ΜΑΘΗΜΑΤΙΚΑ.— **Morse functions on differentiable manifolds,** *by*
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Let M be a closed C^∞ differentiable manifold and $f: M \rightarrow R$ be a C^∞ differentiable function. Then, f is said to be a *Morse function* if f has only nondegenerate critical points. Morse theory studies the connection between the topological structure of a differentiable manifold and the Morse functions defined on it. The terminology used in the sequel can be found in the references.

The present paper is concerned with the Morse-Smale characteristic of a closed C^∞ differentiable manifold and its relation to the structure of manifolds.

Definition. The *Morse-Smale characteristic* of M is the minimal number of critical points any Morse function on M can assume. We denote it by $\mu(M)$. More precisely,

$$\mu(M) = \min_{f \in \Omega} \sum_{i=0}^{n} c_i(M, f), \quad n = \dim M$$

where Ω is the space of Morse functions on M, and $c_i(M, f)$ is the number of critical points of index i of $f \in \Omega$.

Note, the Morse-Smale characteristic $\mu(M)$ is invariant in the category of C^∞ differentiable manifolds and differentiable maps, i. e., $\mu(M)$ is a differential topological invariant.

Theorem 1. *Let* M *be a closed* C^∞ *differentiable* 3*-dimensional manifold whose fundamental group is nonabelian. Then, the Morse-Smale characteristic* $\mu(M) \geqslant 6$.

Theorem 2. *The Morse-Smale characteristic of a closed, orientable,* 3*-dimensional manifold* M *is homotopy invariant provided* M *is homeomorphic to the connected sum of closed, orientable, irreducible,* 3*-dimensional manifolds with fundamental group being a nontrivial amalgamated free product.*

* Γ. ΡΑΣΣΙΑ, Συναρτήσεις Morse ἐπὶ διαφορισίμων πολλαπλοτήτων.

Theorem 3. *Let* $H^3 \times H^3$ *be the product of homotopy 3-dimensional spheres. Then, the Morse-Smale characteristic* $\mu(H^3 \times H^3) = 4$.

Remark. It is not known yet, whether the Morse-Smale characteristic of the product of any two closed C^∞ differentiable manifolds M, N equals to the product of the Morse-Smale characteristics of M and N, i.e., whether $\mu(M \times N) = \mu(M) \cdot \mu(N)$. If this were the case, then $\mu(H^3) = 2$ and so the Poincaré conjecture would be true since $\mu(H^3 \times H^3) = 4$ and $\mu(H^3 \times H^3) = \mu(H^3) \cdot \mu(H^3)$. If $\mu(H^3 \times H^3) \neq \mu(H^3) \cdot \mu(H^3)$ for some homotopy 3-dimensional sphere H^3 then the Poincaré conjecture would be false since $\mu(H^3) \neq 2$.

If $\mu(H^3) \leqslant \mu(H^3 \times H^3)$, then $\mu(H^3) = 2$ since $\mu(H^3 \times H^3) = 4$ and $\mu(H^3)$ is an even non-negative integer. If $\mu(H^3) > \mu(H^3 \times H^3)$, it follows that $\mu(H^3) > 4$. Thus, the Poincaré conjecture is true, if and only if, $\mu(H^3) \leqslant \mu(H^3 \times H^3)$.

Theorem 4. *(a) The Morse-Smale characteristic of any closed* C^∞ *differentiable odd-dimensional manifold is an even integer* $\geqslant 2$.

(b) The Morse-Smale characteristic of any 2-connected closed C^∞ *differentiable 6-dimensional manifold is an even integer* $\geqslant 2$.

(c) The Morse-Smale characteristic of any open (i.e., noncompact without boundary) manifold equals to zero.

(d) Let M *be a closed* C^∞ *differentiable manifold and* $p : \widetilde{M} \to M$ *be a k-fold covering space. Then,* $\mu(\widetilde{M}) \leqslant k \cdot \mu(M) - 4(k-1)$.

Theorem 5. *(a) Let* M *be a closed* C^∞ *differentiable manifold,* $\dim M = n > 4$. *Then, the Morse-Smale characteristic*

$$\mu(M^n) = \min_{f \in \Omega} \sum_{i=0}^{n} c_i(M^n, f) = \sum_{i=0}^{n} \min_{f \in \Omega} c_i(M^n, f)$$

if and only if, the Whitehead torsion $\tau(W; M^n) = 0$, *where* $(W; M^n, M^n)$ *is any h-cobordism, i.e.,* $\partial W = M^n + (-M^n)$ *and* $M^n \to W$ *is a homotopy equivalence.*

(b) Let M *be a closed* C^∞ *differentiable manifold* $\dim M = n < 4$. *Then,*

$$\mu(M^n) = \min_{f \in \Omega} \sum_{i=0}^{n} c_i(M^n, f) = \sum_{i=0}^{n} \min_{f \in \Omega} c_i(M^n, f).$$

Proof. (a) Assume that

$$\min_{f\in\Omega} \sum_{i=0}^{n} c_i(M^n, f) = \sum_{i=0}^{n} \min_{f\in\Omega} c_i(M^n, f)$$

$$\dim M = n > 4 \quad \text{and} \quad \tau(W; M^n) \neq 0.$$

Then, by the s-cobordism theorem (see Kervaire [**1**]) we have that W is not homeomorphic to $M \times I$, where $I = [0, 1]$. Now, following the ideas of the proof of the h-cobordism theorem (see Milnor [**3**] or Smale [**7**]) we conclude that W admits a Morse function f with a nonzero number of critical points (Note: $f : (W; M^n, M^n) \to (I; 0, 1)$, $f^{-1}(0) \approx M^n$, $f^{-1}(1) \approx M^n$). So, we have that

$$\min_{f\in\Omega} \sum_{i=0}^{n} c_i(M^n, f) > 0.$$

We claim, however, that

$$\min_{f\in\Omega} c_i(M^n, f) = 0$$

for each $0 \leqslant i \leqslant n$, because of the following reason.

At first by a suitable deformation of f, we can cancel the critical points of index 0, n. Then, for each critical point of index i, $1 \leqslant i \leqslant n-1$ we can insert a pair of auxiliary critical points of index $i+1$ and $i+2$, and we cancel the critical points of index i against the auxiliary critical points of index $i+1$. In this way, the critical points of index i are «traded» for an equal number of critical points of index $i+2$. Therefore,

$$\min_{f\in\Omega} c_i(M^n, f) = 0 \quad \text{for each} \quad 0 \leqslant i \leqslant n$$

and hence

$$\min_{f\in\Omega} \sum_{i=0}^{n} c_i(M^n, f) > \sum_{i=0}^{n} \min_{f\in\Omega} c_i(M^n, f)$$

which is a contradiction to our assumption.

Now, assume $\tau(W; M^n) = 0$. Then, by the s-cobordism theorem, W is homeomorphic to $M \times I$ and therefore W admits a Morse function

$$f : (W; M^n, M^n) \to (I; 0, 1), \quad f^{-1}(0) \approx M^n, \quad f^{-1}(1) \approx M^n$$

having no critical points. Thus,

$$\min_{f \in \Omega} \sum_{i=0}^{n} c_i(M^n, f) = 0.$$

However

$$\sum_{i=0}^{n} \min_{f \in \Omega} c_i(M^n, f) = 0$$

by the previous argument and so,

$$\min_{f \in \Omega} \sum_{i=0}^{n} c_i(M^n, f) = \sum_{i=0}^{n} \min_{f \in \Omega} c_i(M^n, f).$$

(b) We show that if $\dim M = n < 4$, then

$$\min_{f \in \Omega} \sum_{i=0}^{n} c_i(M^n, f) = \sum_{i=0}^{n} \min_{f \in \Omega} c_i(M^n, f).$$

If $n = 1$, it is obviously true. If $n = 2$, it can also be proved that the equality holds.

If $n = 3$, then there exists a Morse function f on M^3 having a single critical point of index 0, and a single one of index 3, i.e.,

$$c_0(M^3, f) = c_3(M^3, f) = 1.$$

However, the Euler characteristic of M^3, equals to zero. Thus, $c_1(M^3, f) = c_2(M^3, f)$ and so

$$\min_{f \in \Omega} \sum_{i=0}^{3} c_i(M^3, f) = \min_{f \in \Omega} \left(2 + 2c_1(M^3, f)\right) =$$

$$= 2 + 2 \min_{f \in \Omega} c_1(M^3, f) = \sum_{i=0}^{3} \min_{f \in \Omega} c_i(M^3, f).$$

Q. E. D.

Question. Is it true that

$$\min_{f \in \Omega} \sum_{i=0}^{4} c_i(M^4, f) = \sum_{i=0}^{4} \min_{f \in \Omega} c_i(M^4, f).$$

Remark. If M^n, $n > 4$ is simply-connected, then the Whitehead group of M^n, $Wh(M^n) = 0$, and so $\tau(W; M^n) = 0$. Therefore

$$\mu(M^n) = \min_{f \in \Omega} \sum_{i=0}^{n} c_i(M^n, f) = \sum_{i=0}^{n} \min_{f \in \Omega} c_i(M^n, f).$$

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ΠΕΡΙΛΗΨΙΣ

Ἡ παροῦσα ἐργασία ἀναφέρεται εἰς τὴν μελέτην ὡρισμένων ἰδιοτήτων τῆς χαρακτηριστικῆς τῶν Morse - Smale ὡς ἐπίσης τῆς σχέσεως αὐτῆς τῆς χαρακτηριστικῆς μὲ τὸ πρόβλημα τῆς δομῆς τῶν διαφορικῶν πολλαπλοτήτων καὶ εἰδικῶς μὲ τὴν εἰκασίαν τοῦ Poincaré (Poincaré conjecture).

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