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ΕΦΗΡΜΟΣΜΕΝΑ ΜΑΘΗΜΑΤΙΚΑ - ΔΥΝΑΜΙΚΗ.— **An efficient and simple approximate technique for solving nonlinear initial - value problems,** *by Anthony N. Kounadis**, διὰ τοῦ 'Ακαδημαϊκοῦ κ. Περ. Θεοχάρη.

ABSTRACT

A very efficient and simple to use technique for the solution of nonlinear initial-value problems associated with nonlinear ordinary differential equations of any order and variable coefficients, is presented. Convergence and uniqueness of solutions obtained by the successive approximations scheme of the proposed technique, are thoroughly established. Error upper bound estimates of the obtained solutions are also assessed leading to significant conclusions regarding the improvement of convergence for large time solutions. The iteration scheme can be also successfully extended to nonlinear boundary-value problems.

The proposed technique is demonstrated by an illustrative example of a nonlinear initial-value problem for which available results exist.

STATEMENT OF THE PROBLEM

Consider the general form of a nonlinear initial-value problem associated with the nonlinear ordinary differential equation of n^{th} order ($n=1, 2, \dots$)

$$G(x, y, y', \dots, y^{(n)})=0, \qquad x\varepsilon I \qquad (1)$$

subject to the initial conditions

$$y(x_0)=c_1,\ y'(x_0)=c_2,\ \dots,\ y^{(n-1)}(x_0)=c_n \quad , \quad x_0\varepsilon I \qquad (2)$$

where $y(x)$ defined on a real x interval I (including x_0) must possess n continuous derivatives, i.e. $y(x)\varepsilon C^n$ on I; the nonlinearities of eq .(1) may be coupling

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terms and / or powers of the function $y(x)$ and its derivatives $y^{(i)}=d^iy/dx^i$ $(i=1, 2,..., n\text{-}1)$.

The subsequent development assumes that eq. (1), after moving all nonlinear terms to its right-hand side, can be written in the form

$$L(y)=f, \qquad x\varepsilon I \qquad (3)$$

where $L=D^n+a_1(x)D^{n-1}+...+a_n(x)D^o$ with $D^ky=y^{(k)}$ $(k=1, 2,...,n)$ and $D^oy=y$; $a_i(x)$ $(i=1, 2,...,n)$ are continuous functions on I (they might also be constant or zero); $f=f(x, y, y',...,y^{(n-1)})$ is assumed as a real continuous function along with its first-partial derivatives in the $(n+1)$-dimensional rectangle

$$R: |x\text{-}x_o|\leq a, \ |y^{(k-1)}-c_k|\leq b_k, \ k=1, 2,...,n \qquad (x, x_o\varepsilon I) \qquad (4)$$

in which $y^{(o)}=y$ and a, b_k are positive numbers. Clearly, the point $(x_o, c_1, c_2,...,c_n)$ lies in R.

In the next section a brief description of the proposed method is given for the solution of the initial-value problem defined by eqs (3) and (2) under the assumptions stated above.

Description of the method

Setting the right-hand side of eq. (3) equal to zero (i.e. $f=0$), we obtain the following homogeneous (or reduced) linear differential equation

$$L(y)=0 \quad , \qquad x\varepsilon I \qquad (5)$$

In the sequel it is assumed that the solution of the linear homogeneous eq. (5) subject to the initial conditions (2) is known (or at least can be readily obtained). Eq. (5) due to eq. (3) can be written under the form

$$y^{(n)}(x)=-\sum_{i=1}^{n} a_i(x)y^{(n-i)}(x) \qquad (i=1, 2,...,n) \qquad (6)$$

Regarding the last equation one can observe that the conditions of the existence and uniqueness proof[1] are satisfied in the neighbourhood of any initial conditions (2). Indeed, denoting the right-hand side of eq. (6) by $\breve{f}$ we remark that it is continuous with respect to all arguments and there exist first-partial derivatives $\partial\breve{f}/\partial y^{(j)}=-a_{n-j}(x)$ $(j=0, 1,...,n\text{-}1)$ bounded in absolute value, since $a_{n-j}(x)$ are continuous on the finite interval I. Let y_o be the solution of the linear initial-value problem associated with eqs (5) and (2). Inserting y_0 in the right-hand side of the nonhomogeneous differential eq. (3), we obtain the following linear differential equation

$$L(y)=f(x,\ y_0,\ y'_0,...,y_0^{(n-1)}), \qquad x\varepsilon I \tag{7}$$

If y_p is a particular integral of the last equation, its general integral $y=y_0+y_p$ constitutes a first approximate solution $y=y_{(1)}$ of the nonlinear initial-value problem defined by eqs (3) and (2) or (1) and (2). Inserting now $y_{(1)}$ in the right-hand side of eq. (3) we obtain the following nonhomogeneous differential equation

$$L(y)=f(x,\ y_{(1)},\ y'_{(1)},...,y_{(1)}^{(n-1)})\ ,\ x\varepsilon I \tag{8}$$

subject to the initial conditions (2).

After determining the particular integral $y_{(1)p}$ corresponding to eq. (8) we obtain the second approximate solution $y_{(2)}=y_0+y_{(1)p}$. Repeating k times the foregoing procedure we obtain the approximate solution of order k, i.e $y_{(k)}=y_0+y_{(k-1)p}$. As will be shown below proceeding in this manner we succeed to improve gradually the approximate solution.

It should be noted that in view of the assumptions made for the functions $a_i(x)$ and f, the initial-value problem associated with each of eqs (7), (8),..., k(=1, 2,...,n) along with conditions (2), has a unique solution for the reasons stated above for the initial-value problem of eqs (6) and (2). Thus, it remains to prove whether: a)each member of the sequence of approximate solutions $y_0,\ y_{(1)},\ y_{(2)},\ y_{(3)},...,y_{(k)}$ satisfies conditions (4), and b) the proposed approximate method converges and if so whether it produces the correct solution.

The difference of two successive approximate solutions

$$y_{(k)}-y_{(k-1)}=y_{(k-1)p}-y_{(k-2)p} \tag{9}$$

being independent of the solution y_0 of the homogeneous differential equation is a function of the difference of the corresponding particular integrals. Thus, the question of convergence of the approximate solution $y_{(k)}$ depends on the convergence of the corresponding particular integral $y_{(k-1)p}$. With the aid of an upper bound estimate for the error of the approximate solution significant conclusions can be drawn regarding the improvement of convergence if large x solutions are required.

Finally, it is worth noticing that when the solution of the linear homogeneous eq. (5) is not known (or cannot be easily obtained) the proposed technique does not fail. Indeed, in such a case one has to move to the right-hand side of eq. (3) those linear terms of L(y) which allow the new equation $\tilde{L}(y)=0$ to be readily solved. The proposed method is still valid if the original equation (1) or (3) is replaced by the equation

$$y^{(n)}(x)=g(x,\ y,\ y',\dots,y^{(n-1)}),\qquad x\varepsilon I \tag{10}$$

The analysis which follows is an extension to that presented in ref. [2] referring to nonlinear initial-value problems associated with second order differential equations.

Convergence and uniqueness

Differential eq. (3) can be reduced to the following system of first-order differential equations

$$\left.\begin{aligned}
&y'_1=y_2 \quad (y_1=y)\\
&y'_2=y_3\\
&\dots\dots\dots\dots\\
&y'_{n-1}=y_n\\
&y'_n+\sum_{i=1}^{n} a_i(x)y_{n+1-i}=f(x,\ y_1,\ y_2,\dots,y_n)
\end{aligned}\right\} \tag{11}$$

which is defined for $(x,\ y)=(x,\ y_1,\ y_2,\dots,y_n)\varepsilon R$ and varying x on I. Thus, eqs (3) and (2) are equivalent to the system of eqs (11) and (2) along with the assumptions made for eq. (3). The initial-value problem defined now by eqs (11) and (2) can also be written in a vector-matrix form as follows

$$\left.\begin{aligned}
&dy/dx+A(x)y=F(x,\ y)\\
&y(x_0)=C
\end{aligned}\right\} \tag{12}$$

where y, F and C are n-dimensional vectors and A(x) is a square matrix of order n, whose expressions are given by

$$y=\begin{bmatrix} y_1\\ y_2\\ \cdot\\ \cdot\\ \cdot\\ y_n \end{bmatrix},\ F=\begin{bmatrix} 0\\ 0\\ \cdot\\ \cdot\\ \cdot\\ f(x,y) \end{bmatrix},\ C=\begin{bmatrix} c_1\\ c_2\\ \cdot\\ \cdot\\ \cdot\\ c_n \end{bmatrix},\ A(x)=\begin{bmatrix} 0 & -1 & 0 \dots & & \\ 0 & 0 & -1 & 0\dots & \\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots\dots & 0 & -1\\ a_n & a_{n-1} & \dots\dots & a_2 & a_1 \end{bmatrix} \tag{13}$$

Note that the vector-functions y and F(x,y) as well as the matrix function A(x) are continuous, since their respective components are continuous functions of x on I.

Moreover, F(x,y) due to the assumptions made for f(x,y) has bounded first-partial derivatives with respect to $y_1,\ldots,y_n$ in

$$R: |x-x_0|\leq a, \quad |y_k-c_k|\leq b_k \qquad (k=1, 2,\ldots,n) \tag{14}$$

For measuring the magnitudes of a vector y and a matrix A the following norms for convenience are chosen

$$\left.\begin{aligned} \|y\| &= \sum_{i=1}^{n} |y_i| \\ \|A\| &= \sum_{i,j=1}^{n} |a_{ij}| \end{aligned}\right\} \tag{15}$$

The distance of two vectors y and $\breve{y}$ is the norm of the difference vector $y-\breve{y}$, i.e.

$$\|y-\breve{y}\| = \sum_{i=1}^{n} |y_i-\breve{y}_i| \tag{16}$$

In view of the assumptions mentioned above the vector function F(x,y) is bounded, that is

$$\| F(x,y\| \leq M \qquad \text{(for some } M>0) \tag{17}$$

on R which is a subset of n-space consisting of all vectors y satisfying

$$\| y-C\|\leq b \qquad \text{(for some } b>0) \tag{18}$$

such that for

$$| x-x_0|\leq a \qquad \text{(for some } a>0) \tag{19}$$

every y remains in R; clearly C is a point also in R.

Since F(x,y) is a continuous vector-function with bounded first-partial derivatives in each $y_i(i=1, 2,\ldots,n)$ it can be proved by using the mean-value theorem (for functions of several variables) that F(x,y) is a Lipschitz vector function[3], i.e.

$$\| F(x,y)-F(x,\breve{y}) \|\leq K\|y-\breve{y}\| \tag{20}$$

for (x,y) and $(x,\breve{y})$ in R and some constant $K>0$.

It can be readily shown that a solution y of eqs (12) must also satisfy the integral equation

$$y(x)=y_0(x)+\int_{x_0}^{x} Y(x)Y^{-1}(\xi)F(\xi,y(\xi))d\xi \tag{21}$$

where $y_0(x)=CY(x)$ is the solution corresponding to the homogeneous equation (12) such that $Y(x_0)=C$, whereas $Y(x)$ is the n-by-n fundamental matrix which satisfies the matrix differential equation

$$\left.\begin{array}{l} dY/dx+A(x)Y=0 \\ \text{such that } Y(x_0)=J \end{array}\right\} \tag{22}$$

J is the identity matrix.

The determinant of the matrix $Y(x)$ satisfies the following identity of Jacobi (generalized Liouville's theorem)

$$|Y(x)|=\exp\left(-\int_{x_0}^{x} \operatorname{tr} A(\xi)d\xi\right) \tag{23}$$

As is known[2] a necessary and sufficient condition that a matrix solution Y of eqs (22) be a fundamental matrix is that $|Y(x)|\neq 0$ for varying x on I.

Note that $\exp(-\int_{x_0}^{x} A(\xi)d\xi)$ is a solution of equation $y'+A(x)y=0$ only if $A(x)$ and $\int_{x_0}^{x} A(\xi)d\xi$ commute, which occurs when $A(x)$ is either constant or diagonal. In such a case we have $Y(x)=e^{-xA}$ and $y_0(x)=Ce^{(x_0-x)A}$.

Following the approximate method outlined above and introduction of the solution $y_0(x)$ into the integral of eq. (21) yields the first approximate solution

$$y_{(1)}(x)=y_0(x)+\int_{x_0}^{x} Y(x)Y^{-1}(\xi)F(\xi,y_0(\xi))d\xi \tag{24}$$

and by iteration it is deduced that

$$y_n=y_0+\int_{x_0}^{x} Y(x)Y^{-1}(\xi)F(\xi,y_{n-1}(\xi))d\xi \tag{25}$$

for $n=1,2,\ldots$

It will be shown below that every «point» $(x,y_n(x))$ $(n=1,2,\ldots)$ including $(x,y_0(x))$-lies in R(i.e. satisfies relation (18) for $|x-x_0|\leq\varepsilon$, where $0\leq\varepsilon\leq a$. Before doing this we observe that since $A(x)$ is a continuous matrix-function on a finite interval I of x, then $\forall$ N such that

$$||Y(x)Y^{-1}(\xi)|| \leq N \qquad (N>0) \tag{26}$$

for every fixed $x\varepsilon I$ and varying ξ on I. Thus, for $\xi = x_0$ due to the second of relations (22) inequality (26) yields

$$||Y(x)|| \leq N \tag{27}$$

For the point $(x, y_0(x))$ due to the second of relations (15) and relations (21) and (27) we have

$$||y_0 - C|| = ||C(Y-J)|| \leq ||C||(||Y||-n) \leq (N-n)||C|| \tag{28}$$

On the other hand for the point $(x, y_1(x))$ using equations (25) we can write the inequality

$$||y_1 - C|| \leq ||y_0 - C|| + \int_{x_0}^{x} ||Y(x)Y^{-1}(\xi)|| \cdot ||F(\xi, y_0(\xi))|| d\xi$$

or due to relations (17), (19), (26) and (28)

$$||y_1 - C|| \leq (N-n)||C|| + NM|x-x_0| = [(N-n)||C|| + NMa]\varepsilon < b \tag{29}$$

where

$$\varepsilon = \min [a, b / ((N-n)||C|| + NMa)] \tag{30}$$

From inequality (29) it follows also that

$$||y_0 - C|| \leq (N-n)||C|| < b \tag{28'}$$

Similarly, for the point $(x, y_n(x))$ we have

$$||y_n - C|| \leq ||y_0 - C|| + NM|x-x_0| < b \tag{31}$$

Hence condition (18) is satisfied for every point $(x, y_n(x))$ $(n=1, 2, ...)$ including $(x, y_0(x))$ and thus all approximate vectors y_n belong to R for varying x on the interval $(x_0-\varepsilon, x_0+\varepsilon)$, where ε is given in relation (30).

The sequence of vectors y_n is said to be convergent if it is convergent with respect to the distance function of each member vector $y_k (k=1,2,...)$ from the vector y which constitutes the correct solution of eqs (12). Such a distance according to relation (16) is equal to

$$||Y_k - y|| = |y_k - y| + |y'_k - y'| \qquad (k=1,2,...,n) \tag{32}$$

Writing eq. (25) for n=1 by virtue of relations (17) and (26) we have

$$||y_1 - y_0|| \leq \int_{x_0}^{x} ||Y(x)Y^{-1}(\xi|| . ||F(\xi, y_0(\xi))|| d\xi \leq MN|x-x_0| \tag{33}$$

Eq. (25) for n=2 yields

$$||y_1 - y_0|| \leq \int_{x_0}^{x} ||Y(x)Y^{-1}(\xi)|| . ||F(\xi, y_1(\xi)) - F(\xi, y_0(\xi))|| d\xi$$

or due to inequalities (20), (26) and (33)

$$||y_1-y_0||\leq N^2KM\,\frac{(x-x_0)^2}{2}=\frac{M}{K}\cdot\frac{[NK(x-x_0)]^2}{2!} \tag{34}$$

and by induction using relation (19) it follows

$$||y_n-y_{n-1}||\leq\frac{M}{K}\cdot\frac{[NK(x-x_0)]^n}{n!}\leq\frac{M}{K}\cdot\frac{(NKa)^n}{n!} \tag{35}$$

Hence for $n\rightarrow\infty$ the vector-function y_n tends uniformly to a limit vector-function y for varying x on I. This vector-function is continuous as a uniform limit of continuous vectors and satisfies inequality

$$||y_n-C||\leq b \quad (b>0) \tag{36}$$

for x varying in the interval $(x_0-\varepsilon,\ x_0+\varepsilon)$, where ε is given in relation (30).

Due to the continuity of the vector-function F(x,y) it is also deduced

$$\lim_{n\rightarrow\infty}F(\xi,y_n(\xi))=F(\xi,y(\xi)) \tag{37}$$

The foregoing solution obtained by the above scheme of successive approximations is unique. Supposing that besides y there is also another solution $\tilde{y}$, such a solution must satisfy the integral equation (21) and hence we can write

$$||y-\tilde{y}\,||\leq\int_{x_0}^{x}||Y(x)Y^{-1}(\xi)||.||F(\xi,y(\xi))-F(\xi,\tilde{y}(\xi))||d\xi$$

or due to inequality (20)

$$||y-\tilde{y}||\leq K\int_{x_0}^{x}||\,Y(x)Y^{-1}(\xi)||.||y-\tilde{y}||d\xi \tag{38}$$

Since $\tilde{y}$-as well as y-is continuous on a finite interval I of x due to the continuity of its components, we can assume

$$\mu=\max||y-\tilde{y}|| \text{ for } x\varepsilon I \tag{39}$$

and thus inequality (38) becomes

$$||y-\tilde{y}||\leq\mu K\int_{x_0}^{x}||\,Y(x)Y^{-1}(\xi)||d\xi \tag{40}$$

Introducing this expression into the integral of inequality (38), we obtain

$$||y-\tilde{y}||\leq\mu K^2\int_{x_0}^{x}||\,Y(x)Y^{-1}\,(\xi)||(\int_{x_0}^{\xi}||Y\,(x)Y^{-1}(s)\,||\,ds)d\xi$$

or

$$||y-\tilde{y}|| \leq \frac{\mu}{2!}\left(K\int_{x_0}^{x} ||Y(x)Y^{-1}(\xi)||d\xi\right)^2 \tag{41}$$

Introducing the last expression into the integral of inequality (38) we obtain

$$||y-\tilde{y}|| \leq \frac{\mu}{3!}\left(K\int_{x_0}^{x} ||Y(x)Y^{-1}(\xi)||d\xi\right)^3 \tag{42}$$

Thus by iteration it follows

$$||y-\tilde{y}|| \leq \frac{\mu}{n!}\left(K\int_{x_0}^{x} ||Y(x)Y^{-1}(\xi)||d\xi\right)^n \tag{43}$$

Letting $n \to \infty$ it is deduced that

$$||y-\tilde{y}|| \leq 0 \tag{44}$$

and hence $y=y$.

Error estimate and convergence improvement[2]

An upper bound for the error in approximating the correct vector solution y by the n^{th} approximation y_n is given by

$$||y-y_n|| \leq \sum_{m=n}^{\infty} ||y_{m+1}-y_m|| \leq \frac{M}{K}\sum_{m=n+1}^{\infty}\frac{[NK(x-x_0)]^n}{m!} \leq \frac{M}{K}\cdot\sum_{m=n+1}^{\infty}\frac{(NKa)^m}{m!}$$

or

$$||y-y_n|| \leq \frac{M}{K}\cdot\sum_{m=n+1}^{\infty}\frac{(NKa)^m}{m!} < \frac{M}{K}\frac{(KNa)^{n+1}}{(n+1)!}\cdot\sum_{m=0}^{\infty}\frac{(KNa)^m}{m!} =$$

$$= \frac{M}{K}\frac{(KNa)^{n+1}}{(n+1)!}.e^{NKa} \tag{45}$$

From the last inequality we observe that for a given number of iterations, an increase of the parameter a-defining the half-range of variation of x-implies an exponential increase in the error upper bound. This means that a very good approximate solution obtained by the above procedure is expected for the initial stages of variation of x. On the other hand if the interval 2a of variation of x is kept constant the increase in the number of iterations reduces the magnitude of error; clearly, the greater the number of iterations is the more significant the reduction of error becomes. From the last observations one can draw two important conclusions:

a. For the initial stages of the independent variable a couple of successive approximations usually leads to sufficiently reliable results within the scope of engineering accuracy. This is very important for nonlinear boundary-value problems of postbuckling response, where the initial postbuckling path is of practical importance. This is clearly shown in refs [4-8] in which the proposed technique has been successfully employed by the author for solving postbuckling response problems associated with highly nonlinear systems of differential equations, b. If the dynamic response of a system is needed for large values of the independent variable (i.e for long periods of time), one has: 1) to divide the given interval of the independent variable into an appropriate number of subintervals (depending on the desired accuracy) and 2) to employ the first or second approximation for each subinterval with initial conditions the end conditions of the preceding subinterval. In this way starting from the given initial conditions we can cover the entire interval of the independent variable.

Since the maximum error occurs at the end of the entire interval the accuracy of the foregoing solution is checked by comparing it with a new solution (at that point) which is obtained by reducing the length of subintervals (or equivalently by increasing the number of subintervals). The number of the same significant figures corresponding to these two solutions indicates that up to this number of significant figures the obtained solution is correct. A better solution can be achieved by further reducing the length of subintervals until the obtained solution reaches the desired accuracy.

Illustrative example

Let us consider the nonlinear differential equation with variable (time-dependent) coefficients[2]

$$\ddot{y}(\xi)+\Omega\left[(1-\frac{e^2}{4}-\xi)y(\xi)+\frac{1}{4}y^3(\xi)-e\right]=0 \qquad (46)$$

subject to the initial conditions

$$y(0)=e, \quad \dot{y}(0)=0 \qquad (47)$$

where the nondimensionalized time ξ varies in the interval (0,7), while Ω and e are given dimensionless parameters.

This nonlinear initial-value problem has been presented by Hoff[9] and refers to the dynamic mid-span deflection of an initially curved, simply supported, column subjected at its movable support to a constant (compressive) velocity. Note that the magnitude of the nonlinear term of eq. (46) increases very rapidly with increasing time. Hoff has succeeded to solve this problem only after using a lengthy, cumbersome and time-consuming procedure. Following this procedure he obtained by means of Bessel functions an analytic solution valid for a very small interval of time ($0 \leq \xi \leq 1$); then by employing modified Bessel functions he succeeded to obtain a solution valid, however, for $1 \leq \xi \leq 2.1$. Beyond this value a series solution of low convergence became inevitable.

According to the approximate method proposed herein by moving the nonlinear term of eq. (46) to its right-hand side, we obtain

$$\ddot{y}(\xi) + \Omega \left\{ (1 - \frac{e^2}{4} - \xi) y(\xi) - e \right\} = - \frac{\Omega}{4} y^3(\xi) \tag{48}$$

Since the solution of the linear homogeneous equation corresponding to eq. (48) due to the variable coefficient of $y(\xi)$ is not available it is more convenient to transfer all terms of this equation to its right hand side; hence eq. (46) is written as follows

$$\ddot{y}(\xi) = \varphi(\xi, y(\xi)) = \Omega \left[e - (1 - \frac{e^2}{4} - \xi) y(\xi) - \frac{1}{4} y^3(\xi) \right] \tag{49}$$

subject to conditions

$$y(0) = e, \quad \dot{y}(0) = 0 \tag{50}$$

Eq. (49) is equivalent to the following integral equation

$$y(\xi) = y_o(\xi) + \int_0^\xi (\xi - s) \varphi(s, y(s)) ds \tag{51}$$

where $y_o(\xi)$ is the solution of the homogeneous equation (resulting by setting $\varphi = 0$)

$$\ddot{y}_o(\xi) = 0 \tag{52}$$

which after integration due to conditions (50) yields

$y_o = e$.

Introducing $y_o(\xi)$ into the integral of eq. (51) we obtain the first approximate solution

$$y_1(\xi) = y_o(\xi) + \int_0^\xi (\xi - s) \varphi(s, y_o(s)) ds \tag{53}$$

By iteration we can readily obtain

$$y_n(\xi) = y_o(\xi) + \int_0^\xi (\xi - s) \varphi(s, y_{n-1}(s)) ds \tag{54}$$

The second approximation

$$y_2(\xi) = e + \frac{e\xi^3}{6} - \frac{\Omega^2 e}{120} (1 + \frac{e^2}{2}) \xi^5 + \frac{\Omega^2 e}{180} \xi^6 - \frac{\Omega^3 e^3}{140} (\frac{3}{56} + \frac{\Omega \xi^3}{660}) \xi^8 \tag{55}$$

gives very accurate results for ξ varying in the interval (0,1.7). For larger values of ξ we must divide the original interval into subintervals and employ for each subinterval the first or the socond approximate solution taking as initial conditions the final (end) conditions of the preceding subinterval. Dividing, for instance, the original interval (ξ_o,ξ_n) into n subintervals

$$\xi_o\leq\xi\leq\xi_1,\ldots,\xi_i\leq\xi\leq\xi_{i+1},\ldots,\xi_{n-1}\leq\xi\leq\xi_n \tag{56}$$

we can write for the subinterval $(\xi_i,\ \xi_{i+1})$ the following equation

$$y_n(\xi)=y_o(\xi-\xi_i)+\int_{\xi_i}^{\xi}(\xi-s)\varphi(s,y_i(s))ds,\ (\xi_i\leq\xi\leq\xi_{i+1}) \tag{57}$$

Introducing for convenience the new variable

$$\tau=\xi-\xi_i \tag{58}$$

eq. (57) becomes

$$y_n(\tau)=y_o(\tau)+\int_0^{\tau}(\tau-s)\varphi(s,y_{n-1}(s))ds \tag{59}$$

The first or second approximation (n=1 or n=2) of the last formula gives very reliable results if the length of each subinterval is relatively small. By means of eq. (59) starting from the given initial conditions we can progressively evaluate the initial and final conditions of each subinterval and thus covering the entire original interval.

We can compare the proposed solution with the Hoff's solution for the specific case where $\Omega=2.25$ and $e=0.25$. By taking the length of each subinterval equal to 0.10 $(=\xi_1-\xi_o=\xi_2-\xi_1=\ldots=\xi_n-\xi_{n-1})$ we have obtained a solution to four significant figures of accuracy for $0\leq\xi\leq5$. Such a solution coincides in the first four significant figures with the numerical solution of Runge Kutta. Comparisons of these three solutions are shown in a graphical form in Fig. 1 from which it is obvious thet Hoff's solution starts to become inaccurate for $\xi>4.5$. From Table 1, one can also compare numerical solutions obtained by the proposed technique and the Runge Kutta's scheme. Clearly, the latter is time consuming for multiparametric discussions compared to the this technique which uses the same formula (corresponding to the first or second approximation) for each subinterval.

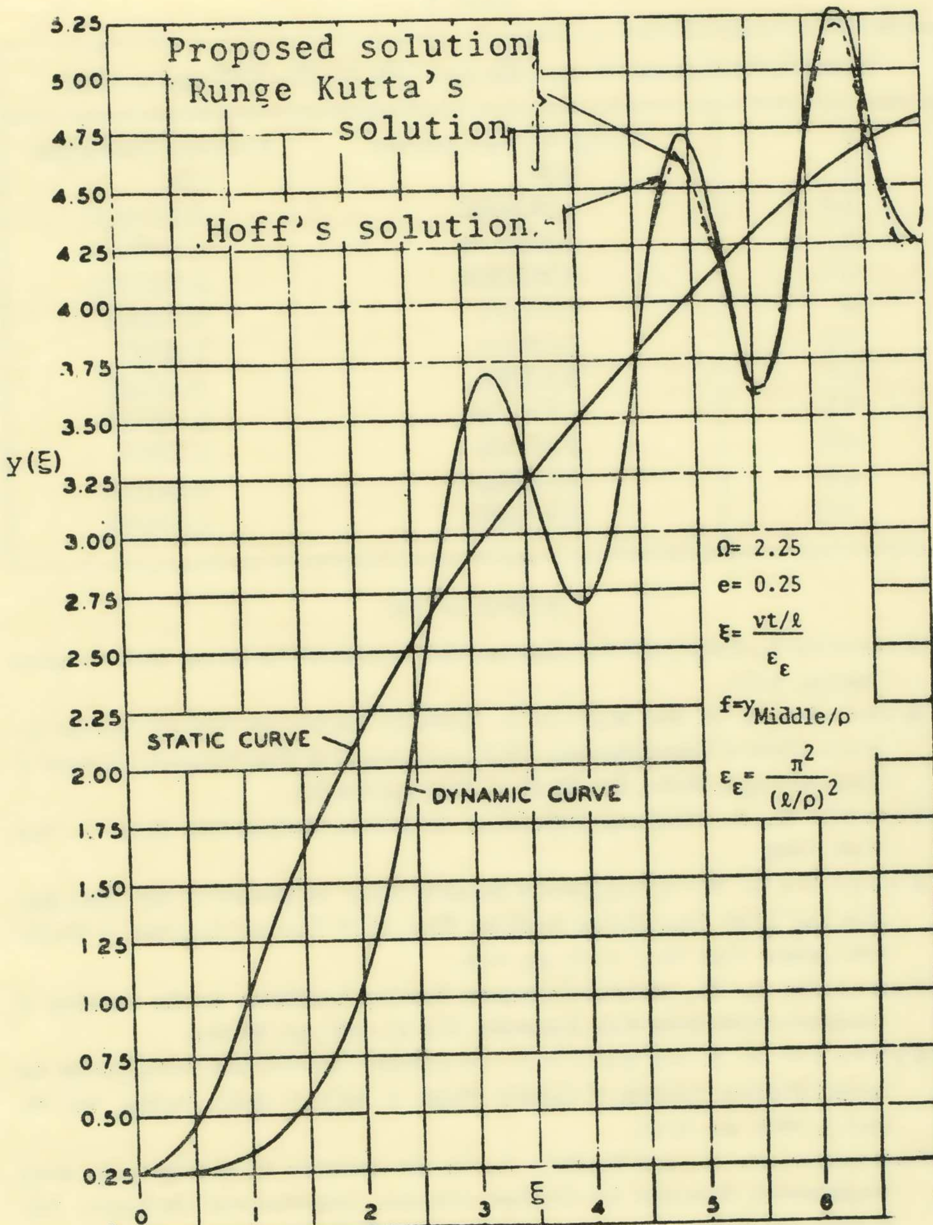

Fig. 1. Nondimensional lateral displacement amplitude $y(\xi)$ as function of nondimensional time ξ in very Rapid loading.

Table 1

Numerical solutions of eqs (46) and (47) for $\Omega=2.25$ and $e=0.25$

ξ	Runge Kutta's scheme	Proposed Technique
0	0.25	0.25
0.5	0.2614907	0.2614907
1	0.3399305	0.3399304
1.5	0.5620209	0.5620208
2	1.077164	1.077164
2.5	2.160111	2.16012
3.0	2.538290	3.538292
3.5	3.321439	3.321428
4.0	2.728861	2.72856
4.5	3.767951	3.767968
5.0	4.514586	4.514562

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ΠΕΡΙΛΗΨΗ

Μία ἀποτελεσματικὴ καὶ ἁπλὴ μέθοδος γιὰ τὴν ἐπίλυση μὴ γραμμικῶν προβλημάτων ἀρχικῶν τιμῶν

Εἶναι γνωστὸν ὅτι σὲ μὴ γραμμικὰ προβλήματα συνοριακῶν τιμῶν ποὺ σχετίζονται μὲ προβλήματα ἐλαστικῆς ἀστάθειας τὸ ἀρχικὸ τμῆμα τοῦ μεταλυγισμικοῦ δρόμου ἰσορροπίας εἶναι ἰδιαίτερης σημασίας γιὰ τὸ σχεδιασμὸ τῶν κατασκευῶν. Ὁ ὑπολογισμὸς δὲ τοῦ τμήματος αὐτοῦ μπορεῖ εὔκολα νὰ γίνει μέσω ἀναπτύγματος μὲ ἕνα-δύο ὅρους, ἐὰν εἶναι γνωστὴ ἡ λύση τοῦ προβλήματος στὴ στάθμη τῆς κρισίμου φορτίσεως, ἰδιαίτερα ἂν αὐτὴ συνδέεται μὲ σημεῖον διακλαδώσεως. Ἐπὶ τῶν πάρα πάνω παρατηρήσεων στηρίζεται ἡ βασικὴ ἰδέα τῆς προτεινομένης τεχνικῆς ποὺ εἶναι ὁ μετασχηματισμὸς τοῦ μὴ γραμμικοῦ προβλήματος συνοριακῶν τιμῶν σὲ γραμμικὸ πρόβλημα συνοριακῶν τιμῶν (γνωστῆς ἢ εὔκολα ἐπιτυγχανομένης λύσεως) μέσω μιᾶς ἢ δύο ἐπαναλήψεων, ἀντίθετα ἀπὸ ἄλλες τεχνικὲς ποὺ ἀπαιτοῦν ἀρκετὲς ἐπαναλήψεις. Ἐφαρμογὲς τῆς μεθόδου εὑρίσκουμε σὲ πολλὲς δημοσιευμένες ἐργασίες τοῦ συγγραφέως, ἐκ τῶν ὁποίων ἐνδεικτικὰ ἀναφέρονται οἱ ἐργασίες ὑπ' ἀριθμ. [4-8] καὶ ἰδιαίτερα ἡ ὑπ' ἀριθμ. 6, ὅπου ἐκτίθεται ἡ μαθηματικὴ θεμελίωση τῆς μεθόδου (σύγκλιση, μοναδικότης λύσεως, ἐκτίμηση σφάλματος) γιὰ τὸ συγκεκριμένο πρόβλημα ποὺ διαπραγματεύεται.

Ἀκολούθως προτείνεται, μετὰ ἀπὸ κατάλληλες προσαρμογές, ἡ ἐφαρμογὴ τῆς μεθόδου σὲ μὴ γραμμικὰ προβλήματα ἀρχικῶν τιμῶν, τὰ ὁποῖα, ὅπως καὶ στὴν προηγουμένη περίπτωση, συνδέονται μὲ συνήθεις μὴ γραμμικὲς διαφορικὲς ἐξισώσεις οἱασδήποτε τάξεως καὶ μὲ μεταβλητοὺς συντελεστές. Ἡ πρώτη σχετικὴ ἐργασία ἀνακοινώθηκε τὸ 1985 (βλ. Πρακτικὰ Α' Ἐθνικοῦ Συνεδρίου Μηχανικῆς 1986, σελ. 1-10). Στὸ σημεῖο αὐτὸ θὰ πρέπει νὰ διευκρινισθεῖ ὅτι, ἀντίθετα ἀπὸ τὰ προβλήματα συνοριακῶν τιμῶν ποὺ ἀπαιτοῦν μίαν ἢ τὸ πολὺ δύο ἐπαναλήψεις, στὰ προβλήματα ἀρχικῶν τιμῶν ἡ εὐστάθεια κινήσεως πρέπει νὰ ἐρευνηθεῖ σὲ μεγάλα χρονικὰ διαστήματα (Stability in the large). Τοῦτο εἶναι ἰδιαίτερα ἐπιβεβλημένο λόγω τῆς ἐνδεχομένης παρουσίας ἰδιοτύπων ἑλκτῶν (strange attractors), βασικὰ χαρακτηριστικὰ τῶν ὁποίων εἶναι ἡ εὐαισθησία στὶς ἀρχικὲς συνθῆκες καὶ ἐνίοτε ἡ ἀπροσδόκητη δράση τους, ὕστερα ἀπὸ σημαντικὸ χρονικὸ διάστημα ἀπόλυτης ἠρεμίας. Ἔτσι λόγω τοῦ πλήθους τῶν λαμβανομένων λύσεων (σὲ μὴ γραμμικὰ προβλήματα ἀρχικῶν τιμῶν) θὰ πρέπει κανεὶς νὰ εἶναι πολὺ προσεκτικὸς στὸν καθορισμὸ ἐκείνης τῆς λύσεως ποὺ ἀντιστοιχεῖ στὸ φυσικὸ πρόβλημα. Μία δεύτερη ἐρευνητικὴ ἐργασία (ὑπ' ἀριθμ. 2 στὴ βιβλιογραφία), συνδεομένη μὲ συνήθεις μὴ γραμμικὲς ἐξι-

σώσεις δευτέρας τάξεως μὲ μεταβλητοὺς (χρονικὰ ἐξηρτημένους) συντελεστές, ἀναφέρεται σὲ δυναμικὰ εὐσταθὲς σύστημα. Ἡ ἐπαναληπτικὴ αὐτὴ μέθοδος συνοδεύεται μὲ ἀπόδειξη τῆς συγκλίσεως καὶ τῆς μοναδικότητος τῆς λύσεως, ὡς καὶ μὲ ἐκτίμηση ἄνω φράγματος τοῦ σφάλματος τῆς προσεγγιστικῆς λύσεως.

Εἰς τὴν παροῦσα ἐργασία γίνεται ἐπέκταση τῆς προηγουμένης δημοσιεύσεως σὲ μὴ γραμμικὰ προβλήματα ἀρχικῶν τιμῶν ποὺ συνδέονται μὲ ὁποιασδήποτε τάξεως συνήθεις διαφορικὲς ἐξισώσεις μὲ μεταβλητοὺς συντελεστές. Ἡ προτεινομένη μέθοδος ὡς πρὸς μὲν τὶς λαμβανόμενες προσεγγιστικὲς λύσεις μπορεῖ νὰ ἐνταχθεῖ στὶς μεθόδους τῆς τεχνικῆς τῶν διαταραχῶν, ἐνῶ ὡς πρὸς τὴ βασική της σύλληψη ἀποτελεῖ οὐσιώδη τροποποίηση τῆς ἐπαναληπτικῆς μεθόδου Picard.

Μὲ βάση τὴν ἐκτίμηση τοῦ σφάλματος τῆς προσεγγιστικῆς λύσεως διατυπώνεται μία ἀποτελεσματικὴ διαδικασία βελτιώσεως τῆς συγκλίσεως, κατάλληλη γιὰ δυναμικὰ συστήματα τῶν ὁποίων ἡ εὐστάθεια κινήσεως εἶναι ἀναγκαῖο νὰ ἐρευνηθεῖ σὲ μεγάλα χρονικὰ διαστήματα, ἐν ὄψει καὶ τῶν ἐνδεχομένων εὐσταθῶν ὁριακῶν κύκλων ἢ ἐπιρροῶν ἰδιοτύπων ἑλκτῶν. Σύμφωνα μὲ τὴν τεχνικὴ αὐτὴ ὑποδιαιρεῖται τὸ ἐπιλεγόμενο ἀρχικὸ χρονικὸ διάστημα εἰς ὑποδιαστήματα, γιὰ κάθε ἕνα τῶν ὁποίων ἐφαρμόζεται ἡ πρώτη ἢ ἡ δεύτερη προσέγγιση λύσεως, μὲ ἀρχικὲς τιμὲς τὶς τιμὲς ποὺ ἀντιστοιχοῦν στὸ τέλος τοῦ προηγουμένου ὑποδιαστήματος.

Τέλος, παρατίθεται ἀριθμητικὸ παράδειγμα, τὸ ὁποῖο ἐπιλύεται τόσο μὲ τὴ μέθοδο Hoff, ὅσο καὶ μὲ τὸ 4ης τάξεως ἀριθμητικὸ σχῆμα τοῦ Runge Kutta μὲ βῆμα 0.02. Ἀπὸ τὸ παράδειγμα αὐτὸ διαπιστώνεται τὸ πλεονέκτημα τῆς προτεινομένης μεθόδου ἰδιαίτερα σὲ πολυπαραμετρικὰ συστήματα.