ΜΑΘΗΜΑΤΙΚΑ.— **On continuous homomorphisms between topological tensor algebras,** *by Anastasios Mallios**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φιλ. Βασιλείου.

The purpose of the present paper is to give an abstract treatment, within the context of the general theory of topological tensor (product) algebras, concerning certain particular features of (continuous algebra) homomorphisms between «generalized group algebras» consisting of vector-valued functions, as these algebras have been considered, for instance, in Ref. [**1**]. Now, the later algebras constitute a particular instance of topological tensor product algebras [**3**], and the present discussion is essentially founded upon the basic formula (decomposition) relating the spectrum of an abstract (topological) tensor (product) algebra to the spectra of the factor algebras [**5**], as well as its refinements (cf., for instance, Ref. [**4**], p. 104, Theorem 2.1). On the other hand, the main feature of the results contained herein, is an analogous decomposition of a continuous algebra homomorphism, between suitable tensor product algebras, in case the (algebra) homomorphism considered preserves, in an appropriate sense, the first factor algebra of its domain of definition (cf. Theorems 2.1 and 2.2 below). Besides, the results obtained specialize to those of A. Hausner in [**1**], whose paper has also been the motive to this study.

1. The algebras considered in the following are linear associative ones over the complex number field. On the other hand, the topological spaces involved are supposed to be Hausdorff. We use in the sequel the terminology of [**4**] concerning the general theory of topological tensor product algebras. Besides, we also refer to [**7**] regarding, in particular, the class of the locally m-convex topological algebras.

Now, given the topological algebras E, F we denote by *H*om (E, F) the set of all continuous (algebra) homomorphisms between them, which is also considered as a topological space, denoted by Hom_s (E, F), the corresponding topology on it being that of the simple convergence in E.

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On the other hand, we denote by L_s (E, F) the corresponding space of continuous linear maps between the topological vector spaces indicated, topologized as above.

We start with the following lemma which will be used in the sequel (cf. Theorem 2.1 below). Its proof being plausible is omitted. Thus, we have.

Lemma 1.1. Let E, F, G be topological algebras and let $u \in \mathcal{H}om$ (E, F). Moreover, let

(1.1) $$^{t}u : L_s(F, G) \to L_s(E, G)$$

be the corresponding «transpose map» of u, with respect to (the topological vector space) G. Then, one has

(1.2) $$\mathrm{Im}\,({}^{t}u\,|_{\mathrm{Hom}\,(F,\,G)}) \subseteq \mathcal{H}om\,(E, G).$$

Besides, we also need the following.

Lemma 1.2. Let E, F be (commutative) semi-simple topological algebras and let $E \hat{\otimes}_{\tau} F$ be the respective complete topological tensor (product) algebra, under a «faithful» topology τ on $E \otimes F$ [**3**]. Moreover, let z be an element of $E \hat{\otimes}_{\tau} F$ with $z \neq 0$, in such a way that one has the relation

(1.3) $$\hat{z} = \hat{x}\varrho,$$

with $x \in E$, concerning the corresponding Gel'fand transforms of the elements indicated, and ϱ being a continuous complex-valued function on M(F) (: the *spectrum* [**4**] of the topological algebra F). Then, there exists an element $y \in F$ such that one has $z = x \otimes y$.

Proof: If $x \otimes y \in E \otimes F$, one defines a map.

(1.4) $$\varphi_{x,\,y} : M(E) \to F : f \to \varphi_{x,\,y}(f) := f(x)y,$$

which is obviously continuous, so that for every $z \in E \hat{\otimes} F$, one defines a continuous map $\varphi_z : M(E) \to F$, extending (1.4) by linearity and then by continuity. Now, if $x \neq 0$, since E is semi-simple there exists an $f_0 \in M(E)$ such that $\hat{x}(f_0) = f_0(x) \neq 0$. On the other hand, consider the relation:

(1.5) $$y = (1/\hat{x}(f_0))\,\varphi_z(f_0) \in F.$$

Now, for every $g \in M(F)$, one obtains by (1.5),

$$g(y) = (1/\hat{x}(f_0))\, g(\varphi_z(f_0)) = (1/\hat{x}(f_0))\, \hat{z}(f_0 \otimes g)$$

$$= (1/\hat{x}(f_0))\, (\widehat{x\varrho})(f_0 \otimes g) = (1/\hat{x}(f_0))\, \hat{x}(f_0)\, \varrho(g) = \varrho(g),$$

that is, we obtain $g(y) = \hat{y}(g) = \varrho(g)$, for every $g \in M(F)$, and hence one has $\varrho = \hat{y}$. Therefore, $\hat{z} = \widehat{x\varrho} = \hat{x}\hat{y} = \widehat{x \otimes y}$, so that since $E \hat{\otimes}_\tau F$ is semisimple (cf., for instance, [3; p. 252, Theorem 2.1]), one obtains $z = x \otimes y$, and this finishes the proof of the lemma.

We conclude this section with the following theorem whose one half will be used in the sequel (cf. Theorem 2.1 below), and which also has an independent, interest per se. Thus, we have:

Theorem 1.1. Let E, F be topological algebras with locally equicontinuous spectra $M(E)$, $M(F)$ respectively, and let $E \hat{\otimes}_\tau F$ be the completion of the corresponding tensor (product) algebra $E \otimes F$ under an «admissible» topology τ [**3**]. Then, the following assertions are equivalent:

1) The Gel'fand maps of the algebras E and F are continuous.

2) The Gel'fand map of the algebra $E \hat{\otimes}_\tau F$ is continuous.

Proof: 1) $\Longrightarrow$ 2): By [4; p. 104, Theorem 2.1], one has, concerning the spectra of the topological algebras involved, the relation $M(E \hat{\otimes}_\tau F) = M(E) \times M(F)$, within a homeomorphism. Hence, if $K \subseteq M(E \hat{\otimes}_\tau F)$ is a compact subset, then $K \subseteq pr_1(K) \times pr_1(K)$, where $pr_1(K) \subseteq M(E)$ and $pr_2(K) \subseteq M(F)$ are compact subsets of the spectra indicated, so that they also are equicontinuous subsets of the same spaces by hypothesis and Ref. [5; p. 305, Theorem 3.1]. Theorefore (cf. also [3; p. 247, Definition 1.1]), $pr_1(K) \otimes pr_2(K) \subseteq M(E) \otimes M(F) = M(E \otimes_\tau F) = M(E \hat{\otimes}_\tau F)$ is an equicontinuous subset of $M(E \hat{\otimes}_\tau F)$ and a fortiori of K, which proves the assertion (cf. [5], p. 305, Theorem 3.1).

2) $\Longrightarrow$ 1): We shall prove that the map $g: E \rightarrow C_c(M(E))$ is continuous. Indeed, let (x_δ) be a net of elements of E converging to $0 \in E$. Now, if $K \subseteq M(E)$ is compact, there exist elements $y \in F$ and $g \in M(F)$ with $\hat{y}(g) \neq 0$, so that, since the net $(x_\delta \otimes y)$ converges to $0 \in E \hat{\otimes}_\tau F$,

one concludes by hypothesis that the net $\widehat{(x_\delta \otimes y)}$ converges to 0 in $C_c(\mathcal{M}(E \hat{\otimes}_\tau F))$, so that one has that it «finally» admits a given arbitrary bound on the compact set $K \times \{g\} \subseteq \mathcal{M}(E) \times \mathcal{M}(F) = \mathcal{M}(E \hat{\otimes}_\tau F)$, and hence one obtains the analogous conclusion for the net $(\hat{x}_\delta)$ in $C_c(\mathcal{M}(E))$ on the compact set K, which proves the assertion. An analogous argument can be provided for the corresponding Gel'fand map of the algebra F, and the proof of the theorem is completed.

2. The present section contains the main results of this paper, which also motivated the whole material presented herein. Thus, we start with the following.

Theorem 2.1. Let E, F, G, H be topological algebras such that E has an approximate identity, F has an identity element 1_F, G is complete and semi-simple with a locally equicontinuous spectrum such that the corresponding Gel'fand map is continuous, and the algebra H is semi-simple, it has an identity element 1_H and a locally equicontinuous spectrum such that the respective Gel'fand map is continuous. Moreover, suppose that the following condition holds true, concerning the algebras E, F, G :

(1) For any $T \in Hom(E \hat{\otimes}_\tau F, G)$ and $\varphi \in Hom(E, G)$ with $T(x \otimes 1_F) = \varphi(x)$, for every $x \in E$, there exists an $f \in \mathcal{M}(F)$ such that $T = \varphi \otimes f$, where τ denotes an «admissible» topology on the respective tensor product algebra [**3**].

Then, for any $T \in Hom(E \hat{\otimes}_\tau F, G \hat{\otimes}_\sigma H)$ and $\varphi \in Hom(E, G)$, with $T(x \otimes 1_F) = \varphi(x) \otimes 1_H$, for every $x \in E$, there exists an element $\varrho \in Hom(F, H)$ such that $T = \varphi \otimes \varrho$, where σ denotes a «faithful» topology [**3**] on the tensor product algebra $G \otimes H$.

Scholium 2.1. The class of the topological algebras considered in the preceding theorem is to be specified in such a way that the results exhibited, for instance, in Ref. [**4**] to be valid. In particular, one can apply locally m-convex topological algebras [**7**].

On the other hand, concerning the cond. (1) of the same theorem, we remark that this is automatically verified if, in particular, the algebras E and G have also identity elements and the sets $\mathcal{H}om\,(E, G)$ and $\mathcal{H}om\,(F, G)$ are locally equicontinuous subsets of the respective spaces of linear maps, so that the assertion is now a consequence of Theorem 3.1 in Ref. [6; p. 80].

Proof of Theorem 2.1. Let (u_α) be an approximate identity of the algebra E. Then, for any $x \in E$ and $y \in F$, one has

$$x \otimes y = \big(\lim_{\alpha}(u_\alpha \otimes y)\big)\,(x \otimes 1_F), \tag{2.1}$$

so that by hypothesis for T, one obtains:

$$T(x \otimes y) = \big(\lim_{\alpha} T(u_\alpha \otimes y)\big)\,T(x \otimes 1_F) = \big(\lim_{\alpha} T(u_\alpha \otimes y)\big)\,(\varphi(x) \otimes 1_H). \tag{2.2}$$

Now, by considering the respective Gel'fand transforms of the preceding relation and by taking into account the hypothesis for the corresponding Gel'fand maps and Theorem 1.1 in the preceding, we have.

$$\widehat{T(x \otimes y)} = (\widehat{\varphi(x) \otimes 1_H})\,\big(\lim_{\alpha} \widehat{T(u_\alpha \otimes y)}\big), \tag{2.3}$$

so that one may consider the last relation as being of the form

$$\widehat{T(x \otimes y)} = \widehat{\varphi(x)} \cdot \psi, \tag{2.4}$$

where ψ denotes a complex-valued continuous function on the spectrum of H defined by the relation

$$\psi(h) = \big(\lim_{\alpha} \widehat{T(u_\alpha \otimes y)}\big)\,(g, h), \tag{2.5}$$

for a given element $g \in \mathcal{M}(G)$, and for every $h \in \mathcal{M}(H)$. Therefore, by Lemma 1.2 in the foregoing, there exists an element $b \in H$ such that one has $\widehat{T(x \otimes y)} = \widehat{\varphi(x) \otimes b}$, so that by the semi-simplicity of the algebra $G \hat{\otimes}_\sigma H$ (cf. also [3; p. 252, Theorem 2.1] and [4; p. 104, § 3, 1)]), one obtains

$$T(x \otimes y) = \varphi(x) \otimes b. \tag{2.6}$$

On the other hand, it is evident from the relation (2.5) that the definition of ψ is independent of the approximate identity (u_α) and the element $x \in E$. Thus, for every $g \in \mathcal{M}(G)$, one obtains a map

$$\varrho_g : F \to H : y \to \varrho_g(y) := b. \tag{2.7}$$

Now, we shall show that the element $b \in H$, as defined above, is actually independent of the element $g \in \mathcal{M}(G)$: Indeed, let $g_1, g_2 \in \mathcal{M}(G)$ with $g_1 \neq g_2$ such that $b_1 = \varrho_{g_1}(y)$ and $b_2 = \varrho_{g_2}(y)$. Then, for every $h \in \mathcal{M}(H)$, one obtains:

$$h(b_1) = h(b_2) = f(y), \tag{2.8}$$

where $f \in \mathcal{M}(F)$: This is a consequence of the following.

Scholium 2.1. By keeping fix the notation applied in the foregoing, let id_G denote the identity map of the algebra G, and let $h \in \mathcal{M}(H)$. Now, if T is the map given by the statement of Theorem 2.1, consider the map

$$\chi = (id_G \otimes h) \circ T : E \hat{\otimes}_\tau F \to G, \tag{2.9}$$

where its range is actually the algebra $G \otimes_{\tau'} \mathbf{C}$, with $\mathbf{C}$ denoting the algebra of complex numbers, and τ' the topology of G making it a topological algebra, so that it is trivially compatible with the structure of the tensor (product) algebra $G \otimes \mathbf{C} = G$, this relation being valid within an algebraic (onto) isomorphism and hence, by the completeness of the algebra G, one gets as the range of the map x (actually of its extension by continuity) the same algebra G. Now, by the relation (2.9) above, one obtains, for every $x \in E$, the relation:

$$\begin{aligned}\chi(x \otimes 1_F) &= (id_G \otimes h)\,(T(x \otimes 1_F)) = (id_G \otimes h)\,(\varphi(x) \otimes 1_H)\\ &= \varphi(x)\,h(1_H) = \varphi(x),\end{aligned}$$

so that, by the condition (1) of Theorem 2.1 above, there exists an element $f \in \mathcal{M}(F)$ such that one has the relation

$$\chi = (id_G \otimes h) \circ T = \varphi \otimes f. \tag{2.10}$$

End of the proof of Theorem 2.1: Now, by the preceding relation (2.10), one obtains, for every $g \in \mathcal{M}(G)$ and for any elements $x \in E$ and $y \in F$, the relation:

$$\begin{aligned}[(id_G \otimes h)\,(T(x \otimes y))]^{\wedge}(g) &= \widehat{\chi(x \otimes y)}\,(g) = (\varphi \otimes f)\,\widehat{(x \otimes y)}\,(g)\\ &= \widehat{\varphi(x) f(y)}\,(g) = f(y)\,.\,\widehat{\varphi(x)}\,(g),\end{aligned}$$

so that by the relations (2.7), (2.8) above, one has

$$\begin{aligned}g((id_G \otimes h)\,(T(x \otimes y))) &= g((id_G \otimes h)\,(\varphi(x) \otimes \varrho_g(y)))\\ &= g(\varphi(x)\,h(\varrho_g(y))) = h(\varrho_g(y))\,\varphi(x)\,(g),\end{aligned}$$

and hence, by the preceding, one finally gets the relation

$$h(\varrho_g(y))\,\varphi(x)(g) = f(y)\,\varphi(x)(g),$$

for any elements $x \in E$ and $g \in \mathcal{M}(G)$, so that we have

$$h(\varrho_g(y)) = f(y), \tag{2.11}$$

and this proves the relation (2.9) in the preceding. Now, by the same relation, one has $\hat{b}_1(h) = \hat{b}_2(h)$, for every $h \in \mathcal{M}(H)$, that is, $\hat{b}_1 = \hat{b}_2$, so that by the semi-simplicity of the algebra H, one obtains $b_1 = b_2$, and this proves the assertion, concerning the definition of the element $\varrho_g(y) = b \in H$. Hence, one has a continuous (algebra) homomorphism

$$\varrho : F \to H : y \to \varrho(y) := \varrho_g(y), \tag{2.12}$$

for an arbitrary $g \in \mathcal{M}(G)$, this relation being actually independent of the particular element g considered, as it has been proved before. Therefore, by the relations (2.6), (2.12), as above, one obtains the relation

$$T(x \otimes y) = \varphi(x) \otimes \varrho(y) = (\varphi \otimes \varrho)(x \otimes y), \tag{2.13}$$

for every decomposable tensor $x \otimes y \in E \otimes F$, so that by extending (2.13) by linearity and then by continuity, one finally gets the relation

$$T = \varphi \otimes \varrho, \tag{2.14}$$

and this completes the proof of the theorem.

In particular, we have the following.

Corollary 2.1. Suppose that the conditions of the preceding Theorem 2.1 are satisfied. Moreover, let that the (continuous algebra) homomorphisms T and φ as defined therein are bijections, in such a way that the restriction of T to the algebra $E \otimes F$ is an onto map, i.e. one has $T(E \otimes F) = G \otimes H$. Then, the map ϱ, as defined by the same theorem, is also a bijection.

Proof: Let y_1, y_2 be elements of F with $y_1 \neq y_2$, and let $x \in E$ with $\varphi(x) \neq 0$. Then, $x \otimes y_1 \neq x \otimes y_2$, so that, since T is an injection, one has $T(x \otimes y_1) \neq T(x \otimes y_2)$, so that by the relation (2.14) above one obtains $\varphi(x) \otimes \varrho(y_1) \neq \varphi(x) \otimes \varrho(y_2)$, and hence, since $\varphi(x) \neq 0$, one gets $\varrho(y_1) \neq \varrho(y_2)$, that is *the map* ϱ *is an injection.* On the other hand, ϱ *is an onto map*: Indeed, if $b \in H$, consider the element $\varphi(x) \otimes b \in G \otimes H$, with

$\varphi(x) \neq 0 \in G$ as above. Then, since T is an onto map, there exists an element $z \in E \otimes F$ such that $T(z) = \varphi(x) \otimes b$. Now, if $g \in \mathcal{M}(G)$ with $g(\varphi(x)) \neq 0$, then for every $h \in \mathcal{M}(H)$, one gets the relation

$$k(T(z)) = (g \otimes h)(\varphi(x) \otimes b) = g(\varphi(x))\, h(b), \tag{2.15}$$

where one has $k = g \otimes h \in \mathcal{M}(G \hat{\otimes}_{\sigma} H)$ (cf. also [**4**], p. 104, Theorem 2. 1). Thus, there exists an element $y \in F$ such that $h(b) = h(\varrho(y))$, for every $h \in \mathcal{M}(H)$, so that by the semi-simplicity of the algebra H, one concludes the relation $b = \varrho(y)$, with $y \in F$ as above, which proves the assertion, and this completes the proof.

On the other hand, we get the following result, by which we also conclude the present discussion. Thus, one has:

Theorem 2. 2. Let E, F, G, H be topological algebras with locally equicontinuous spectra [**5**], in such a way that the spectra of the algebras E and G are, moreover, connected and the spectrum of the algebra F totally disconnected. Moreover, suppose that the algebras F and H have identity elements 1_F and 1_H respectively, and the algebras G and H are semi-simple. Finally, let the following continuous (algebra) homomorphisms be given:

$$T \in Hom\,(E \hat{\otimes}_{\alpha} F,\ G \hat{\otimes}_{\beta} H) \text{ and } \varphi \in Hom\,(E, G), \text{ such that one has}$$
$$T(x \otimes 1_F) = \varphi(x) \otimes 1_H,$$

for every $x \in E$, where by α, β, we mean a «compatible» topology, respectively a «faithful» one on the tensor (product) algebras indicated [**3**]. Then, there exists a continuous (algebra) homomorphism $\varrho \in Hom$ (F, H) such that one has the relation

$$T = \varphi \otimes \varrho,$$

that is, $T(x \otimes y) = (\varphi \otimes \varrho)(x \otimes y) = \varphi(x) \otimes \varrho(y)$, for every decomposable tensor $x \otimes y \in E \otimes F$, the last relation being extended by (linearity and) continuity to the completed algebra $E \hat{\otimes}_{\alpha} F$.

Proof: By hypothesis $\mathcal{M}(E)$ is connected and $\mathcal{M}(F)$ totally disconnected, so that the connected components of $\mathcal{M}(E \hat{\otimes}_{\alpha} F) = \mathcal{M}(E) \times \mathcal{M}(F)$ (the equality being valid within a homeomorphism [**4**]) are exactly of the form $\mathcal{M}(E) \times \{f\}$, with $f \in \mathcal{M}(F)$.