

ΕΦΗΡΜΟΣΜΕΝΑ ΜΑΘΗΜΑΤΙΚΑ.— **Remarks on stability concepts of solutions of dynamical systems, by Demetrios G. Magiros***.

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INTRODUCTION

The study of the stability situation of physical and social phenomena, which are modeled as dynamical systems, is based on a variety of stability concepts, and this variety makes the study complicated and the stability results questionable in many cases.

In this paper, we will give a set of remarks on stability concepts, which may permit a better understanding of the difficulties of current stability problems.

The stability concepts may come from sources of different nature.

Examining the stability of a motion in its orbit and of the orbit of a motion, one can distinguish two basic stability concepts, which contain many other specialized concepts as special cases.

The manner in which a state of a system approaches another state, or deviates from it, the way the perturbations act on a system, or the way one measures their norm and their effect on the system, the type of the mathematical model of the system, etc., are sources for stability concepts of different nature.

All these different stability concepts can be «unified» into the same «stability relationships», and this «unification» of the stability concepts brings a natural simplicity in the understanding of subjects concerning stability, and gives rise to new results.

1. REMARKS ON PERTURBATIONS

It is necessary to make remarks concerning the perturbations of the systems and their solutions. A variety of stability concepts comes from the manner the perturbations act on the system, the way the norm

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of perturbations is taken, and the way their effect on the system is measured.

The perturbations, which can be considered as minor disturbing forces, may act on a system either «momentarily», when only the initial conditions are perturbed, or «permanently», when the system itself is disturbed and the perturbations, during their action, must enter the equations of the system explicitly. The «sudden» and «persistent» perturbations characterize two different classes of stability concepts.

The «norm» of the perturbations can be taken in different ways, and each of them characterizes a special stability concept under persistent perturbations. The perturbations may depend on deviating arguments, when the stability concepts will be related to the deviations of the arguments.

The «effect» of the perturbations is a change of certain quantities pertaining to the original motion and/or to its orbit, and this effect can be visualized by the change of the orbit S of the unperturbed motion $x_i(t)$ into the orbit \bar{S} of the perturbed motion $\bar{x}_i(t)$.

Given a state of a dynamical system, that is a point P on S , if, as a result of the perturbations, \bar{S} is the new orbit, and the point \bar{P} on \bar{S} corresponds to the point P of S , the distance $\varrho = P\bar{P}$ can be taken as the magnitude of the effect of the perturbations at P , Fig. 1.

To a given point P of S , one may make to correspond different points \bar{P} on \bar{S} , and each correspondence characterizes a specific stability concept of the motion. Any such correspondence presupposes an assumption, and each assumption comes from a physical reason.

One can distinguish two, the most physical, correspondences between P and \bar{P} , when two important stability concepts result, namely, the stability concept in the sense of Liapunov and that in the sense of Poincaré, by using the stability distances $\varrho = P\bar{P}$, called Liapunov and Poincaré distances. In Fig. 2 «Liapunov distances» are shown P and \bar{P} correspond to each other at the «same time». In Fig. 3, P and \bar{P} correspond in such a way that :

$$\varrho = \varrho(P, \bar{S}) = P\bar{P} = \min \left\{ \sum_{i=1}^n |\bar{x}_i - x_i|^2 \right\}^{1/2}$$

and $\varrho = P\bar{P}$ is «Poincaré distance».

If the model of the system is expressed by differential equations with deviating arguments, e. g. by differential equations with retarded arguments of retardation $\tau_i(t)$, the «initial functions» $\varphi_i(t)$ of S and $\bar{\varphi}_i(t)$ of \bar{S} define the «initial distance» $q_{01} = P_{01}\bar{P}_{01} = |\bar{\varphi}_i(t) - \varphi_i(t)|$ taken over the interval $t_0 - \tau_i \leq t \leq t_0$, Fig. 2.

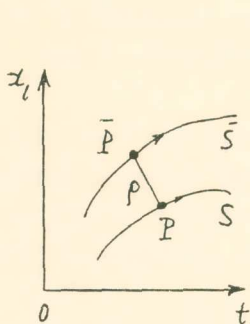


Fig. 1.

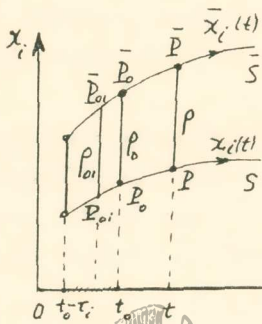


Fig. 2.

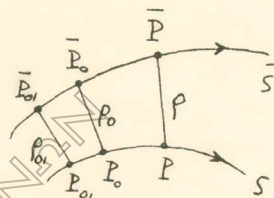


Fig. 3.

The above includes the ordinary differential equations, where $\tau_i = 0$, $q_{01} = q_0 = P_0\bar{P}_0$ at $t = t_0$, Fig. 2.

In case the retardations $\tau_i(t)$ are perturbed, one has a new stability concept characterized by the distance: $q_\tau = |\bar{\tau}_i(t) - \tau_i(t)|$, where $\bar{\tau}_i$ the perturbed regards (1).

2. REMARKS ON THE RELATIONSHIPS OF THE STABILITY CONCEPTS

Following considerations on «physical stability» and using the preceding remarks and notations, any stability concept can be described quantitatively by the «stability conditions»:

$$(a): q_{01} < \delta_1, \quad (b): q < \varepsilon, \quad (c): \lim_{t \rightarrow \infty} q = 0, \quad (d): \|p_i\| < \delta_2, \quad (e): q_\tau < \delta_3 \quad (1)$$

which, then, «unify» all the stability concepts. $\varepsilon, \delta_1, \delta_2, \delta_3$ are positive constants, p_i the perturbations, and, in general, δ_1 and δ_2 depend on t_0 and ε .

By a suitable combination of these relationships, by an appropriate interpretation of the distances involved, and by some restrictions of some quantities of these relationships, one can express any stability concept.

The following remarks may help to clarify the above statements.

The inequality 1(a) is valid for t in the initial interval $E_{t_0} : t_0 - \tau_i \leq t \leq t_0$, while the inequalities 1(b), (d), (e) for t in $t \geq t_0$, $t_0 \geq 0$.

In case of «sudden perturbations», the inequality 1(d) is meaningless, and in this case 1(a), (b) express the «stability», while 1(a), (b), (c) the «asymptotic stability».

In case of «permanent perturbations», when 1(d) is meaningful, the selection of the kind of the norm of perturbations specifies the stability concept, so, one may have «total stability» or «integral stability», or «stability in the mean», under suitable norm of perturbations.

By restricting δ_1 and δ_2 to depend only on ε and not on t_0 , we have the «uniform stabilities», as this happens in periodic systems, or in autonomous systems.

Restriction on t_0 to have a minimum, $\min t_0 = \alpha$, implies «eventual stabilities».

If in (1) the distances ρ are interpreted as Liapunov or Poincaré distances, we have stabilities in the sense of Liapunov or Poincaré.

In case of equilibrium points of a system, when the orbit S shrinks to a point, the distinction between stabilities in the sense of Liapunov and Poincaré is meaningless.

The inequality 1(e) has a meaning in case of perturbed retardations.

In case of periodic motions, when S is a closed curve, the stability concept in Liapunov sense is a narrow concept compared to the stability concept in Poincaré sense. The «isochronism», that is the «constancy of the frequency», characterizes the Liapunov stability, while this notion does not enter in the Poincaré stability. Further we may have that :

- . A motion stable in Liapunov sense is also stable in Poincaré sense ;
- . A motion unstable in Poincaré sense is also unstable in Liapunov sense ;
- . A motion unstable in Liapunov sense may be stable or unstable in Poincaré sense, and

- . A motion stable in Poincaré sense may be stable or unstable in Liapunov sense.

The stability is a property of the solution different from its boundedness property, although in some cases there may exist regions where these properties are equivalent and one implies the other. The boundedness of the solution is characterized by the boundedness of $OP = \rho_1 = |x_i|$, where O the origin of the coordinate system and P point of the orbit; but the stability is characterized by the boundedness of $\rho = |\bar{x}_i - x_i|$, and it is possible for the orbits x_i and \bar{x}_i to be unbounded as $t \rightarrow \infty$, when the stability distance ρ gets the form $(\infty - \infty)$, when ρ will be either infinite, or constant, or zero, and the unbounded solution x_i will be either unstable or stable.

All the stability concepts included in the relationships (1) are of mathematical type, and the results, theorems or criteria, based on them, may not interpret the reality. Also, one and the same phenomenon may be, mathematically speaking, stable or unstable depending on the stability concept employed in the discussion of the stability of the phenomenon when the selection of the stability concept, appropriate for the phenomenon, arises.

The mathematical stability concepts and the stability criteria based on them, represent a possible functioning of the physical system, and in order all these to have a practical usefulness, and to agree with «practical stability», which must be the ultimate purpose of the stability investigations, appropriate modifications, changes, supplements of the mathematical stability concepts must accompany the investigations.

The notion of «practical stability» is a subject not yet completely studied, but in many cases is characterized by the knowledge of: (2)

- . The size of deviation of the state acceptable for a satisfactory operation of the system;
- . The size of permitted initial conditions that can be controlled;
- . The size of permitted perturbations;
- . The finite time T for the stability investigation.

The nonlinearities of the system play a decisive role for practical stability.

3. AN EXAMPLE

We terminate the discussion on stability remarks by using as an example the investigation of the «stability of some precessional phenomena», by which some of the above statements may be clarified.

The rotational motion of a rigid body around its axis of symmetry is governed by the Euler's (ordinary differential) equations, of which the stability of the solutions has been examined under sudden perturbations, then by employing the first three stability relationships (1).

If $\underline{L} = (L_1, L_2, L_3)$ is the external torque vector acting on the body, $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ the angular velocity vector, which characterizes the precession of the body, $\underline{\omega}_0 = (\omega_{01}, \omega_{02}, \omega_{03})$ the initial angular velocity vector, and $I_1, I_2 = I_3 = I$ the moments of inertia, the precessional motion of the body in the following two cases are given by:

$$\left. \begin{aligned} \text{(a) } \omega_1 = \omega_{10} = \text{constant, } \omega_2 = A \cos Q_1, \omega_3 = A \sin Q_1, Q_1 = \frac{(I_1 - I) \omega_{10} t}{I} \\ \text{(in case, } L_1 = L_2 = L_3 = 0; \quad I_1, I_2, A = (\omega_{02}^2 + \omega_{03}^2)^{1/2} \text{ constants)} \\ \text{(b) } \omega_1 = \frac{L_1}{I_1} t, \omega_2 = A \cos Q_2, \omega_3 = A \sin Q_2, Q_2 = Q_1 + \frac{(I_1 - I) L_1 t^2}{2 I_1 I} \\ \text{(in case: } L_1 = \text{constant, } L_2 = L_3 = 0) \end{aligned} \right\} (2)$$

The «regular precession» 2(a) is bounded, while the «helicoid precession» 2(b) is unbounded as $t \rightarrow \infty$. The results for their stability situation are the following:

- (i) The regular precession 2(a) is «stable» but «not asymptotically stable» in Poincaré sense (orbitally). In Liapunov sense it is «stable» but «not asymptotically stable», if ω_{01} is not affected by the perturbation; and it is «unstable», if ω_{01} is affected by the perturbations.
- (ii) The helicoid precession 2(b) is «asymptotically stable» in Poincaré sense, but it is «unstable» in Liapunov sense.

- (iii) The stability situation of the above example in Poincaré sense is preferred, because in this case it is proved that the requirements for «practical stability» are satisfied.

We remark that the stability of a system, by using any of the previous concepts, depends, in general, on the selection of the major variables of the system, and on the transformation of the variables. This will be subject of a next paper.

Π Ε Ρ Ι Λ Η Ψ Ι Σ

1. Εἰς τὴν παροῦσαν ἐργασίαν δίδονται παρατηρήσεις ἐπὶ τῶν ἀντιλήψεων περὶ εὐσταθείας εἰς φυσικὰ καὶ κοινωνικὰ φαινόμενα, τὰ ὅποια μαθηματικοποιοῦνται ὡς δυναμικὰ προβλήματα. Ἡ ποικιλία τῶν ἀντιλήψεων εὐσταθείας κάμνει τὰ προβλήματα εὐσταθείας πολὺ πεπλεγμένα καὶ τ' ἀποτελέσματα τῆς ἐρεῦνης βάσει αὐτῶν ὄχι δεκτὰ ἐνίστε.

2. Ἀναφέρομεν μερικὰς πηγὰς, ἀπὸ τὰς ὁποίας δυνάμεθα νὰ ἔχωμεν ποικίλιαν διαφόρου φύσεως ἀντιλήψεων περὶ εὐσταθείας:

—Ὁ τρόπος μὲ τὸν ὁποῖον μία κατάσταση ἐνὸς συστήματος πλησιάζει μίαν ἄλλην κατάστασιν ἢ ἀπομακρύνεται ἀπὸ αὐτήν.

—Ὁ τρόπος μὲ τὸν ὁποῖον προκαλοῦνται καὶ δροῦν αἱ διαταραχαὶ ἐνὸς συστήματος, ὁ τρόπος μὲ τὸν ὁποῖον μετροῦμεν τὴν ἔντασιν τῶν διαταραχῶν, καθὼς καὶ τὴν τῶν ἀποτελεσμάτων τῶν ἐπὶ τοῦ συστήματος.

—Ἡ δομὴ τοῦ μαθηματικοῦ μοντέλου τοῦ συστήματος κλπ.

3. Ὅλαι αἱ ἀντιλήψεις εὐσταθείας, ἂν καὶ διαφόρου φύσεως, δύνανται ν' ἀναχθοῦν εἰς τὰς αὐτὰς μαθηματικὰς «σχέσεις εὐσταθείας», αἱ ὁποῖαι δίδουν μίαν «ἐνοποίησιν» τῶν ἀντιλήψεων εὐσταθείας, αὐτὴ δὲ ἡ ἐνοποίησις ὑποβοηθεῖ τὴν κατανόησιν τῶν ἀντιστοίχων προβλημάτων, δύνανται δὲ νὰ ὀδηγήσῃ εἰς νέα συμπεράσματα. Μὲ κατάλληλον συνδυασμὸν τῶν «σχέσεων εὐσταθείας», καὶ κατάλληλον ἐρμηνεϊαν ἢ περιορισμὸν τῶν ποσοτήτων τῶν σχέσεων αὐτῶν, δύνανται νὰ προκύψῃ ὁποιαδήποτε ἀντίληψις εὐσταθείας.

4. Αἱ ἀντιλήψεις εὐσταθείας, ποὺ περιλαμβάνονται εἰς τὰς «σχέσεις εὐσταθείας» ὑποδεικνύουν ἐνδεχομένην λειτουργίαν τοῦ συστήματος, τὰ δὲ συμπεράσματα, θεωρήματα ἢ κριτήρια, ποὺ βασίζονται ἐπ' αὐτῶν, ἐνδέχεται νὰ μὴ ἐρμηνεύουν τὴν πραγματικότητα κατὰ ἱκανοποιητικὸν τρόπον, ἢ ἐνδέχεται νὰ ἔχωμεν διὰ τὸ αὐτὸ φαινόμενον διαφόρους καταστάσεις εὐσταθείας, ὅποτε γεννᾶται τὸ πρόβλημα τῆς ἐκλογῆς τῆς καταλλήλου καταστάσεως διὰ τὸ φαινόμενον.

5. Αί μαθηματικά αντίληψεις περί ευσταθείας, καθὼς καὶ τὰ βάσει αὐτῶν συμπεράσματα, διὰ νὰ ἐρμηνεύουν τὴν πραγματικότητα κατὰ ἱκανοποιητικὸν τρόπον, πρέπει νὰ συμφωνοῦν μὲ τὰ πορίσματα τῆς «πρακτικῆς ευσταθείας», πρὸς τοῦτο δὲ χρειάζονται κατάλληλον τροποποίησιν καὶ συμπλήρωσιν. Ἡ πρακτικὴ ευστάθεια δὲν ἔχει πλήρως σπουδασθῆ, ὅμως κύρια χαρακτηριστικά της δύναται νὰ εἶναι ἡ γνῶσις :

- τοῦ μεγέθους τῆς ἀποκλίσεως δεκτῶν καταστάσεων τοῦ συστήματος πρὸς ἱκανοποιητικὴν λειτουργίαν τοῦ συστήματος,
- τοῦ μεγέθους τῶν ἀρχικῶν συνθηκῶν, αἱ ὅποια δύναται νὰ ἐλεγχθοῦν,
- τοῦ μεγέθους τῶν ἐπιτρεπομένων διαταραχῶν,
- τοῦ χρόνου, πὸν μελετῶμεν τὴν ευστάθειαν.
- Αἱ μὴ γραμμικότητες τοῦ συστήματος παίζουν ἀποφασιστικὸν ρόλον διὰ τὴν πρακτικὴν ευστάθειαν, καὶ δὲν δύναται νὰ ἀμεληθοῦν.

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Ὁ Ἀκαδημαϊκὸς κ. Ἰωάννης Ξανθάκης, παρουσιάζων τὴν ἀνωτέρω ἀνακοίνωσιν, εἶπε τὰ ἑξῆς :

Εἰς τὴν ἐργασίαν ταύτην τοῦ κ. Μαγείρου, τὴν ὁποίαν ἔχω τὴν τιμὴν νὰ παρουσιάσω εἰς τὴν Ἀκαδημίαν, ἐκτίθενται ὠρισμένα ἐνδιαφέρονσαι παρατηρήσεις ἐπὶ τῶν ἀντιλήψεων περὶ ευσταθείας εἰς φυσικὰ καὶ κοινωνικὰ φαινόμενα, πὸν ἐκφορᾶζονται μαθηματικῶς ὡς δυναμικὰ προβλήματα.

Αἱ ἀντιλήψεις περὶ εὐσταθείας προέρχονται ἀπὸ πηγὰς διαφόρου φύσεως. Ἡ μελέτη τῆς σταθερότητος μιᾶς κινήσεως ἐπὶ τῆς τροχιάς της ἢ ἡ μελέτη τῆς τροχιάς μιᾶς κινήσεως μᾶς παρέχει δύο διακεκριμένας βασικὰς ἀντιλήψεις περὶ εὐσταθείας, αἱ ὁποῖαι περιέχουν πλῆθος ἄλλων ειδικευμένων ἀντιλήψεων ὡς εἰδικὰς περιπτώσεις. Ὁ τρόπος μὲ τὸν ὁποῖον ἡ κατάστασις ἑνὸς συστήματος πλησιάζει πρὸς μίαν ἄλλην, ἢ παρεκκλίνει ἐξ αὐτῆς, ὁ τρόπος μετρήσεως τῆς ἐντάσεως τῶν διαταραχῶν ἑνὸς συστήματος, ἡ δομὴ τοῦ μαθηματικοῦ μοντέλου ἑνὸς συστήματος καὶ ἄλλα εἶναι πηγαὶ ἀντιλήψεων εὐσταθείας διαφόρου φύσεως. Ἡ ποικιλία αὕτη τῶν ἀντιλήψεων εὐσταθείας κάμνει τὰ προβλήματα λίαν πολύπλοκα καὶ τὰ ἀποτελέσματα τῶν ἐρευνῶν βάσει αὐτῶν ἐνίοτε δὲν εἶναι γενικῶς ἀποδεκτά.

Ὅλαι αἱ ἀντιλήψεις περὶ εὐσταθείας, ἂν καὶ διαφόρου φύσεως, δύνανται νὰ ἀναχθῶν, κατὰ τὸν κ. Μάγειρον, εἰς τὰς αὐτὰς μαθηματικὰς «σχέσεις εὐσταθείας», αἱ ὁποῖαι παρέχουν μίαν ἐνοποίησιν τῶν διαφόρων περιστάσεων. Ἡ ἐνοποίησις αὕτη ὑποβοηθεῖ εἰς τὴν πληρεστέραν κατανόησιν τῶν ἀντιστοιχῶν προβλημάτων.

Τὰ συμπεράσματα, θεωρήματα ἢ κοιτήρια, τὰ στηριζόμενα εἰς τὰς μαθηματικὰς σχέσεις ἀντιλήψεων εὐσταθείας, εἶναι δυνατόν νὰ μὴ ἐρμηνεύουν κατὰ ἰκανοποιητικὸν τρόπον τὴν πραγματικότητα, ὅποτε παρίσταται ἀνάγκη τροποποιήσεων ἢ συμπληρώσεων τοῦ μαθηματικοῦ προτύπου, οὕτως ὥστε ἡ μαθηματικὴ διατύπωσις τῆς ἀντιλήψεως εὐσταθείας νὰ πλησιάζῃ, ὅσον τὸ δυνατόν περισσότερον, πρὸς τὴν λεγομένην «Πρακτικὴν Εὐστάθειαν», ἣτις ὅμως δὲν ἔχει ἀκόμη πλήρως μελετηθῆ.