to the classical anisotropic elasticity theory (Lekhnitskii, 1968), the stress concentration factor K_T in the presence of an elliptic crack in a transversely isotropic plate loaded in tension along the strongest material direction is given by (Theocaris and Philippidis, 1989c):

$$K_T = 1 + \left[2\left(\frac{E_L}{E_T}\right)^{1/2} + 2\left(\frac{E_L}{2G_L} - \nu_L\right) \right]^{1/2} \frac{a}{b} \tag{27}$$

in which b/a denotes the ratio of the elliptic crack semi-axes.

Therefore, the dependence between the eigenangle ω_p and the longitudinal elastic modulus G_L was sought for, by plotting the relationship between the eigenangle ω_p and the ratio $E_L/2G_L$ in Fig. 3. It is clear from the graph that the ratio $E_L/2G_L$ rises to very high values as the value of the eigenangle ω_p tends to either 90^o or 180^o. On the contrary, as the eigenangle ω_p approaches the limiting value 135^o corresponding to an isotropic body, the ratio $E_L/2G_L$ is decreased, holding a very steady value, for eigenangles belonging in the interval [110^o, 170^o].

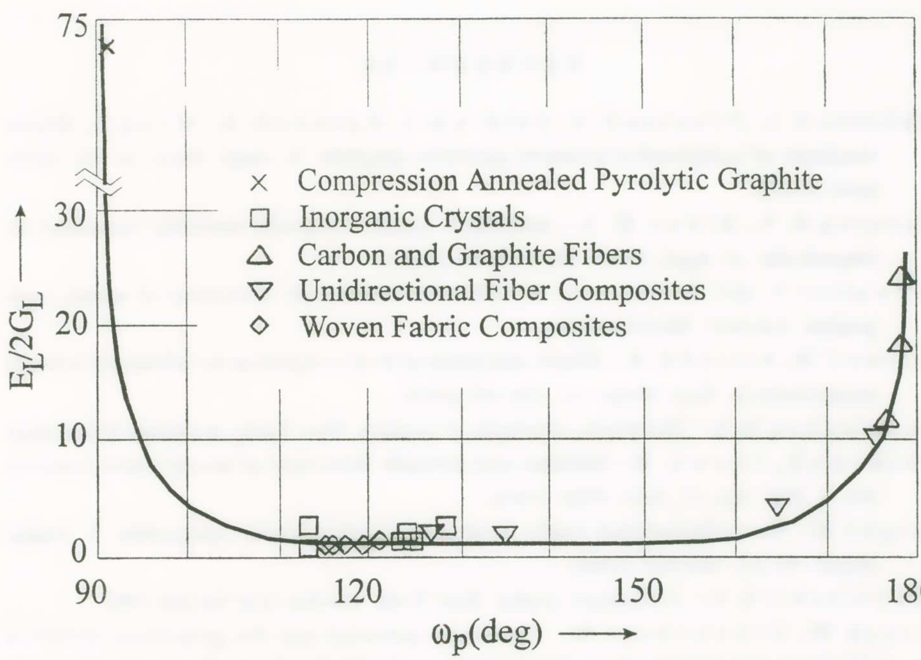

Fig. 3. Phenomenological functional dependence of ratio $E_L/2G_L$ on the plane eigenangle ω_p.

The continuous curve of Fig. 3 is supposed to represent the mean behavior of $E_L/2G_L$ with respect to the eigenangle ω_p for all the experimental points. In the interval [90^o, 135^o] lie all the weak-axis and woven fabric composites, whereas in the region [135^o, 180^o] which is quite similar, if not identical, to the behavior in the interval [90^o, 135^o] lie all the strong axis fiber reinforced materials.

It should be noted that the results mentioned above are in accordance with what was found for the spectral decomposition of the compliance tensor in the principal stress space (Theocaris and Philippidis, 1990). Therefore, once more, it is concluded that departure of the eigenangle ω_p from the value corresponding to the isotropic body leads to an increase in the value of the ratio $E_L/2G_L$. On the contrary, if G_L itself increases, while the other elastic constants remain the same, the eigenangle ω_p approaches the value 135°, leading to enhanced fracture toughness. In conclusion, the eigenangle ω_p serves as a single parameter characterising the elasticity and toughness of transversely isotropic materials on the principal stress plane (σ_1, σ_3).

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ΠΕΡΙΛΗΨΙΣ

Φασματικὴ ἀνάλυσις τῶν τανυστῶν ἐνδόσεως S εἰς ἐπιπέδους πλάκας ἀνισοτρόπων ὑλικῶν

Ἡ θεωρία τῆς φασματικῆς ἀναλύσεως τοῦ τανυστοῦ ἐνδόσεως **S** ἐπεκτείνεται εἰς τὸ ἄρθρον αὐτὸ δι' ἐπίπεδα προβλήματα τάσεων. Δημιουργεῖται τοιουτοτρόπως ἡ δυνατότης διὰ τὸν χαρακτηρισμὸν τῶν ἐλαστικῶν ἰδιοτήτων τῶν ἀνισοτρόπων μέσων εἰς ἐπιπέδους πλάκας ὑπὸ συνθήκας ἐπιπέδου ἐντατικῆς καταπονήσεως. Θεωροῦμεν πρὸς τοῦτο καρτεσιανὸν σύστημα συντεταγμένων πρὸς τὸ ὁποῖον ἀναφέρονται οἱ τανυσταὶ τάσεως, **σ**, καὶ παραμορφώσεως, **ε**, καὶ τοῦ ὁποίου αἱ διευθύνσεις ταυτίζονται μὲ τὰς κυρίας διευθύνσεις τοῦ μέσου, μὲ τὸν ἄξονα 33 ὡς τὸν ἰσχυρὸν ἄξονα τοῦ μέσου, κάθετον ἐπὶ τοῦ ἰσοτρόπου (ἐγκαρσίου) ἐπιπέδου.

Ὑπολογίζονται αἱ ἐκφράσεις τῶν τριῶν ἰδιοτιμῶν τοῦ **S** συναρτήσει τῶν ἐλαστικῶν μέτρων καὶ τοῦ λόγου Poisson τοῦ μέσου. Ἐν συνεχείᾳ δίδεται τὸ σύνολον τῶν τανυστῶν ($\mathbf{E}_M$), οἱ ὁποῖοι ἀναλύουν τὸν μοναδιαῖον τανυστὴν **I**. Αὐτοὶ ὁρίζονται ἀπὸ τοὺς συμμετρικοὺς τανυστὰς **f** καὶ **g**, οἱ ὁποῖοι ἐν συνεχείᾳ ὁρίζονται ἀπὸ τοὺς συμμετρικοὺς τανυστὰς **a** καὶ **b**. Τοιουτοτρόπως, ὁρίζεται πλήρως ἡ φασματικὴ ἀνάλυσις τοῦ τανυστοῦ ἐνδόσεως **S**. Τέλος, ὁρίζεται «νέα» ἀδιάστατος παράμετρος ω_p, ἡ ὁποία ὀνομάζεται **ἐπίπεδος ἰδιογωνία**, ἡ ὁποία ἐκφράζεται συναρτήσει τῶν ἐλαστικῶν σταθερῶν τοῦ μέσου. Ἡ ἰδιογωνία αὐτή, μετὰ τῶν ἰδιοτιμῶν τοῦ **S** ἀποτελοῦν τὰς ἀναγκαίας παραμέτρους διὰ τὴν ἀναλλοίωτον περιγραφὴν τῆς ἐλαστικῆς συμπεριφορᾶς τῶν ἐγκαρσίως ἰσοτρόπων πλακῶν.

Ἐν συνεχείᾳ, ὁρίζονται οἱ τρεῖς ἰδιοτανυσταὶ $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ καὶ $\boldsymbol{\sigma}_3$ συναρτήσει τῶν συνιστωσῶν τῶν τάσεων σ_1, σ_3 καὶ σ_{13}, καθὼς ἐπίσης καὶ τῆς ἐπιπέδου ἰδιογωνίας ω_p. Ἀποδεικνύεται ὅτι οἱ ἰδιοτανυσταὶ $\boldsymbol{\sigma}_1$ καὶ $\boldsymbol{\sigma}_2$ ἐξαρτῶνται μονοσημάντως ἐκ τῆς ἰδιογωνίας ω_p. Ἀντιθέτως ὁ τρίτος ἰδιοτανυστὴς $\boldsymbol{\sigma}_3$ εἶναι ἀποκλίνων καὶ παραμένει σταθερὸς δι' ὅλα τὰ ἐγκαρσίως ἰσότροπα σώματα. Ἀποδεικνύεται ἐπιπλέον ὅτι ὁ ἰδιοτανυστὴς $\boldsymbol{\sigma}_3$ συνδέεται ἀποκλειστικῶς μὲ τὴν στροφικὴν ἐλαστικὴν ἐνέργειαν, ἐνῶ οἱ ὑπόλοιποι δύο ἰδιοτανυσταὶ $\boldsymbol{\sigma}_1$ καὶ $\boldsymbol{\sigma}_2$ ἐκφράζονται μὲ συνδυασμοὺς τῆς στροφικῆς καὶ διογκωτικῆς ἐνεργείας.

Μελετῶνται τόσον ἡ μεταβολὴ τῆς ἰδιογωνίας ω_p, ὅσον καὶ τὰ ὅρια τῶν τιμῶν τῶν ἀνισοτρόπων λόγων τοῦ Poisson, ὑπακούοντα εἰς τοὺς νόμους τῆς θερμοδυναμικῆς. Εὑρίσκεται ὅτι ἡ τιμὴ τῆς ἰδιογωνίας ω_p, ἡ ἀντιστοιχοῦσα εἰς τὸ ἰσότροπον σῶμα ἰσοῦται μὲ 135^o, ἐνῶ, γενικῶς, ἡ γωνία αὐτὴ διὰ τὰ ἀνισότροπα σώματα μεταβάλλεται μεταξὺ τῶν τιμῶν τῶν γωνιῶν $\omega_p = 0^o$ καὶ 180^o. Μελετῶνται αἱ τιμαὶ τὰς ὁποίας λαμβάνει ἡ ἰδιογωνία ω_p, διὰ σειρὰν ὅλην ἐγκαρσίως ἰσοτρόπων ὑλικῶν καὶ παρατηρεῖται ὅτι μὲ τὴν αὔξησιν τῆς ἀνισοτροπίας τοῦ ὑλικοῦ, ἡ τιμὴ τῆς ἰδιογωνίας ω_p προσεγγίζει ἀντιστοίχως τὰ ὅρια 90^o καὶ 180^o, ἐνῶ ἀντιθέτως, μείωσις τῆς ἀνισοτροπίας ὁδηγεῖ εἰς τιμὰς τῆς ἰδιογωνίας ω_p, πλησίον τοῦ ὁρίου τῶν 135^o. Εἰς τὴν περίπτωσιν αὐτήν, ὁ ἰδιοτανυστὴς $\boldsymbol{\sigma}_1$, προσεγγίζει τὴν ὑδροστατικὴν φόρτισιν καὶ ἑπομένως ὁ ἀντίστοιχος τανυστὴς τῆς παραμορφώσεως προσεγγίζει τὸν σφαιρικὸν τανυστήν. Συνεπῶς τὰ ὑλικὰ αὐτὰ παρουσιάζουν σχεδὸν ἀπεριόριστον ἀντοχὴν ὑπὸ θλιπτικὴν ὑδροστατικὴν φόρτισιν. Τέλος ἀποδεικνύεται ὅτι ἡ ἐπίπεδος ἰδιογωνία ω_p εἶναι ἱκανὴ νὰ χρησιμοποιηθῇ διὰ τὴν πλήρη περιγραφὴν μονοπαραμετρικῶς τῆς ἀνισοτροπίας τοῦ ὑλικοῦ ἐγκαρσίως ἰσοτρόπων ἐπιπέδων πλακῶν.