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ΕΦΗΡΜΟΣΜΕΝΑ ΜΑΘΗΜΑΤΙΚΑ.— **On the linearization of nonlinear models of phenomena. Second part: Linearization by approximate methods,** *by Demetrios G. Magiros* *. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ἰ. Ξανθάκη.

## INTRODUCTION

In a previous paper, contained in this volume, we discussed exact linearization techniques for solving nonlinear differential equations, when we get exact general solutions of NLDE in a closed form.

These ideal methods can be applied to only a few cases, when approximate linearization methods are suggested for approximate particular solutions of NLDE.

By a variety of examples, we point out different approximate linearization methods, cases where the linearization is not permitted, and cases where the linearization gives useful results.

In many fields of research, we see advantages and disadvantages of the linearization methods, as well as the importance and the influence of the nonlinearities.

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* Δ. Γ. ΜΑΓΕΙΡΟΥ, **Γραμμικοποίησις μὴ γραμμικῶν μαθηματικῶν μοντέλων τῶν φαινομένων. Μέρος ΙΙ: Γραμμικοποίησις μὲ κατὰ προσέγγισιν μεθόδους.**— Scientific Consultant, General Electric Company, (RESD) Philadelphia, PA., U. S. A.

## LINEARIZATION BY APPROXIMATE METHODS

Whenever exact methods are not applicable to NLDE, and this case is the usual one in applications, approximate methods are suggested, by which one may get approximations of particular solutions.

The approximate methods are either geometrical, or analytical, or numerical, and the concept of linearization may enter to any of these methods.

The linearization by approximate methods leads to results, which are acceptable in some cases, but not acceptable in some other cases, this depending mainly on the nature of the problem which is associated with the linearization. In some linearization approximate methods the concepts of the «error» and the «norm», which measures the error, play essential role and characterize the methods, but in some other methods the concepts of the «error» and «norm» are not necessary. By the following examples and appropriate remarks we try to clarify the above statements.

E x a m p l e 1. Let us pose the following problem:

«Find a curve such that the product of the distances from two different points E and E′ to anyone of its tangents is a non-zero constant, $b^2$». We take the line EE′ as x - axis, the middle point of EE′ as the origin, and $EE' = 2c$. The property of the curve of the problem leads to DE:

$$(y - cy')^2 - c^2y'^2 = b^2(1 + y'^2)$$

then we have:

$$y = xy' \pm (b^2 + \alpha^2 y'^2)^{1/2}, \qquad \alpha^2 = b^2 + c^2 \tag{1}$$

The equation (1) is of Clairaut type and by exact methods (not exact linearization) its solution can be found to be:

General Solution: $$y = cx \pm (b^2 + \alpha^2 c^2)^{1/2} \tag{1.1}$$

Singular Solution: $$\frac{x^2}{\alpha^2} + \frac{y^2}{b^2} = 1 \tag{1.2}$$

The true solution of the problem is the ellipse (1.2), and not any particular solution coming from the general solution (1.1).

If we try to solve the NLDE (1) by linearization, the only way is to omit the nonlinear term in (1), when the corresponding linear equation is $y = xy' + b$ of which the solution is $y = cx\ b$, that is the family of straight lines in the x, y - plane, but not the solution (1.2) (ellipse).

By this example we see that the linearization by omitting the nonlinearities leads in general to unaccepted results.

Example 2. The NLDE.

$$y' = (x^2 + \varepsilon^2 y^2)^{1/2}, \quad \varepsilon = \text{parameter} \tag{2}$$

can not be solved by exact methods, but by a linearization coming from an «appropriate restriction of the variables», which leads to an approximate geometrical method, called «method of isoclines».

The linearization of (2) by omitting the nonlinearity leads to the linearized equation: $y' = \pm x$, of which the general solution is the family of parabolas $y = c \pm \frac{1}{2} x^2$, $c$ = parameter. Such a linearization is not accepted.

The linearization of (2) by restricting the variables x and y according to the restriction:

$$(x^2 + \varepsilon^2 y^2)^{1/2} = k, \quad k = \text{constant} \tag{2.1}$$

gives the linearized equation:

$$y' = k \tag{2.2}$$

with general solution:

$$y = kx + c \tag{2.3}$$

The restriction (2.1) of the variables of (2) is written in the form:

$$\frac{x^2}{k^2} + \frac{y^2}{(k/\varepsilon)^2} = 1 \tag{2.4}$$

The above linearization of (2) is equivalent to taking curves in the x, y - plane through any point of which the unknown solution of (2) has the same slope, that is the same inclination angle with x-axis.

This idea gives a «graphical method» for construction of the solution of a NLDE, called «method of isoclines».

The «isocline curves» of (2) are the ellipses (2.4), by which one can get approximately the solution of (2) in the form of a «directed field».

Example 3. The study of the multivibrator, a basic electronic circuit, with nonlinear resistance and in the absence of external forces, leads to the famus Van der Pol equation :

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0 \tag{3}$$

which is equivalent to the normal system :

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \varepsilon(1 - x_1^2)x_2 - x_1 \tag{3.1}$$

The time t is the independent variable.
This equation can not be solved by exact methods.
Van der Pol found graphically that this equation has one isolated periodic solution, its «limit cycle».

We can linearize the system (3.1) by considering $x^2$ as an infinitesimal stronger than x, that is by taking the «nonlinearity condition» :

$$x^2 = o(x) \tag{3.2}$$

when the linearized system is :

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \varepsilon x_2 \tag{3.3}$$

The origin of (3.3) is an «unstable equilibrium» for $\varepsilon > 0$, and it is a «globally asymptotically stable» for $\varepsilon < 0$.

According to a theorem of Liapunov, the stability situation of the origin of (3.1) is «topologically» the same with the stability situation of the origin of the linearized system (3.3), and, then, if the point of interest in our investigation is the stability situation of the origin of the NLS (3.1), the above linearization by the condition (3.2), is permitted.

But, the NLS (3.1) has one «limit cycle», while the linearized system (3.3) has no such solution. In addition, if the origin of (3.3) is «asymptotically stable», it will be «globally asymptotically stable», while the origin of (3.1) will be «asymptotically stable» in a finite region, the «region of attraction», of which the determination of the boundary needs the knowledge of the nonlinearity of (3.1). Therefore, the linearization, from the above point of interest, is not permitted, and the nonlinearity is necessary to be taken into account.

**Linearization and the Stability of Equilibrum Points of Dynamical Systems.**

The results of the previous example (the Van der Pol equation) can be generalized and supplemented by considering the general system of NLDE in its normal form :

$$\dot{x}_i = f_i(t, x_1, \ldots, x_n); \quad i = 1, \ldots, n \tag{a}$$

where the functions $f_i$ have the properties that guarantee the existence and uniqueness of the solution of (a) through any point $x_o$ in the region of the validity of (a) :

$$D: \quad 0 \leqslant t, \quad -\infty < x_1, \ldots, x_n < \infty \tag{a. 1}$$

In a large number of cases, the system (a) can be written in the form :

$$\dot{x} = Ax + X \tag{a. 2}$$

where $x$, $\dot{x}$, $X$ are n-column matrices, $A$ is a $(n \times n)$ - matrix either constant or time-dependent; $X$ does not contain linear terms in the variables $x_1, \ldots, x_n$ then $X$ represents the set of the nonlinearities of (a), and $X(t, 0) \equiv 0$. The system :

$$\dot{x} = Ax \tag{a. 3}$$

is the linear part of (a. 2), that is its «first approximation».

For the applications, especially of engineering kind, the deduction of the NLDE (a. 2) to the LDE (a. 3), that is the linearization of (a. 2), has been universally prevailed in the past. This lasted until some decades ago, when stringent requirements and serious demands of new phenomena, as, e. g., phenomena of vacuum tubes, made clear the inadequacy of the linearization and the importance of the nonlinearities [**1**].

To study properties of the solutions of the NLDE (a. 2), (e. g., their continuity, periodicity, boundedness, stability, oscillation) the linearization of (a. 2) helps only in a few cases, and its acceptance is permitted only «under restrictions of the nature of the matrix A and of the nature and smallness of the nonlinearities X».

The study of the stability of the trivial solution of (a. 2) gives a typical area of research for a verifi cation of the above statements and remarks. Let us first state some results and apply them to appropriate examples in order to show possibilities for a linearization and the influence of the nonlinearities.

The systems (a. 2) and (a. 3) may be either «critical» or «noncritical». In the «noncritical systems» the real parts of the eigenvalues of the matrix A are all nonzero; and in the «critical systems» there are eigenvalues of A with zero real part.

## a. Criteria of the First Approximation.

The following classical «Liapunov criteria of the first approximation» are based on the nature of the eigenvalues of the system.

Criterion a. «If in a noncritical linear system (a. 3) all eigenvalues have negative real part, then this system is «asymptotically stable» at the origin. If, in addition, the nonlinearity X of the system (a. 2) satisfies the condition:

$$\lim_{|x| \to 0} \frac{|X|}{|x|} = 0 \qquad \text{(a. 4)}$$

then, (a. 2) is also «asymptotically stable» at the origin».

Criterion b: «If in a noncritical linear system (a. 3) there are eigenvalues, at least one, with positive real part, then (a. 3) and (a. 2) are «unstable» at the origin, even if the nonlinearity condition (a. 4) holds».

Criterion c: «In a critical system one can not decide about the stability or instability on the system at the origin without taking into account the nonlinearities of the system».

As we can see, the linearization of the system in the first two cases gives results accepted for the nonlinear system, but in the third case, the «undecided case», the linearization is not permitted.

The above criteria are referred to «autonomous systems». In «nonautonomous systems», the linearization in the case of the first criterion necessitates nonlinearity conditions different than (a. 4).

E. g., if the matrix A is either constant or periodic function of time t, and $X = X(t, x)$, the «nonlinearity condition» will be:

$$\lim_{|x| \to 0} \frac{|X(t, x)|}{|x|} \leqslant m e^{\alpha t} |x|^{b} \qquad (a.5)$$

where α, b, m are positive constants and $t \geqslant 0$.

In implicit equations, e. g., if $X = X(x, \dot{x})$, the «nonlinearity condition» will be:

$$\lim_{\substack{|x| \to 0 \\ |\dot{x}| \to 0}} \frac{|X(x_1 \dot{x})|}{|x| + |\dot{x}|} = 0 \qquad (a.6)$$

There are «nonlinearity conditions» of integral type.

## b. Criteria by Using a Liapunov Function.

The use of Liapunov functions, which gives the «second Liapunov method» in stability theory, answers the question of the stability situation of the origin in critical systems.

A function $V = V(t, x_1, \ldots, x_n)$, of which the time derivative is calculated by using the NLDE (a):

$$\dot{V} = \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} \frac{\partial x_i}{\partial t} = \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_1} f_i \qquad (a.7)$$

is called a «Liapunov function».

To find a «Liapunov fuction» for a nonlinear system is a very important problem. The following criteria by using Liapunov functions consist of the classical «second group of Liapunov criteria».

Criterion d: «If in a region D, where the origin of the system is included, there is a Liapunov function V such that V is positive and its time derivative $\dot{V}$ is negative, except at the origin where both V and $\dot{V}$ are zero, then the origin (as an equilibrium point of the system) is «asymptotically stable».

Criterion e: «If in a region D, which includes the origin, there exists a Liapunov V such that V is positive in D, except at the origin where it is zero, and $\dot{V}$ is either zero everywhere in D or negative in D with the exception of some points of D (the origin included) where it is zero, then the origin is «stable» in D».

Criterion f: «If V and $\dot{V}$ are both positive in D, but both zero at the origin, then the origin is «unstable» in D».

Of all the above criteria there are modifications and extensions, but these criteria are sufficient to show possibilities for a successful linearization of nonlinear systems, the influence of the nonlinearities for the stability of the origin of the systems, and indicate an approximate calculation of the «stability regions» of the origin.

By the following examples we analyze and accomplish all these [**3**].

Example 4. The Duffing equation:

$$\ddot{x} + \alpha_1\dot{x} + \alpha_2 x + bx^3 = 0 \tag{4}$$

which in a normal form, can be written as:

$$x = x_1, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -\alpha_2 x_1 - \alpha_1 x_2 - bx_1^3 \tag{4.1}$$

or in a matrix form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha_2 & -\alpha_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ -bx_1^3 \end{pmatrix} \tag{4.2}$$

has three equilibrium points, among which is the origin of which we ask for the stability situation.

The nonlinearity condition (a. 4) is satisfied, for

$$\lim_{|x|\to 0} \frac{|X|}{|x|} = \lim_{|x_1|\to 0} \frac{|b|\,|x_1^3|}{|x_1|} = \lim_{|x_1|\to 0} |b|\,x_1^2 = 0 \tag{4.3}$$

The eigenvalues are given by:

$$\lambda = \frac{1}{2}\left(-\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_2}\right) \tag{4.4}$$

then the origin of (4), that is its state $(x, \dot{x})$ at the origin, is

(a): «asymptotically stable», if $\alpha_1 > 0$, $\alpha_2 > 0$

(b): in the «undecided case», if $\alpha_1 = \alpha_2 = 0$, or $\alpha_1 = 0$, $\alpha_2 > 0$, or $\alpha_1 \neq 0$, $\alpha_2 = 0$

(c): «unstable» in the other cases of $\alpha_1$ and $\alpha_2$.

E x a m p l e 5. The equation :

$$\ddot{x} + \dot{x} + x - \sqrt{x} = 0, \qquad x > 0 \tag{5}$$

can be written in the form :

$$x = x_1, \qquad \dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - x_2 + \sqrt{x_1} \tag{5.1}$$

or :

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{x_1} \end{pmatrix} \tag{5.2}$$

The equilibrium points of this system are the origin and ($x = 1$, $x_2 = 0$), and we are interested in the stability of the origin.

The eigenvalues are :

$$\lambda = \frac{1}{2}(-1 \pm \sqrt{3}\, i), \qquad i = \sqrt{-1} \tag{5.3}$$

The nonlinearity condition (a. 4) is not satisfied, for :

$$\lim_{|x| \to 0} \frac{|X|}{|x|} = \lim_{|x_1| \to 0} \frac{\sqrt{x_1}}{|x_1|} = \lim_{|x_1| \to 0} \frac{1}{\sqrt{x_1}} = \infty \tag{5.4}$$

Then, the state ($x = 0$, $\dot{x} = 0$) is «asymptotically stable» for the linear system corresponding to (5), but this state is «unstable» for the system (5) because of its nonlinearity.

E x a m p l e 6. The linear equation :

$$\ddot{x} + k\dot{x} + (\omega^2 + \varepsilon \cos t)\, x = 0 \tag{6}$$

with k positive constant, and $\varepsilon$ small parameter, is equivalent to:

$$\dot{x} = y, \qquad \dot{y} = -(\omega^2 + \varepsilon \cos t)\, x - ky \tag{6.1}$$

Its eigenvalues are:

$$\lambda = \frac{1}{2}\left(-k \pm \sqrt{k^2 - 4(\omega^2 + \varepsilon \cos t)}\right) \tag{6.2}$$

and the origin of (6) is «asymptotically stable».

If we add to (6) a nonlinearity $X = X(x)$ such that:

$$\lim_{|x| \to 0} (|X| / |x|) = 0$$

then the state $x = 0$, $\dot{x} = 0$ will be «asymptotically stable».

Also, if we add a nonlinearity $X = X(t, x)$ such that:

$$\lim_{|x| \to 0} (|X(t, x)| / |x|) = 0$$

holds uniformly in t for $t \geqslant 0$, then the state $(x, \dot{x})$ is «asymptotically stable» at the origin.

Example 7.

$$\ddot{x} + \alpha x - e^t x^2 = 0 \tag{7}$$

The nonlinearity $X = -e^t x^2$ satisfies the condition (a. 5) for $m = \alpha = b = 1$, for:

$$\lim_{|x| \to 0} \frac{-e^t x^2}{|x|} = \lim_{|x| \to 0} e^t |x| = 0 \tag{7.1}$$

The eigenvalues are $\lambda = \pm \sqrt{-\alpha}$, then we have two cases:

(a): for $\alpha > 0$, $\lambda = \pm \sqrt{\alpha}\, i$, (b): for $\alpha < 0$, $\lambda = \pm \sqrt{\alpha}$

Therefore, although the nonlinearity is favorable for stability of the origin, if $\alpha > 0$ we have the «undecided case», and if $\alpha < 0$ the origin is «unstable».

Example 8. We consider the nonlinear singular equation:

$$\ddot{x} + \left(\frac{1}{1+x}\right)\dot{x} + \left(\frac{1}{1+x}\right)x = 0 \tag{8}$$

For the discussion of this equation by linearization, we need to separate in it the linear and nonlinear parts. We can write (8) as:

$$\ddot{x} + \left(1 - \frac{x}{1+x}\right)\dot{x} + \left(1 - \frac{x}{1+x}\right)x = 0$$

when:

$$\ddot{x} + \dot{x} + x - \left[\frac{x}{1+x}(x + \dot{x})\right] = 0 \tag{8.1}$$

The nonlinearity: $X = -\left[\frac{x}{1+x}(x + \dot{x})\right]$ satisfies the condition needed for stability of the origin, for:

$$\lim_{\substack{|x| \to 0 \\ |\dot{x}| \to 0}} \frac{\frac{x}{1+x}\,(|x| + |\dot{x}|)}{|x| + |\dot{x}|} = \lim_{|x| \to 0} \frac{|x|}{|1+x|} = 0 \tag{8.2}$$

The eigenvalues of the equation (8.1) are :

$$\lambda = \frac{1}{2}(-1 \pm \sqrt{3}\, i) \tag{8.3}$$

then the state $(x, \dot{x})$ is «asymptotically stable» at the origin.

The previous examples are treated by using the criteria of the «first approximation», that is, by using the nature of the eigenvalues. In the following two examples we use the criteria of the «Liapunov second method», which are related to Liapunov functions.

Example 9.

$$\left.\begin{aligned} \dot{x} &= -y + \alpha x^3 \\ \dot{y} &= \phantom{-}x + \alpha y^3 \end{aligned}\right\} \tag{9}$$

The constant $\alpha$ characterizes the nonlinearity of the system. The origin is an equilibrium point, and the eigenvalues are $\lambda = \pm i$, then the stability of the origin can not be decided by linearization. We use a Liapunov function and apply the «criteria of the second group».

The function :

$$V = x^2 + y^2 \tag{9.1}$$

can be taken as a Liapunov function.

(a) : For the linear system ($\alpha = 0$),

$$\dot{x} = -\dot{y}, \qquad y = x \tag{9.2}$$

the derivative of V becomes :

$$\dot{V} = 2(x\dot{x} + y\dot{y}) = 2(-xy + xy) = 0 \tag{9.3}$$

Both V and $\dot{V}$ are zero at the origin, V is positive in the x, y - plane, and $\dot{V}$ is everywhere zero, then the origin is «stable» (but not «asymptotically stable») for the linear system (9.2).

(b): The function $\dot{V}$ for the nonlinear system (9) becomes:

$$\dot{V} = 2\{x(-y+\alpha x^3) + y(x+\alpha y^3)\} = 2\alpha(x^4+y^4) \qquad (9.4)$$

Both V and $\dot{V}$ are zero at the origin, $V > 0$; and $\dot{V} > 0$ for $\alpha > 0$, $\dot{V} < 0$ for $\alpha < 0$.

Then, for $a > 0$ the origin is «unstable» for the system (9), and for $\alpha < 0$ it is «asymptotically stable».

This example shows the great influence of the nonlinearity, even if it is very small in magnitude. The stability of the origin of a linear system is completely changed if we add to the linear system a small nonlinearity, and in the nonlinear system which results, the nature of the stability depends not only on the magnitude of the nonlinearity but also on the sign of the nonlinearity.

Example 10. The linearization is unable to help the calculation of the «region of asymptotic stability» of the origin of a nonlinear system. The nonlinearities play a decisive role in this problem.

The Liapunov functions help to find an approximation to the region. The Van der Pol equation can be used as an example.

The Van der Pol equation, written in a normal form, is:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \varepsilon(1-x_1^2)x_2 \qquad (10)$$

and the origin of this system is, for $\varepsilon < 0$, «asymptotically stable». An appropriate Liapunov fuction for this system is:

$$V = \frac{1}{3}(x_1^2 + x_2^2) \qquad (10.1)$$

when, by using (10), its derivative is:

$$\dot{V} = \varepsilon x_2^2(1-x_1^2) \qquad (10.2)$$

The function V is zero at the origin and positive everywhere in the x, y - plane. The function $\dot{V}$ is zero at the origin; and since $\varepsilon < 0$, it is $\dot{V} < 0$ for $1 - x_1^2 > 0$, that is inside the unit circle:

$$x_1^2 + v_2^2 = 1 \qquad (10.3)$$

This circle is a part of the unknown «region of attraction» of the origin of the Van der Pol equation, that is this circle is an approximation of this region.

Example 11. Linearization of Almost Linear Systems.

In the previous examples, we saw different types of approximate linearization methods, which are not explicitely related to the concept of the «error of the approximation». In this example of «linearization of almost linear systems» we have a case of linearization which is characterized by the «error of approximation».

The nonlinear systems of the special form:

$$m\ddot{x} + kx = \varepsilon f(t, x, \dot{x}) \tag{11}$$

where m and k are positive constants, ε a small parameter, and f a nonlinear function, is called «almost linear» or «quasi - linear» system.

It appears in many fields and its treatment is very difficult.

By the «method of Krylor-Bogaliubov» one can replace this system by an «equivalent linear system» [**2**].

The linearization by this method is succeeded by the use of a combination of restrictions and transformations of variables and coefficients, and introducing some integrator operators.

This method can be successfully applied to modern control systems, where certain linear loops are interlinked with nonlinear ones.

The starting point of the method are formulae known from the «first approximation» of the system.

Let us see the essentials of the method.

By a series of mathematical operations during the process of linearization two parameters $\lambda_1$ and $\lambda_2$ are introduced, and, by using them, the equations (11) can be reduced to the linear system:

$$m\ddot{x} + \lambda_1\dot{x} + \lambda_2 x = 0(\varepsilon^2) \tag{11.1}$$

The solution of (11.1) gives the solution of (11) with accuracy of $\varepsilon^2$, and (11.1) and (11) are «almost equivalent» with error $\varepsilon^2$.

$\lambda_1$ and $\lambda_2$ are given in integral form by:

$$\left.\begin{aligned} \lambda_1 &= \frac{\varepsilon}{\pi\alpha\omega}\int_0^{2\pi} f(\alpha\cos\psi, -\alpha\omega\sin\psi)\sin\psi\, d\psi \\ \lambda_2 &= k - \frac{\varepsilon}{\pi\alpha}\int_0^{2\pi} f(\alpha\cos\psi, -\alpha\omega\sin\psi)\cos\psi\, d\psi \end{aligned}\right\} \tag{11.2}$$

where $\alpha = \alpha(t)$ and $\psi = \psi(t)$ are the $\alpha$ and $\psi$ of $x = \alpha \cos\psi$, which is the solution of the first approximation of (11), that is when $\varepsilon = 0$ and $\omega = \sqrt{k/m}$ is the linear frequency.

$\lambda_1$ is the equivalent coefficient of damping, and $\lambda_2$ the equivalent coefficient of restoring force. $\lambda_1$ and $\lambda_2$ depend on the amplitude $\alpha$.

The integral representation (11.2) of $\lambda_1$ and $\lambda_2$ is found formally, but one can give a justification of these formulae by using the «principle of energy balance», and the «principle of harmonic balance».

The calculation of $\lambda_1$ and $\lambda_2$ is the main subject of the above linearization «method of Krylov - Bogaliubov».


## ΠΕΡΙΛΗΨΙΣ

Εἰς προηγουμένην ἐργασίαν, ἡ ὁποία περιέχεται εἰς τὸν παρόντα τόμον, ἔχομεν ἐξετάσει «ἀκριβεῖς μεθόδους γραμμικοποιήσεως» μὴ γραμμικῶν μοντέλων τῶν φαινομένων, αἱ ὁποῖαι δίδουν «γενικὰς λύσεις» τῶν μοντέλων.

Εἰς τὴν παροῦσαν ἐργασίαν ἐξετάζομεν «κατὰ προσέγγισιν μεθόδους γραμμικοποιήσεως», ὁπότε λαμβάνομεν προσεγγίσεις «μερικῶν λύσεων» τῶν μοντέλων. Ἡ σπουδὴ τῶν ἰδιοτήτων τῶν λύσεων μὴ γραμμικῶν διαφορικῶν ἐξισώσεων μόνον εἰς ὀλίγας περιπτώσεις ὑποβοηθεῖται μὲ τὴν γραμμικοποίησιν. Εἰς τὰς περισσοτέρας περιπτώσεις ἡ γραμμικοποίησις ὁδηγεῖ εἰς συμπεράσματα ἀσύμφωνα μὲ τὴν πραγματικότητα. Διὰ μίαν γραμμικοποίησιν, ἡ ὁποία ἐνδέχεται νὰ εἶναι ἐπωφελής, εἶναι ἀναγκαῖον νὰ λαμβάνεται ὑπ᾽ ὄψιν ἡ φύσις τοῦ γραμμικοῦ μέρους τῶν μοντέλων, καθὼς καὶ κατάλληλοι περιορισμοὶ τῶν μὴ γραμμικοτήτων τῶν μοντέλων.

Τὸ πρόβλημα τῆς εὐσταθείας καταστάσεων ἰσορροπίας δυναμικῶν συστημάτων, αἱ ὁποῖαι δύνανται νὰ θεωροῦνται ὡς ἡ ἀρχὴ τῶν συστημάτων ἀναφορᾶς τῶν μοντέλων, δίδει ἓν τυπικὸν παράδειγμα, ποὺ ὑποδεικνύει περιπτώσεις δεκτῆς γραμμικοποιήσεως, καθὼς καὶ τὴν ἐπίδρασιν τῶν μὴ γραμμικοτήτων εἰς τὴν ἐξέτασιν τῶν φαινομένων.

Ἀκόμη καὶ εἰς μὴ γραμμικὰ συστήματα, τὰ ὁποῖα εἶναι «σχεδὸν γραμμικά», ἡ γραμμικοποίησις ἐμφανίζει ἀφαντάστους δυσκολίας.

Ἡ ἀλήθεια τῶν ἀνωτέρω παρατηρήσεων δεικνύεται μὲ κατάλληλα παραδείγματα.

## REFERENCES

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Ὁ Ἀκαδημαϊκὸς κ. **Ἰω. Ξανθάκης**, παρουσιάζων τὴν ἀνωτέρω ἐργασίαν, εἶπε τὰ ἑξῆς:

Ἔχω τὴν τιμὴν νὰ παρουσιάσω ἐργασίαν τοῦ κ. Δημητρίου Μαγείρου ὑπὸ τὸν τίτλον: «Κατὰ προσέγγισιν Μέθοδοι γραμμικοποιήσεως μὴ γραμμικῶν διαφορικῶν ἐξισώσεων».

Εἰς προγενεστέραν ἀνακοίνωσιν ὁ κ. Μάγειρος ἀναφέρεται εἰς ἀκριβεῖς μεθόδους γραμμικοποιήσεως μὴ γραμμικῶν διαφορικῶν ἐξισώσεων προτύπων τῶν φαινομένων, αἱ ὁποῖαι μᾶς ὁδηγοῦν εἰς γενικὰς λύσεις τῶν ἐν λόγῳ προτύπων.

Εἰς τὴν παροῦσαν ἐργασίαν ἐξετάζονται «Κατὰ προσέγγισιν μέθοδοι αἱ ὁποῖαι μᾶς ὁδηγοῦν εἰς μερικὰς λύσεις τῶν ἐν λόγῳ διαφορικῶν ἐξισώσεων». Ἡ σπουδὴ τῶν ἰδιοτήτων τῶν λύσεων μὴ γραμμικῶν διαφορικῶν ἐξισώσεων εἰς ὀλίγας μόνον περιπτώσεις ὑποβοηθεῖται μὲ τὴν γραμμικοποίησιν. Πράγματι εἰς τὰς περισσοτέρας τῶν περιπτώσεων ἡ γραμμικοποίησις μᾶς ὁδηγεῖ εἰς συμπεράσματα ποὺ δὲν συμφωνοῦν μὲ τὴν πραγματικότητα. Εἴς τινας ὅμως περιπτώσεις αἱ κατὰ προσέγγισιν μέθοδοι γραμμικοποιήσεως μᾶς παρέχουν συμπεράσματα ἀποδεκτά.

Ὁ συγγραφεὺς παρέχει μίαν ποικιλίαν ἐφαρμογῶν, ὅπου αἱ κατὰ προσέγγισιν μέθοδοι γραμμικοποιήσεως δὲν εἶναι ἐπιτρεπταί, καθὼς καὶ περιπτώσεις ὅπου αἱ ἐν λόγῳ μέθοδοι μᾶς ὁδηγοῦν εἰς χρήσιμα συμπεράσματα.