ΕΦΗΡΜΟΣΜΕΝΑ ΜΑΘΗΜΑΤΙΚΑ.— **On the separatrices of dynamical systems,** *by Demetrios G. Magiros**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φ. Βασιλείου.

INTRODUCTION

Separatrices are special trajectories or motions of dynamical systems, and play an important role in the study of problems of the systems of current interest, especially when quantitative aspects enter the problems. But there is no general and systematic discussion on the use of the properties and on the determination of the separatrices on nonlinear dynamical systems, this determination being by itself an important problem.

In this paper we will see remarks and results concerning the properties of separatrices and their use for the study of physical problems.

The definition of separatrices given in topology is supplemented as needed in physical problems, some theorems on separatrices are stated, a list of useful properties of separatrices is given, and by selected examples we see the usefulness of the separatrices in the study of physical problems.

1. DEFINITION OF SEPARATRICES

We give the definition of separatrices both from a topological and a dynamical point of view.

We can say that a space W is filled by a collection S of solution curves of a dynamical system, if each solution curve of S lies in W, and each point in W is on exactly one solution curve of S.

The whole space W of the validity of a differential system may be decomposed into subspaces of which the corresponding collection of the solution curves has common properties which characterize each space.

These subspaces are called «canonical regions» of the space W, and

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the paths of the solution curves of the system, which bound these canonical regions, are called «separatrices» of the system [**7**, **10**].

In this topological definition of separatrices the solution curves are considered only as paths, that is as locus of point sets. In the reality, the solution curves of the dynamical systems are time-parametrized curves, that is paths on which the law of the motion of the system is known, when the topological definition of the separatrices, although it helps the investigation in some aspects, is unrealistic.

The time must be included in the concept of separatrices of physical problems. This can be succeeded by accepting the separatrices as «special "limiting" trajectories through special equilibrium states». Supplemented by this property, the topological definition of separatrices satisfies physical requirements and acquires a «physical validity».

By examples which will follow we clarify concepts related to separatrices and emphasize the usefulness of their properties in the investigation of physical problems.

2. THEOREMS RELATED TO SEPARATRICES

The separatrices are intimately related to the singular points of the system. It is the nature of the trajectories at the neighborhood of a singular point which guarantees the existence of separatrices through the singular point. We give, without analysis or proof, statements of theorems concerning singular points and corresponding separatrices, and the formulation of these theorems is given as needed in applications.

Theorem 1. Given a «noncritical linear dynamical system» in its normal form, if m is the number of the solution curves through a point of the space W of its validity, we may have the following cases:

a) For $m = 1$, the point is «regular», but for any other value of m the point is «singular»;

b) For $m = 0$ the singular point is a «center»;

c) For m a finite even integer, the singular point is a «saddle» point, and all the solution curves through this point are «branches of a separatrix»;

d) For $m = \infty$, the singular point is either a «node» or a «spiral» point, and among the infinitely many solution curves through the point some of them may be separatrices.

The above singular points are «elementary singular points» and characterize the linear noncritical systems.

Theorem 2. In «noncritical nonlinear systems» it is the order of magnitude of the nonlinearities of the system which decides on the nature of the singular point and of the corresponding separatrices, and we have the following cases:

a) If the order of magnitude of the nonlinearities is appropriately small, the singular point is «elementary» and the situation of separatrices is as in Theorem A.

b) If the smallness of the magnitude of the nonlinearities can not be restricted appropriately, the singular point is «nonelementary» and the situation of separatrices is a complicated matter.

Theorem 3. We distinguish two cases:

a) In «critical linear systems» the singular point may be elementary or nonelementary, and the separatrices will be in a complicated situation, especially if the system has many singular points.

b) In «critical nonlinear systems», or in «nonlinear systems without linear part», the phase portrait near the nonelementary singular point is very complicated. A small neighborhood around such a point may be devided by separatrices into sectors with this point as the apex. These sectors may be of «nodal» (parabolic), or «elliptic», or «saddle» (hyperbolic) type.

We remark that there are cases very complicated, and only a few results are known today for highly nonelementary singular points and the corresponding separatrices. The following theorem is due to Bendixson [**3**].

T h e o r e m 4. If a system is the x, y-phase plane is given by

$$y' = x^{-m}[\alpha y + bx + B(x, y)] \tag{12.1}$$

where $B(x, y)$ is a polynomial of degree at least two, and $\alpha \neq 0$, we have the following four cases:

a) If $\alpha > 0$ and m = even integer, then there is only one branch of integral curves tending to the origin on the left side of y-axis, $x < 0$, while integral curves on the other side, $x > 0$, constitute a nodal distribution; that is, there is a coalescence of «saddle-nodal» points.

In this case there exists a separatrix through the origin.

b) If $\alpha < 0$ and m = even integer, this case can be transformed to the previous case, and we have a coalescence of «nodal-saddle» points (node at $x < 0$, and saddle at $x > 0$).

A separatrix exists through the origin in this case.

c. If $\alpha > 0$ and m = odd integer, the origin is a nodal point, when a separatrix may exist.

d. If $\alpha < 0$ and m = odd integer, the origin is a saddle point and a separatrix exists.

The analysis of the above statements is based on the definition of the separatrices, on the concepts of the «α-limiting» and «ω-limiting» properties of the separatrices, and on other concepts.

3. SOME PROPERTIES OF SEPARATRICES

Combining the definition of separatrices and results coming from the theorems, one can find properties of separatrices, which are very useful in applications. In the following we list some of these properties.

— The separatrices may be points, lines, surfaces, depending on the dimensions and the structure of the dynamical system.

— There is no separatrix through a center.

— A separatrix through a singular point may be either a «α-limiting» or a «ω-limiting» trajectory, when, starting from a point of the separatrix, the time to reach the terminating point is «infinite».

— Separatrices starting from a singular point may terminate to the same singular point, when they are «closed» separatrices, and they have a finite length. «Non-closed» or «open» separatrices do not start and terminate at the same singular point. They start from a singular point and they may terminate either to another singular point or to infinity. Some of these open separatrices may have finite length.

— Separatrices through a node have at this point a definite tangent, and separatrices terminating to a spiral point move around it spirally and they do not have a definite tangent at this point.

— An «isolated closed path» of a dynamical system, in case all its points are regular, is a «limit-cycle» of the system, when it corresponds to a periodic phonomenon with a fixed period. But, if this closed path is through a singular point, the periodicity disappears and the closed path is not a limit cycle, but it is a «closed separatrix». The limit cycle is a separatrix according to the topological definition; but it is not a separatrice according to the supplemented definition of separatrices.

— The separatrices have important physical significance. We indicate some of them.

They may determine the whole region of the validity of a dynamical system and separate it from the «empty regions» which are without real solutions of the system.

They may be boundary curves of the regions in each of which the solutions are characterized by different stability situations, when the separatrices are in a «neutral stability situation». This property is of paramount importance in contemporary nonlinear control problems.

They may have some other physical meaning.

4. DETERMINATION OF SEPARATRICES

For the determination of the separatrices we see two cases. In case we know the general solution of the mathematical model of the physical problem, the determination of separatrices is identical to the determination of special particular solutions through appropriate singular points of the system.

In case the general solution is not known, approximate methods, either geometrical, or numerical, or analytical, map help the investigation towards the determination of the separatrices.

Remark. We remark that the concept of «index» of singular points, of trajectories in general and of separatrices in particular, introduced by Poincaré, plays an important role for investigation of their nature [**5**, **13**].

5. EXAMPLES

In each of the following examples appropriate remarks are given related to properties of separatrices.

Example 1. The separatrices in this example determine exactly the boundary of the canonical regions, which are regions of the validity of the system where real trajectories exist, and empty regions.

The dynamical system [**4** (a)] :

$$\dot{x}^2 = 1 - x^2, \qquad \dot{y}^2 = 1 - y^2 \tag{1.1}$$

has four singular points, the points $(\pm 1, \pm 1)$, which are points of intersection of the lines $x = \pm 1$, $y = \pm 1$.

The system (1. 1) corresponds in the x, y - phase plane to the DE:

$$y'^2 = \frac{1 - y^2}{1 - x^2}. \tag{1.2}$$

For the reality of the solutions of (1. 2), the $(1 - x^2)$ and $(1 - y^2)$ must have the same sign, and this restriction helps to determine the real regions of the validity of (1. 2). By separating the variables and integrating, one can find the general solution of (1. 2):

$$\left.\begin{array}{lll} \operatorname{arc\,sin} y \pm \operatorname{arc\,sin} x = c\,; & |x| < 1, & |y| < 1 \\ \operatorname{arc\,cos\,h} y \pm \operatorname{arc\,cos\,h} y = c\,; & |x| > 1, & |y| > 1 \end{array}\right\} \tag{1.3}$$

c is the arbitrary constant. Figure 1 shows the phase-portrait of (1. 3).

The separatrices are the lines $x = \pm 1$ and $y = \pm 1$, which separate the whole x, y - plane into nine regions five of which have families of real solutions and four are empty regions.

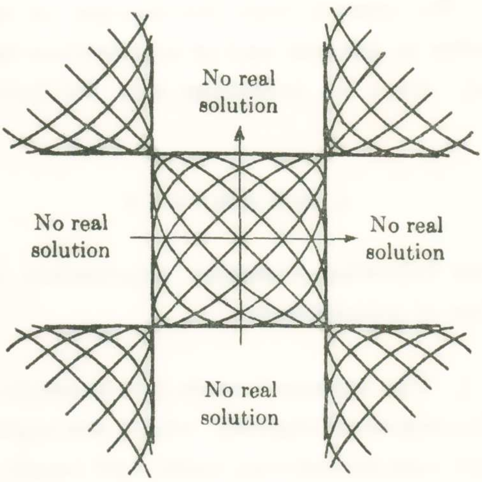


Fig. 1.

Example 2. In this example we see that the separatrices, with the help of some other curves (which are not separatrices), determine the boundaries of the canonical regions, and that the common property of each family of trajectories of the regions is a special stability situation.

The dynamical system:

$$\dot{x} = x(\varepsilon x - 1), \quad x(0) = x_0, \quad t \geqslant 0 \tag{2.1}$$

has as singular points the points $x = 0$ and $x = \frac{1}{\varepsilon}$. The general solution of (2. 1) is:

$$x(t) = \frac{x_0}{\varepsilon x_0 - (\varepsilon x_0 - 1)e^t} \tag{2.2}$$

of which the portrait is shown in Figure 2.

The separatrices in the t, x-plane are the lines $x = 0$, $x = \frac{1}{\varepsilon}$, $t = 0$. These separatrices, with the help of the line:

$$t = \log \frac{\varepsilon x_0}{\varepsilon x_0 - 1} \tag{2.3}$$

separate the half t, x-plane, $t \geqslant 0$, into four regions of which I, II, III are real regions of the validity of (2.1), and IV is the empty region.

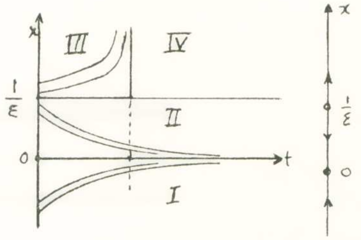


Fig. 2.

E x a m p l e 3. There exist dynamical systems without singular points, then without separatrices, of which the canonical regions are separated by curves of nature different than the separatrices. In this example these separating curves are «asymptotes».

The dynamical system:

$$\dot{x} = 2, \qquad \dot{y} = y^2 - 1 \tag{3.1}$$

is without singular points, then without separatrices. This system corresponds to the DE:

$$y' = \frac{1}{2}(y^2 - 1) \tag{3.2}$$

of which the general solution is:

$$y = \frac{1 + ce^x}{1 - ce^x}. \tag{3.3}$$

The phase-portrait of (3.3) is shown in Figure 3.

The lines $y = \pm 1$ separate the x, y-plane into three canonical regions, and these lines are «asymptotes» of the families of the solutions of the regions. We remark that in the previous example, Figure 2, the separatrix $x = 0$ is an asymptote for the trajectories of the regions I and II.

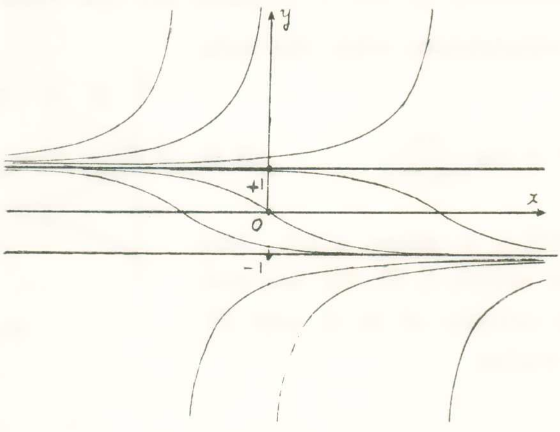


Fig. 3.

Example 4. Here we distinguish the concept of a separatrix from the concept of an envelope of a family of solution curves. Both have the property to separate the region of the validity of the system into canonical regions, but the separatrices are special members of the families of the solutions, while the envelopes have not, in general, this property but they are special singular solutions of the system, tangent to all members of the families of the solutions.

We give appropriate examples.

a. The motion of a projectile in a vacuo.

1. We imagine all trajectories described by projectiles fired from the same point 0 with the same initial velocity $\bar{v}_0$ on a x, y - plane, each trajectory corresponding to a different direction of firing φ, Figure 4. All these trajectories belong to a family of parabolas given by [**6**]:

$$y = mx - \frac{g(1+m^2)x}{2v_0} \tag{4.1}$$

where $m = \tan\varphi$ is the parameter, and g serves as a retardation.

For the envelope of the family (4.1), we eliminate the parameter m between (4. 1) and its derivative with respect to m, when the result:

$$y = \frac{v_0^2}{g^2} - \frac{gx^2}{2v_0^2} \tag{4. 2}$$

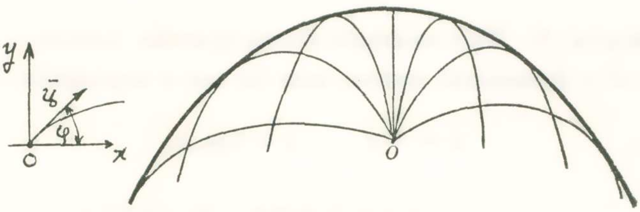


Fig. 4.

a parabola, is the envelope. This envelope separates the points which can be reached from those which can not be reached, and it is not a number of the family (4. 1), Figure 4.

The family (4. 1) has no separatrix.

2. There are other physical problems of the same nature, e. g., the «caustics» in optics are envelopes of light rays reflected by a mirror.

Let us see another example.

b. Consider the DE :

$$y'^2 = \frac{x^2}{1-x^2} \tag{4. 3}$$

which is equivalent to :

$$y' = \pm \frac{x}{\sqrt{1-x^2}} \tag{4. 4}$$

valid in the strip $|x| \leqslant 1$. Its general solution is:

$$f \equiv x^2 + (y+c)^2 - 1 = 0. \tag{4. 5}$$

That is, a family of circumferences with centers on the y-axis and tangent to the lines $x = \pm 1$.

For every «regular» point of the strip two circumferences pass, but this is not the case for the points of the lines $x = 0$, $x = \pm 1$, which are «singular lines». The lines $x = \pm 1$ are boundaries of the strip and they are tangents to every member of the family (4. 5), and these lines are «singular solutions» of (4. 4), envelopes of the family (4. 5). There is no separatrix in the family (4. 5).

We remark that in some systems, as in the Example 11, Figure 12, envelope and separatrix exist and are identical.

Example 5. This example shows that the boundary of the stability regions of a dynamical system may be not a separatrix. The system:

$$\dot{x} = -y, \qquad \dot{y} = f(x, y) \tag{5. 1}$$

with

$$f(x, y) = \begin{cases} -x + 2x^3y^3, & \text{if: } x^2y^2 < 1 \\ -x, & \text{if: } x^2y^2 > 1 \end{cases} \tag{5. 2}$$

has the origin as the singular point, which is in the region $x^2y^2 < 1$, when the appropriate equation in the x, y - plane is:

$$y' = \frac{-x + 2x^3y^2}{-y}. \tag{5. 3}$$

The eigenvalues of (5. 3) are both real and negative, then the origin is a «node», a «regular attractor», when starting from any point of the

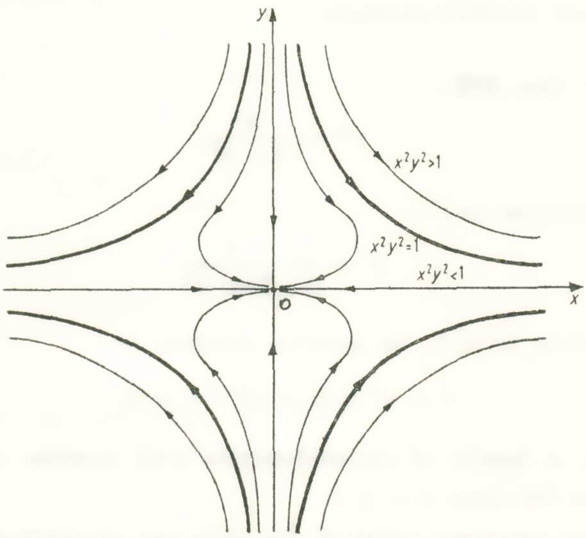


Fig. 5.

region $x^2y^2 < 1$ and following the corresponding trajectory we will terminate to the origin. The order of the magnitude of the nonlinearity of

(5.3) agrees with this result. The phase-portrait of the solutions of (5.3) is shown in Figure 5. The curve $x^2y^2=1$, which consists of four branches, determines the region of attractiveness of the origin. Outside of this region the stability situation is different. The curve $x^2y^2=1$, which is the boundary of different stability situations of the regions, is not a separatrix of the system.

Example 6. By this example we see changes in the nature of the separatrices by putting restrictions to the constants of the system [**9**].

Take the system:

$$\ddot{x} = x(\alpha^2 - x^2) + b\dot{x} \tag{6.1}$$

which has the normal form:

$$\dot{x} = y, \quad \dot{y} = x(\alpha^2 - x^2) + by, \quad \alpha b \neq 0. \tag{6.2}$$

The singular points are $O(o, 0)$, $A_1(\alpha, 0)$, $A_2(-\alpha, o)$.

For the nature of the origin we find the characteristic equation of (6.2):

$$\lambda^2 - b\lambda - \alpha^2 = 0$$

when the eigenvalues are:

$$\lambda_{1,2} = \frac{1}{2}\left(b \pm \sqrt{b^2 + 4\alpha^2}\right) \tag{6.3}$$

and since these eigenvalues are real and of opposite sign, the origin is a saddle point.

For the nature of the points A_1 and A_2, we use the transformations $x = \bar{x} + \alpha$, $y = \bar{y}$, when (6.2) is reduced to a perturbed system of which the origin corresponds to A_1 and A_2, and the characteristic equation of the perturbed system is $\lambda^2 - b\lambda + 2\alpha^2 = 0$, and the eigenvalues are:

$$\lambda_{1,2} = \frac{1}{2}\left(b \pm \sqrt{b^2 - 8\alpha^2}\right). \tag{6.4}$$

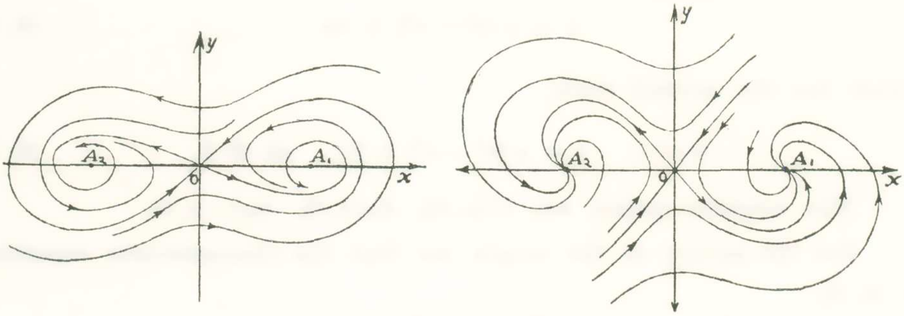
We have two cases.

a. If $b^2 < 8\alpha^2$, λ_1 and λ_2 are complex numbers with real part of sign of b, when A_1 and A_2 are spirals, stable for $b < 0$ unstable for $b > 0$.

Figure 6(a) has been drawn for A_1, A_2 stable spirals. For unstable spirals one merely reverses the arrows in this Figure.

The separatrices in this case connect saddle and spiral points and have infinite length.

b. If $b^2 > 8\alpha^2$, λ_1 and λ_2 are real and of the same sign as b, when A_1, A_2 are nodes, stable for $b<0$, unstable for $b>0$. Figure 6(b) shows the case of stable nodes A_1, A_2, and for unstable nodes we reverse the arrows in this Figure. Two of the separatrices are of finite length and two of infinite length.



(a) (b)

Fig. 6.

E x a m p l e 7. Here we have a physical problem of biology or economics in which the separatrices are calculated as special particular solutions of the general solution of the model of the problem. In addition we see a property of separatrices which is very important in interpreting theoretical results.

There are many assemblies around us of which the elements influence each other through competition and cooperation.

The «p r o b l e m o f p o p u l a t i o n g r o w t h» is a problem of this nature.

We discuss this problem as a biological problem, but, by appropriate changes in the meaning of the variables and the constants involved, the problem can become a problem in other fields, as, e.g., in economics.

We consider two coexisting species of population numbers x and y at time t, both hunters, that is one species kills members of the other species. By using appropriate assumptions, the correspondent mathematical model is the nonlinear differential system [8]:

$$\dot{x} = \alpha x - cxy, \qquad \dot{y} = by - dxy \tag{7. 1}$$

x and y are positive integers, but they can be considered as positive continuous functions of time. The coefficients α, b, c, d have a physical meaning and here are taken as positive integers.

The equilibrium points of (7. 1) are the origin and the point $A\left(\frac{b}{c}, \frac{\alpha}{c}\right)$, and we can check that the origin is a «node», and A a «saddle» point.

The system (7. 1) corresponds in the x, y - phase plane to the DE:

$$y' = \frac{b - dx}{x} \cdot \frac{y}{a - cy} \tag{7. 2}$$

of which the general solution is:

$$y^{\alpha} \cdot e^{-cy} = k \cdot x^{b} \cdot e^{-dx}. \tag{7. 3}$$

The constant k in (7. 3), which corresponds to the point A, is:

$$k = \left(\frac{\alpha}{c}\right)^{\alpha} \cdot \left(\frac{d}{b}\right)^{b} \cdot e^{b-\alpha}. \tag{7. 4}$$

Inserting (7. 4) into (7. 3) one gets the equation of the separatrix through A. For a specific case, let us take $\alpha = 4$, $b = 3$, $c = 2$, $d = 1$, when the point A and the constant k are: A (3, 2), $k \simeq .218$, and the equation of the separatrix through A is:

$$(y^2/x^2) \cdot e^{x-2y} = .218\,. \tag{7. 5}$$

An investigation of (7. 5) leads to the Figure 7, in which the four branches of the separatrix are the courves through the point A, and these branches separate the first quadrant into the four regions I, II, III, and IV.

Starting from any point of any of these regions, we see that, as $t \to \infty$, one of the species tends to vanish asymptotically, while the other species tends to become infinite. In addition, we see that the species y eventually dissapears if the corresponding (x, y) - point is in the region

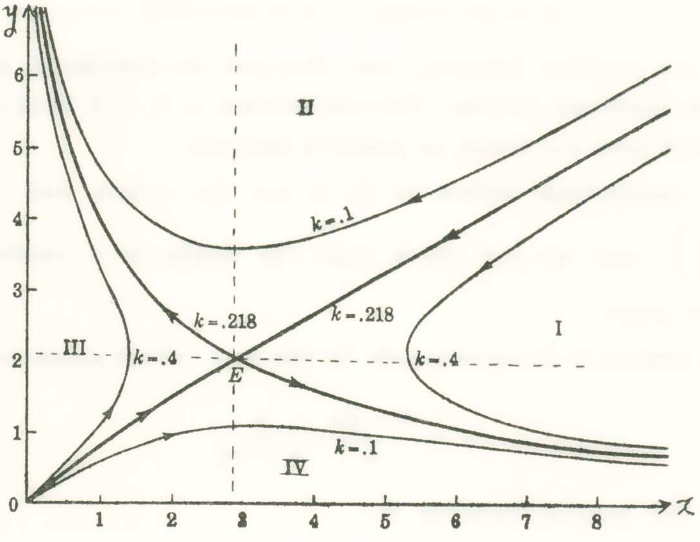


Fig. 7.

I or IV, and we see the opposite situation if the (x, y) - point is in the regions II and III. These results indicate a new property of separatrices, and show how important it is to know the location of the separatrices in the x, y - plane.

The origin 0 is a repulsor in the regions III and VI, and in III the species $x \to 0$, while in IV $y \to 0$.

Of course, due to the over-simplification of the model (7. 1) of the problem, the above results are somehow unrealistic. For better results, the model of the problem must be modified by taking into account other influences for the growth of the species, e.g., the food supply, etc.

We remark that the previous discussion, modified by suitable changes to the problem and appropriate specification on the competitive species and the limiting resources, might be useful for an investigation of a problem of nature different than the above. E. G., one can have a problem in the field of economics if the variables denote the size or extent of commercial enterprizes for a common source and for a common market.

E x a m p l e 8. By this example we see how the separatrices can be calculated in case of coexistence of many singular points, and also that the spearactrices, either closed or open, may be of finite length.

If the system is expressed by the DE [**11**]

$$\ddot{x} + 3x - 4x^3 + x^5 = 0 \tag{8.1}$$

by using $\dot{x} = y$, this equation can be reduced to

$$y' = \frac{1}{y}\left\{x\,(x^2-1)\,(x^2-3)\right\} \tag{8.2}$$

valid in the (x, y) - phase plane. The singular points are the origin and the points $(\pm 1, 0)$ and $(\pm \sqrt{3}, 0)$, and we can check that the origin and $(\pm \sqrt{3}, 0)$ are «centres», while $(\pm 1, 0)$ are «saddle» points.

The general solution of (8.1) in the (x, y) - phase plane is:

$$y^2 = c - 3x^2 + 2x^4 - \frac{1}{3}x^6. \tag{8.3}$$

The value of the arbitrary constant c of (8.3) corresponding to the saddle points $(\pm 1, 0)$ is $c = \frac{4}{3}$, when the separatrix through the points $(\pm 1, 0)$ is:

$$y^2 = \frac{1}{3}\,(4 - 9x^2 + 6x^4 - x^6) \tag{8.4}$$

of which the graph is shown in Figure 8.

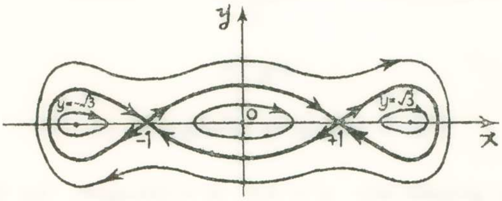


Fig. 8.

The separatrix (8. 4) has four branches all of which have a finite lenght. The branches around the points $(\pm\sqrt{3}, 0)$ are «closed», and that around the origin are «open».

From this and the previous example we see a procedure for the determination of separatrices through saddle points of the system, of which the existence guarantees the existence of the separatrices.

E x a m p l e 9. In these examples we see systems which have infinitely many «open» separatrices with finite or infinite lengths.

a. $$\ddot{x} + \omega^2 \sin x = 0. \tag{9. 1}$$

This is the pendulum equation and it is equivalent to the system:

$$\dot{x} = y, \quad \dot{y} = -\omega^2 \sin x, \tag{9. 2}$$

ω is the proper frequency. The singular points are infinitely may and they are the points $x = n\pi$, $n =$ integer, of the x-axis.

For even n the singular points are centers, and for odd n saddle points. There are infinitely many canonical regions and infinitely many separatrices running from a saddle point to the nearest saddle point. Figure 9 gives the corresponding phase-portrait. The separatrices are open and have finite length.

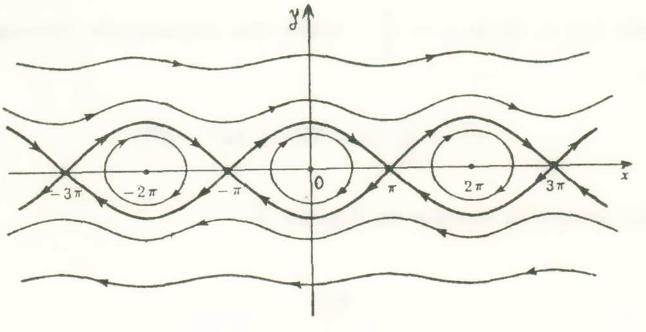


Fig. 9.

b. $$\ddot{x} + k\dot{x}|\dot{x}| + \omega^2 \sin x = 0. \tag{9. 3}$$

The singular points are $x = n\pi$, $n =$ integer, on the x-axis. For even n are spirals, and for odd n are saddle points. The infinitely many separatrices are of infinite length and run from a saddle point to the nearest spiral points, or they run from infinity to saddle points. Figure 10 shows the corresponding phase-portrait.

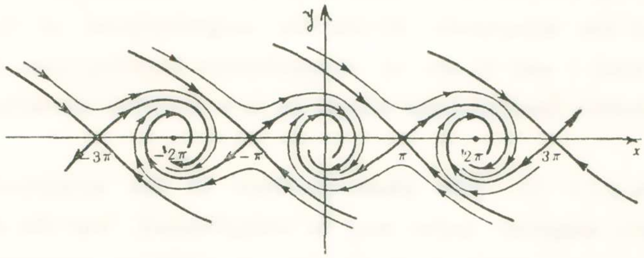


Fig. 10

E x a m p l e 10. In this example we see an nonelementary singular poit of complicated nature.

We take the system in polar coordinates

$$\dot{r} = r(1-r), \qquad \dot{\vartheta} = \sin^2\left(\frac{\vartheta}{2}\right). \tag{10.1}$$

Its singular points are $O(r=0,\ \vartheta=0)$, $O_1(r=1,\ \vartheta=0)$.

The DE in the phase-plane, corresponding to (10.1) is:

$$\frac{dr}{d\vartheta} = \frac{r(1-r)}{\sin^2\left(\frac{\vartheta}{2}\right)} \tag{10.2}$$

which can be integrated, and the family of the solutions $r = r(\vartheta) + c$ is shown in Figure 11.

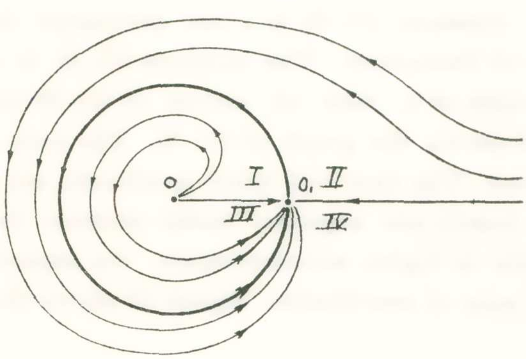


Fig. 11.

The separatrix is the circumference with center O and radius $OO_1 = 1$. The origin is a repulsor or a negative attractor (unstable). The point O_1 is a nonelementary singular point of complicated nature. The line OO_1 and the separatrix divide the neighborhood of O_1 into four sectors of which I and II are of «hyperbolic» (saddle) type, and III and IV of «parabolic» (nodal) type where O_1 is a positive attractor.

E x a m p l e 11. The phase-portrait at the neighborhood of a nonelementary singular point may be complicated, but the separatrices through this point may be very simple ; in addition «separatrices» and «envelopes» may be identical.

This is shown by the present example [**4** (b)].

The system :

$$\dot{x} = x\,(2y^3 - x^3)\,, \qquad \dot{y} = -\,y\,(2x^3 - y^3) \tag{11.1}$$

corresponds to the DE :

$$y' = -\,\frac{y\,(2x^3 - y^3)}{x\,(2y^3 - x^3)}\,. \tag{11.2}$$

The origin is the only singular point and, since the system is without linear part, this point is nonelementary.

The right hand member of (11.2) is a function of the ratio (y/x), when by using the transformation $y = x \cdot u(x)$ one can separate the variables and integrate. The general solution of (11.2) can be found to be

$$x^3 + y^3 - 2cxy = 0, \tag{11.3}$$

c is the arbitrary constant. (11.3) is a one parameter family of curves known as «Folia of Descartes». The equation (11.3) is satisfied at the origin for any value of c, then all curves of (11.3) are through the origin. Figure 12 shows the graph of (11.3). The axes of coordinates are the separatrices. The first and third quadrants are elliptic sectors, the second and fourth are negative nodal sectors. In this example, although the origin is highly nonelementary, the separatrices are very simple lines, the axes of coordinates. Figure 12 shows the phase-portrait of (11.3).

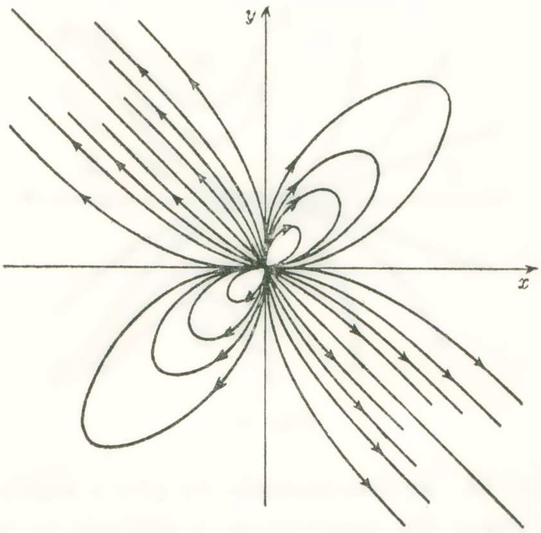


Fig. 12. (Folia of Descartes).

We remark that

The derivative of (11. 3) with respect to the arbitrary constant c gives $x = 0$, $y = 0$, then the axes of coordinates of the above system are «envelopes» of the families of the solutions of the system and are identical with the separatrices of the system.

Example 12. In this example the nonelementary singular point is of «nodal-saddle» type, and we have three separatrices.

The system:

$$\dot{x} = -x^6, \qquad \dot{y} = y^3 - yx^4 \tag{12.1}$$

corresponds to:

$$y' = \frac{yx^4 - y^3}{x^6}. \tag{12.2}$$

The origin is the only singular point which is nonelementary.

The phase portrait, shown in Figure 13, can be found approximately by, say, geometrical methods.

There are four sectors I, II, III, IV and three separatrices which are the y - axis and the curve OO_1 and OO_2. The 180° sector I has negative nodal trajectories.

The origin is a positive attractor in the sector II. The sectors III and IV are of saddle type.

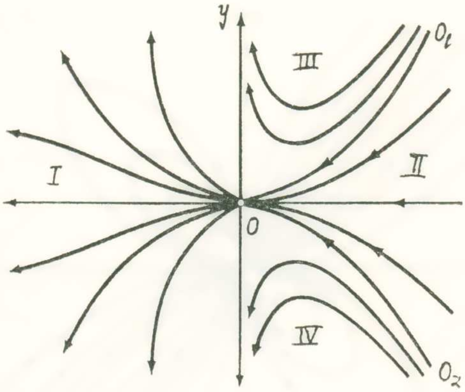


Fig. 13.

Example 13. By this example we give a highly nonelementary singular point, where the separatrices is difficult to be calculated. If the system in its phase-plane is given by [5]:

$$y' = x\left\{\frac{x^2 - xy - xy^2}{x^2 - y^2 - x^2y^2} + \frac{y}{x^2}\right\}. \tag{13.1}$$

the origin is the nonelementary singular point. Figure 14 shows the graph of the solutions of (13.1) at the neighborhood of the origin found

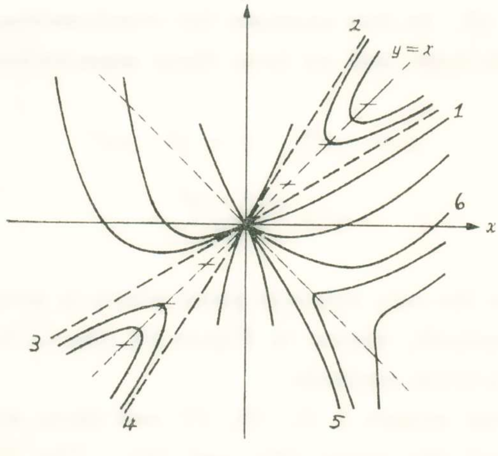


Fig. 14.

by approximate methods. We have six sectors with apex the origin, of which three are of hyperbolic type and three of parabolic type. Two hyperbolic sectors contain the line $y = x$ in the first and third quadrants, and the third hyperbolic sector contains the line $y = -x$ in the fourth quadrant. The six branches of the separatrix are the curves 01, 02, 03, 04, 05, 06 shown in Figure 14.

Example 14. At the neighborhood of the saddle points of the previous examples, either elementary or nonelementary saddle points, the behavior of the trajectories were characterized by the property that these trajectories do not intersect the correspondent separatrices.

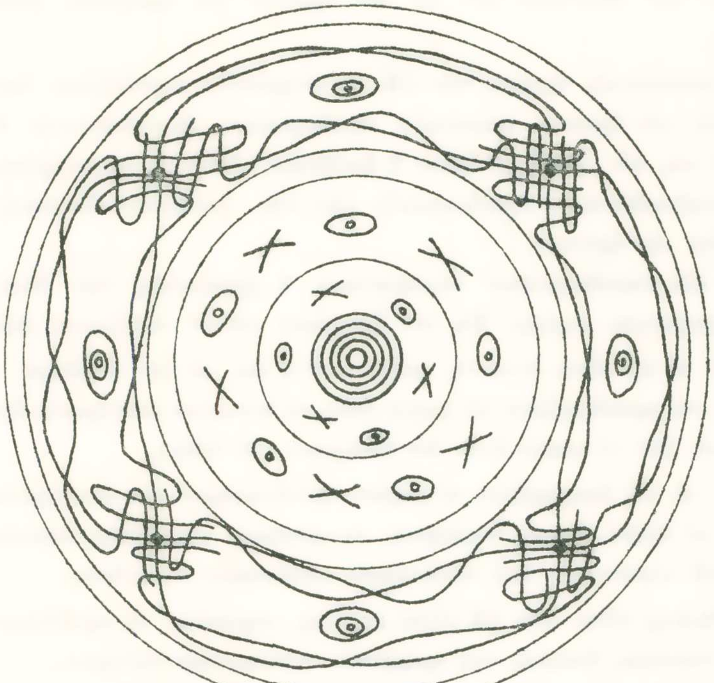


Fig. 15.

There are systems with saddle points at the neighborhood of which the behavior of the trajectories is very complicated. The three-body problem and some problems of dynamics show this complexity, Figure 15 [**1**, **2**, **12**].

ΠΕΡΙΛΗΨΙΣ

Αἱ διαχωριστικαὶ καμπύλαι (ΔΚ) τῶν δυναμικῶν συστημάτων παίζουν σπουδαῖον ρόλον εἰς τὴν ἔρευναν φυσικῶν προβλημάτων τρέχοντος ἐνδιαφέροντος. Ὅμως δὲν ὑπάρχουν σήμερον εἰδικὰ δημοσιεύματα ἀναφερόμενα εἰς τὸν γενικὸν προσδιορισμὸν τῶν ΔΚ, ὅπως καὶ εἰς τὴν χρῆσιν τῶν ἰδιοτήτων των εἰς τὴν ἔρευναν φυσικῶν προβλημάτων ποσοτικοῦ τύπου, ὁ δὲ προσδιορισμός των εἶναι ἀφ' ἑαυτοῦ ἕνα σπουδαῖον πρόβλημα.

Εἰς τὴν παροῦσαν ἐργασίαν δίδονται παρατηρήσεις καὶ εὑρίσκονται συμπεράσματα σχετικὰ μὲ τὴν ὕπαρξιν καὶ τὸν προσδιορισμὸν τῶν ΔΚ, ὅπως καὶ μὲ τὴν χρῆσιν τῶν ἰδιοτήτων των εἰς τὴν ἔρευναν καὶ ἑρμηνείαν φυσικῶν προβλημάτων.

Ὁ τοπολογικὸς ὁρισμὸς τῶν ΔΚ συμπληρώνεται καταλλήλως ὥστε νὰ γίνῃ χρήσιμος εἰς τὴν ἔρευναν πρακτικῶν προβλημάτων, διατυπώνονται θεωρήματα σχετικὰ μὲ τὰς ΔΚ χωρὶς ἀνάλυσιν ἢ ἀπόδειξιν, δίδονται παρατηρήσεις διὰ τῶν ὁποίων ὑποβοηθεῖται ὁ προσδιορισμὸς τῶν ΔΚ, τονίζονται ἰδιότητες τῶν ΔΚ χρήσιμοι διὰ τὴν ἔρευναν.

Διὰ τῶν παραδειγμάτων καταφαίνεται ἡ χρησιμότης τῶν ἰδιοτήτων τῶν ΔΚ εἰς ἐφαρμογὰς κυρίως. Ἐκ τῶν ἰδιοτήτων αὐτῶν τονίζονται δύο κυρίως:

(α): αἱ ΔΚ εἶναι δυνατόν, μόναι των ἢ καὶ μὲ τὴν βοήθειαν καὶ ἄλλων καμπυλῶν, νὰ προσδιορίζουν τὰ χωρία ὅπου τὰ δυναμικὰ συστήματα ἔχουν λύσεις πραγματικὰς ἀπὸ τὰ χωρία ὅπου δὲν ὑπάρχουν κἂν λύσεις,

(β): αἱ ΔΚ διαχωρίζουν τὸ χωρίον προσδιορισμοῦ τῶν συστημάτων εἰς χωρία, ὅπου αἱ λύσεις ἔχουν διαφόρους καταστάσεις εὐσταθείας ἕκαστον χωρίον, ὁπότε αἱ ΔΚ εὑρίσκονται ὑπὸ «οὐδετέραν» κατάστασιν εὐσταθείας.

Ἡ ἰδιότης αὕτη τῶν ΔΚ εἶναι μεγάλης σημασίας εἰς προβλήματα εὐσταθείας τῆς νεωτέρας θεωρίας «μὴ γραμμικῶν συστημάτων ἐλέγχου».

Ἡ παροῦσα ἐργασία θὰ συμπληρωθῇ καταλλήλως πολὺ σύντομα.

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