for anisotropic bodies was done by Hoffman's criterion, which is the Hill criterion, but incorporates also the strength differential effect by adding the missing linear terms in the quadratic expression of Hill's criterion [3]. However, this addition resulted in a destruction of the tensorial invariancy of Hill's criterion, but gained the advantage of describing correctly the yielding behaviour of real engineering materials.

The old generation of yield criteria has sacrified exactness to the altar of simplicity and convenience, by profiting from the advantages of the tensor invariancy, and thus abolished the inclusion in these criteria of the important universal phenomena of the mechanical behaviour of the materials, such as the Bauschinger effect and others based on the strength differential effect.

From these criteria only the Coulomb or internal friction yield and fracture criterion incorporated the strength differential effect absolutely necessary to describe the yielding behaviour of brittle materials where this effect is very important [4].

The new generation of yield criteria was remodelled on Hill's criterion and its modification by Hoffman. They are based on energy balance considerations, and they are mainly the *Tsai-Wu* tensor failure polynomial (TFP) [5] and the elliptic paraboloid failure surface [6]. Both are described by quadric surfaces and present the fundamental property to comply with basic physical laws.

The tensor polynomial criterion is formulated by means of a series of Cartesian components of the stress tensor. It is represented by hypersurfaces in the six-dimensional stress space, which is impossible to be conveniently visualized geometrically in the physical stress space. Only plane sections of this hypersurface were possible to be studied which represent quadric surfaces in the Cartesian (σ_x, σ_y, σ_{xy})-parametric space. However, even these subspaces do not yield a direct interrelation between the externally applied load and the material strength directions, a drawback necessitating meticulous and delicate experiments for its establishment. On the other hand the TFP-criterion is an excellent instrument for evaluating the influence of the change of one failure parameter on the values of the remaining ones. Thus, the tensor polynomial criterion constitutes for the anisotropic bodies what the Mohr circles yield graphically for the isotropic materials.

The elliptic paraboloid failure surface (EPFS) was introduced by the author [7-9] by extending the domain of general failure surface for isotropic materials presenting the strength differential effect, which is defined by the strength differential parameter $R = \sigma_{0C}/\sigma_{0T}$, where σ_{0C} and σ_{0T} are the yield stresses in compression and tension respectively. Then, the paraboloid of revolution failure surface for isotropic materials becomes an elliptic paraboloid surface for orthotropic materials, where six strength parameters, three for tension and three for compression along the principal strength axes of the material define three different strength differential parameters $R_{31} = \sigma_{31C}/\sigma_{31T}$ $R_{32} = \sigma_{32C}/\sigma_{32T}$ and $R_{21} = \sigma_{21C}/\sigma_{21T}$. For transversely isotropic materials where $R_{31} = R_{32}$ (the σ_3-axis is assumed as the strong axis of the transversely isotropic material) the failure surface remains an elliptic paraboloid surface, but it presents some symmetry with respect to the principal strength axes, having as plane of symmetry the principal diagonal plane (σ_3, δ_{12}), where σ_3 is the strong axis and δ_{12} is the bisector of the right angle (σ_1, σ_2) along the plane of isotropy.

A property, which is maintained in this family of yield criteria, that is the isotropic, the transversely isotropic, as well as the orthotropic materials, is that the axis of symmetry of the failure surface is that their axes of symmetry coincide or are parallel to the hydrostatic axis in the stress space, besides the fact that all failure surfaces are paraboloids. This comes from the fact that, independently of the isotropy or anisotropy of the material, the consequences from an arbitrary external loading $(\sigma_1 = \sigma_2 = \sigma_3)$ should be independent of direction and, therefore, the hydtrostatic axis should be an axis of symmetry of the failure surface [10-12].

In this paper the properties of the elliptic paraboloid failure surface are studied for the general orthotropic material. It was shown that now the EPFS ceases to remain symmetric to the principal diagonal plane, but it is shifted and angularly displaced toward the middle strength σ_2 of the orthortopic material. The consequences of this angular displacement and shifting of the EPFS were studied and important results were derived.

THE EPFS FOR THE ORTHOTROPIC BODY

Since any failure criterion for isotropic materials may be considered as a degenerate case of a general condition describing failure modes for anisotropic solids, the fact that the paraboloid of revolution failure surface describes excellently the yield behaviour of any isotropic material presenting additionally the strength differential effect (SDE) ($R \neq 1.0$) constitutes a serious motif to extend its validity for anisotropic bodies.

The main features which must be conserved in this extension are: i) that the yield surface must be described by a quadric equation with some modifications taking into account the contribution of anisotropy and: ii) the axis of symmetry of the new surface must remain parallel to the hydrostatic axis, because of the invariance of the influence of the external loading on the anisotropic material.

Assuming that the principal failure strengths in tension and compression are expressed as σ_{Ti} and σ_{Ci} respectively, with i=1, 2, 3, and the σ_3-axis is the strongest one, the general form of the elliptic paraboloid valid for the orthotropic material is expressed by [12]:

$$\left(\frac{1}{\sigma_{Ti}\sigma_{Ci}}\right)\sigma_{ii}^{2}-\left[\frac{1}{\sigma_{Ti}\sigma_{Ci}}+\frac{1}{\sigma_{T(i+1)}\sigma_{C(i+1)}}-\frac{1}{\sigma_{T(i+2)}\sigma_{C(i+2)}}\right]\sigma_{i}\sigma_{(i+1)} + \left[\frac{1}{\sigma_{Ti}}-\frac{1}{\sigma_{Ci}}\right]\sigma_{ii}=1 \quad (1)$$

where the double index convention is accepted and the indices (i+1) and (i+2) in the second LHS term are to be understood as integers modulo three. Moreover, the following symbolism for simplification is adopted: $\sigma_{11}=\sigma_1$, $\sigma_{22}=\sigma_2$ and $\sigma_{33}=\sigma_3$.

It should be noted here that, since Eq. (1) is valid only when the principal stress directions from the externally applied load coincide with the principal strength directions of the material, care should be taken, to assure this coincidence. Otherwise, for a random orientation of the σ_1, σ_2, σ_3-principal stress axes, relatively to the principal strength directions, there is another elliptic paraboloid with the same properties as the previous one, whose failure strengths σ_{Ti} and σ_{Ci} should be conveniently calculated from the angular displacements of the two systems. In the following we shall assume a general orthotropic material loaded conveniently so that its principal strength directions coincide with the principal loading directions.

We refer now Eq. (1) to a frame Oxyz related to the (σ_2, σ_2, σ_3)-principal stress direction frame by the following relations:

$$\sigma_1 = -\frac{1}{\sqrt{6}}(\sqrt{3}\,x + y - \sqrt{2}\,z) \qquad x = \frac{1}{\sqrt{2}}(-\sigma_1 + \sigma_2)$$

$$\sigma_2 = \frac{1}{\sqrt{6}}(\sqrt{3}x - y + \sqrt{2}\,z) \qquad \text{or} \qquad y = \frac{1}{\sqrt{6}}(-\sigma_1 - \sigma_2 + 2\sigma_3) \qquad (2)$$

$$\sigma_3 = \frac{1}{\sqrt{6}}(2y + \sqrt{2}\,z) \qquad z = \frac{1}{\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3)$$

Equations (2) indicate at once that the Oz-axis coincides with the hydrostatic axis, subtending equal angles with the principal directions ($\cos a_1 = \cos a_2 = \cos a_3 = 1/\sqrt{3}$). Moreover, the Ox-axis lies on the (σ_1, σ_2)-principal plane and the Oy-axis lies on the principal diagonal plane (σ_3, δ_{12}), where δ_{12} is the bisector of the $\sigma_1 \hat{O} \sigma_2$-angle. The Oxyz-system is a tri-orthogonal right-hand system. It may be readily derived that the angles subtended by the Oy-and Oz-axes with the σ_3-axis are equal to 35.26° and 54.76° respectively. Figure 1 presents the mapping of the EPFS in the three-dimensional stress space (σ_1, σ_2, σ_3), as well as in the Oxyz-frame.

Equation (1) referred to the Oxyz-frame is expressed by:

$$\left(\frac{1}{\sigma_{T1}\sigma_{C1}} + \frac{1}{\sigma_{T2}\sigma_{C2}} - \frac{1}{2\sigma_{T3}\sigma_{C3}}\right)x^2 + \frac{3}{2\sigma_{T3}\sigma_{C3}}\,y^2 +$$

$$+\sqrt{3}\left(\frac{1}{\sigma_{T1}\sigma_{C1}} - \frac{1}{\sigma_{T2}\sigma_{C2}}\right)xy + \frac{1}{\sqrt{2}}\left[\left(\frac{1}{\sigma_{T2}} - \frac{1}{\sigma_{C2}}\right) - \left(\frac{1}{\sigma_{T1}} - \frac{1}{\sigma_{C1}}\right)\right]x +$$

$$+\frac{1}{\sqrt{6}}\left[2\left(\frac{1}{\sigma_{T3}} - \frac{1}{\sigma_{C3}}\right) - \left(\frac{1}{\sigma_{T1}} - \frac{1}{\sigma_{C1}}\right) - \left(\frac{1}{\sigma_{T2}} - \frac{1}{\sigma_{C2}}\right)\right]y + \frac{1}{\sqrt{3}}$$

$$\left(\frac{1}{\sigma_{T3}} - \frac{1}{\sigma_{C3}}\right) + \left(\frac{1}{\sigma_{T1}} - \frac{1}{\sigma_{C1}}\right) + \left(\frac{1}{\sigma_{T2}} - \frac{1}{\sigma_{C2}}\right)z = 1 \qquad (3)$$

The presence of both linear terms with respect to the Ox- and Oy-axes indicates that the EPFS for the orthotropic materials has its axis of symmetry eccentrically positioned relatively to the principal diagonal plane lying on the side of the intermediate principal stress, σ_2.

The intersection of the EPFS with the deviatoric plane, $z=0$, shown in Fig. 2, can be readily derived from relation (3) by putting into the general form of the expression for the paraboloid $z=0$. Then we have:

$$ax^2+by^2+2hxy+2gx+2fy+I'z+c=0 \tag{4}$$

where:

$$a=\left(\frac{1}{\sigma_{T1}\sigma_{C1}}+\frac{1}{\sigma_{T2}\sigma_{C2}}-\frac{1}{2\sigma_{T3}\sigma_{C3}}\right)$$

$$b=\frac{3}{2\sigma_{T3}\sigma_{C3}} \quad c=-1$$

$$2h=\sqrt{3}\left(\frac{1}{\sigma_{T1}\sigma_{C1}}-\frac{1}{\sigma_{T2}\sigma_{C2}}\right) \tag{5}$$

$$2g=\frac{1}{\sqrt{2}}\left[\left(\frac{1}{\sigma_{T2}}-\frac{1}{\sigma_{C2}}\right)-\left(\frac{1}{\sigma_{T1}}-\frac{1}{\sigma_{C1}}\right)\right]$$

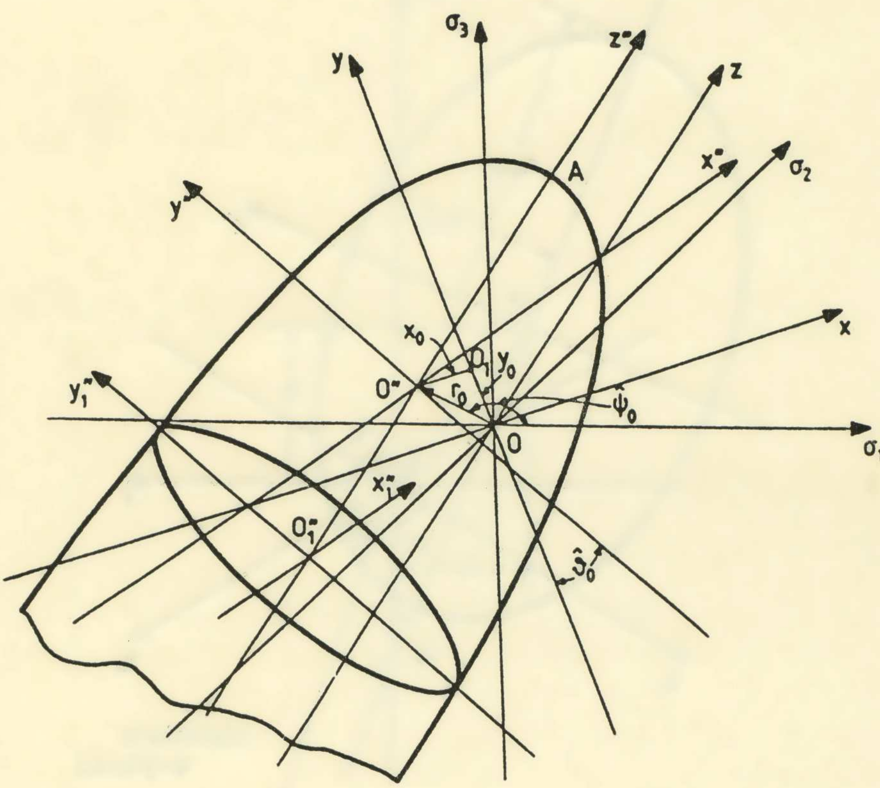

Fig. 1. The elliptic paraboloid failure surface (EPFS) for the general orthotropic material as it appears in the three dimensional (σ_1, σ_2, σ_3)-space and its connection with the Oxyz- and O″x″y″z″-Cartesian frames whose 0z- and O″z″-axes are coincident or parallel to the hydrostatic axis.

$$2f = \frac{1}{\sqrt{2}}\left[2\left(\frac{1}{\sigma_{T3}}-\frac{1}{\sigma_{C3}}\right)-\left(\frac{1}{\sigma_{T1}}-\frac{1}{\sigma_{C1}}\right)-\left(\frac{1}{\sigma_{T2}}-\frac{1}{\sigma_{C2}}\right)\right]$$

and:

$$I' = \frac{1}{\sqrt{3}}\left[\left(\frac{1}{\sigma_{T3}}-\frac{1}{\sigma_{C3}}\right)+\left(\frac{1}{\sigma_{C1}}-\frac{1}{\sigma_{C1}}\right)+\left(\frac{1}{\sigma_{T2}}-\frac{1}{\sigma_{C2}}\right)\right] \tag{6}$$

The coefficients C, G, F defining the position and form of the paraboloid are given by [10]:

$$C=(ab-h^2),\ G=(hf-bg) \text{ and } F=(hg-af) \tag{7}$$

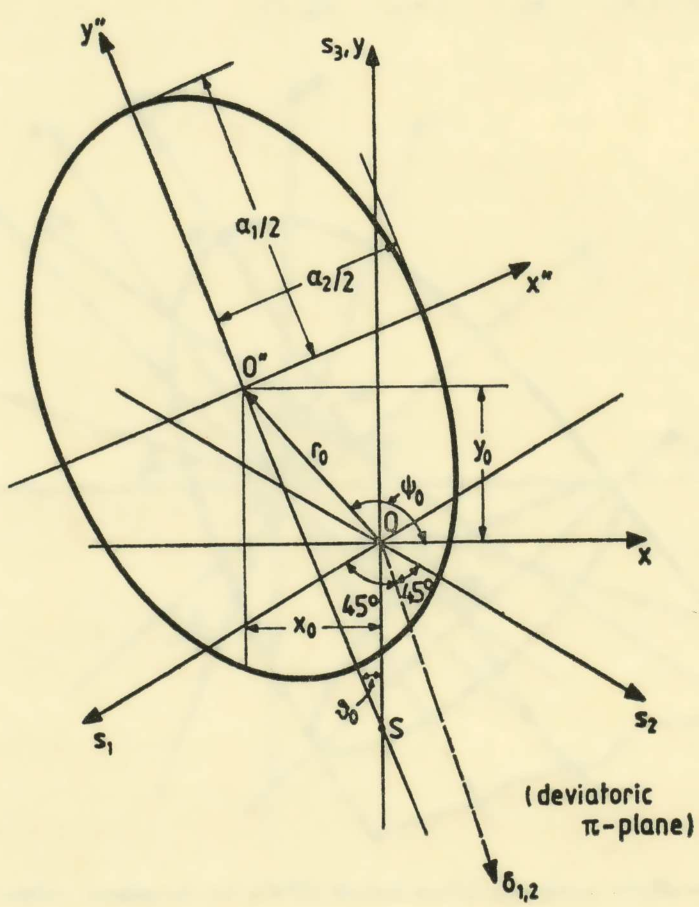

Fig. 2. The intersection of the EPFS by the deviatoric plane $z=z''=0$ and the position of the ellipse of intersection relatively to the $Os_1s_2s_3$-deviatoric system.

Then, the polar distance $r_0=(OO'')$ between the origin O of the initial frame Oxyz and the point O″ of piercing the deviatoric plane, the axis of symmetry of the EPFS, normalized to the maximum failure stress in tension σ_{T3}, is given by:

$$\frac{r_0}{\sigma_{T3}}=\frac{(G^2+F^2)^{1/2}}{C} \tag{8}$$

With the quantities C G E I' and d evaluated from the particular values of a, b, h, g and f for each material, it is a routine work to define the shape and position of the EPFS belonging to some orthotropic material.

We define first the coordinates of the point O″, where the axis of symmetry of the EPFS for the orthotropic material is piercing the deviatoric plane z=0, which contains the Ox- anf Oy-axes. These coordinates x_0, y_0, according to the theory of the paraboloids, are given by:

$$x_0=\frac{G}{C}\text{ and } y_0=\frac{F}{C} \tag{9}$$

where C, G and F are given in the appendix, as well as in ref. [13]. In the appendix are also given the explicit expressions for the coordinates x_0 and y_0 of the center O″ of the ellipse representing the intersection of the EPFS and the π-plane. The angle ψ_0 subtended by the polar radius OO″ and the **Ox**-axis in Fig. 2 is given by:

$$\tan\psi_0=\frac{y_0}{x_0}=\frac{\sqrt{3}}{3}\,\frac{\left\{\left(\frac{1}{\sigma_{T1}}-\frac{1}{\sigma_{C1}}\right)\left[\frac{5}{\sigma_{T2}\sigma_{C2}}-\frac{1}{\sigma_{T1}\sigma_{C1}}-\frac{1}{\sigma_{T3}\sigma_{C3}}\right]+\left(\frac{1}{\sigma_{T2}}-\frac{1}{\sigma_{C2}}\right)\cdot\left[\frac{5}{\sigma_{T1}\sigma_{C1}}-\frac{1}{\sigma_{T2}\sigma_{C2}}-\frac{1}{\sigma_{T3}\sigma_{C3}}\right]-4\left(\frac{1}{\sigma_{T3}}-\frac{1}{\sigma_{C3}}\right)\left[\frac{1}{\sigma_{T1}\sigma_{C1}}+\frac{1}{\sigma_{T2}\sigma_{C2}}+\frac{1}{\sigma_{T3}\sigma_{C3}}\right)\right\}}{\left\{2\left(\frac{1}{\sigma_{T3}}-\frac{1}{\sigma_{C3}}\right)\left(\frac{1}{\sigma_{T1}\sigma_{C1}}\;\frac{1}{\sigma_{T2}\sigma_{C2}}\right)+\left(\frac{1}{\sigma_{T1}}-\frac{1}{\sigma_{C1}}\right)\cdot\left(\frac{3}{\sigma_{T3}\sigma_{C3}}+\frac{1}{\sigma_{T2}\sigma_{C2}}-\frac{1}{\sigma_{T1}\sigma_{T1}}\right)-\left(\frac{1}{\sigma_{T2}}-\frac{1}{\sigma_{C2}}\right)\left[\frac{3}{\sigma_{T3}\sigma_{C3}}+\frac{1}{\sigma_{T1}\sigma_{C1}}-\frac{1}{\sigma_{T2}\sigma_{C2}}\right]\right\}} \tag{10}$$

For the transversely isotropic materials it can be readily found that G=0 and therefore the coordinate x_0 is always zero. This means that the EPFS for transtropic materials is always symmetric to the principal diagonal plane (σ_3, δ_{12}).

The position of point O″ of the center of the ellipse which coincides also with the point where the axis of symmetry of the EPFS is piercing the deviatoric π-plane depends on the values of F- and C- quantities. A full discussion of the position of the center O″ is given in ref. [13].

THE INTERSECTION OF EPFS BY THE π- PLANE

The intersection of the EPFS by the deviatoric plane is derived from relation (4) by putting z=0 and is given by:

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \tag{11}$$

This curve is representing an ellipse under certain conditions discussed in detail in ref. [13], which generally are met for typical fiber composites satisfying the stability condition [12]. In relation (11) the constants a, b, c, h, g, f are given by relations (5).

The equation of the EPFS in the new O″x″y″z″-frame, whise O″x″- and O″y″-axes coincide with the principal axes of the intersection of EPFS and the π-plane, is expressed by:

$$\bar{a}\,x''^2+\bar{b}\,y''^2+\bar{r}\,z''=(1-fy_0-gx_0) \tag{12}$$

where the coefficients $\bar{a}$, $\bar{b}$, $\bar{r}$ are given by:

$$2\,\bar{a}=(a+b)+[(a-b)^2+4h^2]^{1/2} \tag{13}$$

$$2\bar{b}=(a+b)-[(a-b)+^2 4h^2]^{1/2} \tag{14}$$

$$\bar{r}=l'/2 \tag{15}$$

It can be readily shown, after some straightforward calculations, that the equation for the elliptic intersection of the EPFS by the deviatoric π-plane is expressed by:

$$\frac{x''^2}{a_1^2}+\frac{y''^2}{a_2^2}=1 \tag{16}$$

where the quantities a_1, a_2 are given by:

$$a_1 = \left[\frac{1-gx_0-fy_0}{\delta+(\xi-\eta)^{1/2}}\right]^{1/2}$$

$$a_2 = \left[\frac{1-gx_0-fy_0}{\delta-(\xi-\eta)^{1/2}}\right]^{1/2}$$

where g and f are the coefficients given in relations (5) and the quantities δ, ξ, η are expressed by:

$$\delta = \frac{1}{2}\left[\frac{1}{\sigma_{T1}\sigma_{C1}} + \frac{1}{\sigma_{T2}\sigma_{C2}} - \frac{1}{\sigma_{T3}\sigma_{C3}}\right]$$

$$\xi = \frac{1}{2}\left[\frac{1}{\sigma^2_{T1}\sigma^2_{C1}} + \frac{1}{\sigma^2_{T2}\sigma^2_{C2}} + \frac{1}{\sigma^2_{T3}\sigma^2_{C3}}\right] \qquad (18)$$

$$\eta = \left[\frac{1}{\sigma_{T1}\sigma_{C1}\sigma_{T2}\sigma_{C2}} + \frac{1}{\sigma_{T2}\sigma_{C2}\sigma_{T3}\sigma_{C3}} + \frac{1}{\sigma_{T3}\sigma_{C3}\sigma_{T1}\sigma_{C1}}\right] \qquad (18)$$

Moreover the angle ϑ_0 subtended by the O″x″-axis (or O″y″-axis) and the Ox-axis (or Oy-axis) is given by:

$$\theta_0 = \frac{1}{2}\tan^{-1}\left\{\frac{\sqrt{3}\sigma_{T3}\sigma_{C3}(\sigma_{T1}\sigma_{C1}-\sigma_{T2}\sigma_{C2})}{2\sigma_{T1}\sigma_{C1}\sigma_{T2}\sigma_{C2}-\sigma_{T1}\sigma_{C1}\sigma_{T3}\sigma_{C3}-\sigma_{T2}\sigma_{C2}\sigma_{T3}\sigma_{C3}}\right\} \qquad (19)$$

Similar ellipses, but of different sizes, are produced by intersections of the elliptic paraboloid by planes z″=c parallel to the deviatoric plane. The sizes of these ellipses are diminishing as we approach the vertex of the paraboloid.

It is worthwhile pointing out that, depending on the relative values of the individual strengths σ_{Ti}, σ_{Ci}, it is possible for certain orthotropic materials that the EPFS has its apex either on the tension-tension-tension octant of the stress space, or on the compression-compression-compression octant. The first group of materials is called compression strong (C-strong) materials, whereas the second group is called tension strong (T-strong) materials.

Compression strong materials are almost all orthotropic materials, but there are a few exceptions like the paper sheets and the oriented polypropylene, which are T-strong materials. For a complete discussion of the conditions under which one material may be T-or C-strong see ref. [9], where an extensive study for transversely isotropic materials was undertaken.

As soon as the expression for the elliptic paraboloid is established in the O″x″y″z″-frame, it is easy to define the distance d_0'' of the vertex of the elliptic paraboloid from the deviatoric plane. This can be found by putting $x''=y''=0$ in Eq. (12). Then, we derive that:

$$d_0''=\frac{\sqrt{3}\,(1-fy_0-gx_0)}{\left[\left(\frac{1}{\sigma_{T1}}-\frac{1}{\sigma_{C1}}\right)+\left(\frac{1}{\sigma_{T2}}-\frac{1}{\sigma_{C2}}\right)+\left(\frac{1}{\sigma_{T3}}-\frac{1}{\sigma_{C3}}\right)\right]} \tag{20}$$

On the other hand, the distance d_0 between the origin O (Fig. 1) of the initial Oxyz-system and the point where the hydrostatic axis, Oz, is piercing the surface of the elliptic paraboloid is expressed by [12]:

$$d_0=\frac{\sqrt{3}}{\left[\left(\frac{1}{\sigma_{T1}}-\frac{1}{\sigma_{C1}}\right)+\left(\frac{1}{\sigma_{T2}}-\frac{1}{\sigma_{C2}}\right)+\left(\frac{1}{\sigma_{T3}}-\frac{1}{\sigma_{C3}}\right)\right]} \tag{21}$$

THE INTERSECTION OF EPFS BY THE (σ_3, σ_1) — PLANE

The equation expressing the intersection of the EPFS by the (σ_3, σ_1)-principal stress plane is readily derived by putting into Eq. (1) the value $\sigma_2=0$. Then we have:

$$H_{11}\sigma^2_1+H_{33}\sigma^2_3+2H_{31}\sigma_3\sigma_1+h_{11}\sigma_1+h_{33}\sigma_3=1 \tag{22}$$

where:

$$H_{11}=\frac{1}{\sigma_{T1}\sigma_{C1}},\quad H_{33}=\frac{1}{\sigma_{T3}\sigma_{C3}},$$

$$H_{31}=\frac{1}{2}\left[\frac{1}{\sigma_{T3}\sigma_{C3}}-\frac{1}{\sigma_{T1}\sigma_{C1}}-\frac{1}{\sigma_{T2}\sigma_{C2}}\right],$$

$$h_{11}=\left(\frac{1}{\sigma_{T1}}-\frac{1}{\sigma_{C1}}\right),\quad h_{33}=\left(\frac{1}{\sigma_{T3}}-\frac{1}{\sigma_{C3}}\right) \tag{23}$$

Relation (22) represents an ellipse if and only if certain conditions are satisfied which are generally met for strong fiber composites [12].

The center of the ellipse of the intersection of the EPFS and the σ_3, σ_1-plane has the coordinates:

$$(\sigma_{01},\sigma_{03}) = \left\{ \frac{1\,(h_{33}H_{31}-h_{11}H_{11})}{2\,(H_{11}H_{33}-H^2_{31})},\ \frac{1\,(h_{33}H_{31}-h_{11}H_{33})}{2\,(H_{11}H_{33}-H^2_{31})} \right\} \tag{24}$$

The system of Cartesian coordinates $(M\bar{\sigma}_1\,\bar{\sigma}_3)$, to which this ellipse is central, represented by $(\bar{\sigma}_3, \bar{\sigma}_1)$ in Fig. 3, is angularly displaced relatively to the (σ_3, σ_1)-system by an angle θ_1, subtended between either $\bar{\sigma}_3$ and σ_3, or $\bar{\sigma}_1$ and σ_1 and it is given by:

$$\theta_1 = \frac{1}{2}\tan^{-1}\left[\frac{2H_{31}}{H_{33}-H_{11}}\right] \tag{25}$$

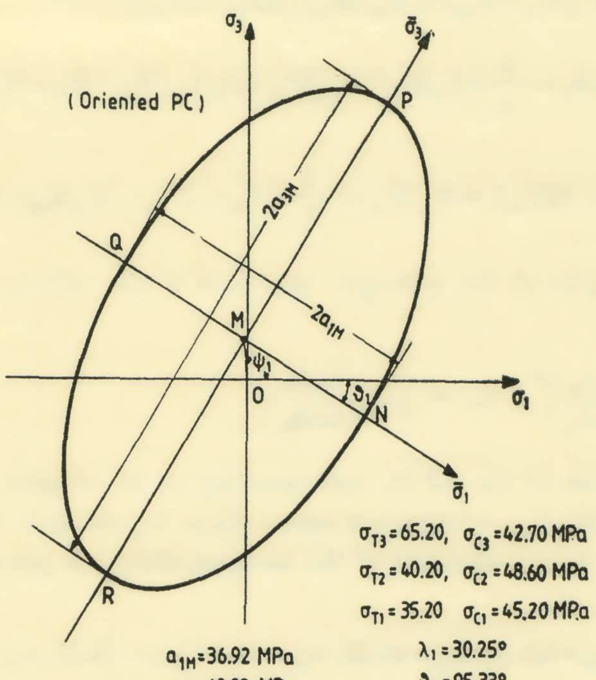

Fig. 3. The intersection of the EPFS by the (σ_3, σ_1)-principal stress plane for an oriented polycarbonate with different strengths along its principal axes.

Figure 3 presents the intersection of the EPFS for an orthotropic material by the (σ_3, σ_1) principal stress plane.

The equation of the ellipse expressed in the new coordinate system $\bar{\sigma}_3$, $\bar{\sigma}_1$ is given by:

$$\bar{a}\,\sigma^2{}_1+\bar{b}\,\sigma^2{}_3=-\frac{\det\mathbf{A}_3}{\det\mathbf{A}_2} \tag{26}$$

where the coefficients $\bar{a}$ and $\bar{b}$ are expressed by:

$$\bar{a}=\frac{1}{2}\left[(H_{11}+H_{33})+[(H_{33}-H_{11})^2+4H^2{}_{31}]^{1/2}\right]$$

$$\bar{b}=\frac{1}{2}\left[(H_{11}+H_{33})-[(H_{33}-H_{11})^2+4H^2{}_{31}]^{1/2}\right] \tag{27}$$

and the determinants $\mathbf{A}_3$ and $\mathbf{A}_2$ are given by:

$$\det\mathbf{A}_2=-\frac{1}{4}\left[(H_{11}-H_{22})^2+H_{33}(H_{33}-2H_{11}-2H_{22})\right]$$

$$\det\mathbf{A}_3=-\det\mathbf{A}_2-\frac{1}{4}\left[H_{11}h^2{}_{33}+H_{33}h^2{}_{11}+h_{11}h_{33}(H_{11}+H_{33}-H_{22})\right]$$

with (28)

$$[(H_{33}-H_{11})^2+4H^2{}_{31}]=2\left\{H^2{}_{11}+\frac{1}{2}H^2{}_{22}+H^2{}_{33}-H_{11}H_{22}-H_{22}H_{33}\right\}^{1/2}$$

The lengths of the principal semi-axes of the ellipse, a_{1M} and a_{3M}, are given by:

$$a_{1M}=\left[-\frac{\det\mathbf{A}_3}{\bar{a}\det\mathbf{A}_1}\right]^{1/2},\quad a_{3M}=\left[-\frac{\det\mathbf{A}_3}{\bar{b}\det\mathbf{A}_2}\right] \tag{29}$$

The intersection of the EPFS corresponding to an oriented polycarbonate material by the (σ_3, σ_1)-principal stress plane is presented in Fig. 3. The characteristic failure strengths of the material along the principal directions are given by:

$\sigma_{T3}=65.20$, $\sigma_{C3}=42.70$; $\sigma_{T2}=40.20$, $\sigma_{C2}=48.60$; $\sigma_{T1}=35.20$, $\sigma_{C1}=45.20$MPa.

The coordinates of the center of the ellipse are:

$a_{1M}=-3.03$MPa, $a_{3M}=10.63$MPa

The polar radius $r_1=(OM)=11.05$MPa, while the center M of the ellipse is lying in the second quadrant of the $\bar{\sigma}_1$, $\bar{\sigma}_3$-plane. The principal semi-axes of the elliptic intersection are given by:

$a_{1M}=39.45$MPa and $a_{3M}=57.98$MPa

while the angle θ_1, subtended between $\bar{\sigma}_1$- and σ_1-axes, is given by $\theta_1=159^o$. Finally the angle λ_1, subtended by the polar radius $r_1=(OM)$ and the σ_1-axis, is $\lambda_1=106.49^o$.

The angle λ_1 subtended by the r_1-polar radius and the σ_1-axis in Fig. 3 is expressed by:

$$\lambda_1=\tan^{-1}\left\{\frac{h_{11}(H_{22}-H_{33})-H_{11}(h_{11}+2H_{33})}{h_3(H_{22}-H_{11})-H_{33}(2H_{11}+h_{33})}\right\} \tag{30}$$

Finally, the angle θ_1 indicated in Fig. 3 is expressed by:

$$\tan 2\theta_1=\frac{\sigma_{T1}\sigma_{C1}\sigma_{T2}\sigma_{C2}+\sigma_{T2}\sigma_{C2}\sigma_{T3}\sigma_{C3}-\sigma_{T1}\sigma_{C1}\sigma_{T3}\sigma_{C3}}{\sigma_{T2}\sigma_{C2}(\sigma_{T1}\sigma_{C1}-\sigma_{T3}\sigma_{C3})} \tag{31}$$

Angle ϑ_1 for the oriented polycarbonate was found to be $\theta_1=159^o$.

The two other intersections of the EPFS by the principal stress planes (σ_2, σ_3) and (σ_1, σ_2) may be found by replacing in relation (22) the coefficients and the principal stresses with indices 1 and 3 by the indices 2 and 3 and, on the other hand, by 1 and 2 and using the respective values for the coefficients H_{ii}, H_{ij} and h_{ij}.

However, since σ_{T2} and σ_{C2} are the intermediate failure strengths the differences between σ_{T3}, σ_{C3} and σ_{T2}, σ_{C2} and, on the other hand, σ_{T1}, σ_{C1} and σ_{T2}, σ_{C2} are always smaller than the respective differences between σ_{T3}, σ_{C3} and σ_{T1}, σ_{C1}. Then, the elliptic intersection in the (σ_3, σ_1)-principal stress plane presents the strongest anisotropy and, therefore, it constitutes the most important intersection concerning the phenomena of anisotropy of the material.

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APPENDIX

The expressions for the quantities C, G and F are give by:

$$C=\frac{3}{2}\Bigg\{\Bigg(\frac{1}{\sigma_{T1}\sigma_{C1}\sigma_{T2}\sigma_{C2}}+\frac{1}{\sigma_{T2}\sigma_{C2}\sigma_{T3}\sigma_{C3}}+\frac{1}{\sigma_{T3}\sigma_{C3}\sigma_{T1}\sigma_{C1}}\Bigg]-\frac{1}{2}\,\frac{1}{\sigma^2_{T1}\sigma^2_{C1}}+\frac{1}{\sigma^2_{T2}\sigma^2_{C2}}+\frac{1}{\sigma^2_{T3}\sigma^2_{C3}}\Bigg]\Bigg\} \tag{A1}$$

$$G=\frac{1}{4\sqrt{2}}\Bigg\{2\left(\frac{1}{\sigma_{T3}}-\frac{1}{\sigma_{C3}}\right)\left(\frac{1}{\sigma_{T1}\sigma_{T1}}-\frac{1}{\sigma_{T2}\sigma_{C2}}\right)+\left(\frac{1}{\sigma_{T1}}-\frac{1}{\sigma_{C1}}\right)\left(\frac{3}{\sigma_{T3}\sigma_{C3}}+\frac{1}{\sigma_{T2}\sigma_{C2}}-\frac{1}{\sigma_{T1}\sigma_{C1}}\right)-\left(\frac{1}{\sigma_{T2}}-\frac{1}{\sigma_{C2}}\right)\left(\frac{3}{\sigma_{T3}\sigma_{C3}}+\frac{1}{\sigma_{T1}\sigma_{C1}}-\frac{1}{\sigma_{T2}\sigma_{C2}}\right]\Bigg\} \tag{A2}$$

$$F=\frac{1}{4\sqrt{6}}\Bigg\{\left(\frac{1}{\sigma_{T1}}-\frac{1}{\sigma_{C1}}\right)\left(\frac{5}{\sigma_{T2}\sigma_{C2}}-\frac{1}{\sigma_{T1}\sigma_{C1}}-\frac{1}{\sigma_{T3}\sigma_{C3}}\right)+\left(\frac{1}{\sigma_{T2}}-\frac{1}{\sigma_{C2}}\right)\left(\frac{5}{\sigma_{T1}\sigma_{C1}}-\frac{1}{\sigma_{T2}\sigma_{C2}}-\frac{1}{\sigma_{T3}\sigma_{C3}}\right)+4\left(\frac{1}{\sigma_{T3}}-\frac{1}{\sigma_{C3}}\right)\left(\frac{1}{2\sigma_{T3}\sigma_{C3}}-\frac{1}{\sigma_{T1}\sigma_{C1}}-\frac{1}{\sigma_{T2}\sigma_{C2}}\right)\Bigg\} \tag{A3}$$

The coordinates x_0, y_0 of the center O″ of the intersection of the EPFS and the deviatoric π-plane are expressed by:

$$x_0=\frac{\sqrt{2}}{3}\times\frac{\Big\{2\left(\frac{1}{\sigma_{T3}}-\frac{1}{\sigma_{C3}}\right)\left(\frac{1}{\sigma_{T1}\sigma_{C1}}-\frac{1}{\sigma_{T2}\sigma_{C2}}\right)+\left(\frac{1}{\sigma_{T1}}-\frac{1}{\sigma_{C1}}\right)\left(\frac{3}{\sigma_{T3}\sigma_{C3}}+\frac{1}{\sigma_{T2}\sigma_{C2}}-\frac{1}{\sigma_{T1}\sigma_{C1}}\right)-\left(\frac{1}{\sigma_{T2}}-\frac{1}{\sigma_{C2}}\right)\left(\frac{3}{\sigma_{T3}\sigma_{C3}}+\frac{1}{\sigma_{T1}\sigma_{C1}}-\frac{1}{\sigma_{T2}\sigma_{C2}}\right)\Big]}{\Big\{2\left[\frac{1}{\sigma_{T1}\sigma_{C1}\sigma_{T2}\sigma_{C2}}+\frac{1}{\sigma_{T2}\sigma_{C2}\sigma_{T3}\sigma_{C3}}+\frac{1}{\sigma_{T3}\sigma_{C3}\sigma_{T1}\sigma_{C1}}\right]-\left[\frac{1}{\sigma_{T3}\sigma_{C3}}-\frac{1}{\sigma_{T1}\sigma_{C1}}\right]^2-\left[\frac{1}{\sigma_{T1}\sigma_{C1}}-\frac{1}{\sigma_{T2}\sigma_{C2}}\right]^2-\left[\frac{1}{\sigma_{T2}\sigma_{C2}}-\frac{1}{\sigma_{T3}\sigma_{C3}}\right]^2\Big\}}$$

$$y_0=\frac{\sqrt{6}}{9}\,\frac{\left\{\left(\frac{1}{\sigma_{T1}}-\frac{1}{\sigma_{C1}}\right)\left[\frac{5}{\sigma_{T2}\sigma_{C2}}-\frac{1}{\sigma_{T1}\sigma_{C1}}-\frac{1}{\sigma_{T3}\sigma_{C3}}\right]+\left(\frac{1}{\sigma_{T2}}-\frac{1}{\sigma_{C2}}\right)\cdot\left[\frac{5}{\sigma_{T1}\sigma_{C1}}-\frac{1}{\sigma_{T2}\sigma_{C2}}-\frac{1}{\sigma_{T3}\sigma_{C3}}\right]-4\left(\frac{1}{\sigma_{T3}}-\frac{1}{\sigma_{C3}}\right)\left[\frac{1}{\sigma_{T1}\sigma_{C1}}+\frac{1}{\sigma_{T2}\sigma_{C2}}+\frac{1}{\sigma_{T3}\sigma_{C3}}\right]\right\}}{\left\{2\left[\frac{1}{\sigma_{T1}\sigma_{C1}\sigma_{T2}\sigma_{C2}}+\frac{1}{\sigma_{T2}\sigma_{C2}\sigma_{T3}\sigma_{C3}}+\frac{1}{\sigma_{T3}\sigma_{C3}\sigma_{T1}\sigma_{C1}}\right]-\left[\frac{1}{\sigma_{T3}\sigma_{C3}}-\frac{1}{\sigma_{T1}\sigma_{C1}}\right]^2-\left[\frac{1}{\sigma_{T1}\sigma_{C1}}-\frac{1}{\sigma_{T2}\sigma_{C2}}\right]^2-\left[\frac{1}{\sigma_{T2}\sigma_{C2}}-\frac{1}{\sigma_{T3}\sigma_{C3}}\right]^2\right\}}$$

ΠΕΡΙΛΗΨΗ

Ὁ ἐλλειπτικὸς παραβολοειδὴς τόπος διαρροῆς τῶν ἀνισοτρόπων (ὀρθοτρόπων) ὑλικῶν

Ἡ γενικὴ περίπτωσις τοῦ ἀνισοτρόπου ὑλικοῦ παρουσιάζοντος ὅμως συμμετρίαν ὀρθοτροπίας παρουσιάζεται εἰς τὴν ἐργασίαν αὐτὴν ὅταν τὸ μέσον δὲν παρουσιάζει οὐδεμίαν μορφὴν συμμετρίας τῆς ἀντοχῆς του. Διὰ τὴν γενικὴν αὐτὴν κατηγορίαν ὑλικῶν ἡ ὑπάρχουσα συμμετρία τῶν ἐγκαρσίως ἰσοτροπων ὑλικῶν, διὰ τὰ ὁποῖα ὁ τόπος διαρροῆς εἶναι ἐπιφάνεια μονόχωνος, συμμετρικὴ ὡς πρὸς τὸ κύριον διαγώνιον ἐπίπεδον τοῦ χώρου τῶν κυρίων τάσεων, τὸ ὁποῖον περιλαμβάνει τὸν ἰσχυρὸν ἄξονα σ_3 τοῦ ὑλικοῦ καὶ διέρχεται διὰ τῆς διχοτόμου τῆς γωνίας τῶν κυρίων ἀξόνων σ_1 καὶ σ_2 τοῦ ἰσοτρόπου ἐπιπέδου, παύει νὰ ἰσχύη.

Διὰ τὰ ὀρθότροπα ὑλικὰ ὁ τόπος διαρροῆς των, ὁ ὁποῖος ἐξακολουθεῖ νὰ παρίσταται ἀπὸ ἐλλειπτικὴν παραβολοειδῆ ἐπιφάνειαν, παύει νὰ παρουσιάζη συμμετρίαν ὡς πρὸς τὸ κύριον διαγώνιον ἐπίπεδον. Τὸ ἐπίπεδον συμμετρίας τοῦ τόπου διαρροῆς τῶν ὀρθοτρόπων ὑλικῶν παρουσιάζει κλίσιν ὡς πρὸς τὴν πλευρὰν τοῦ θετικοῦ ἄξονος τῆς ἐνδιαμέσου κυρίας τάσεως σ_2.

Ἡ τομὴ τοῦ ἐπιπέδου συμμετρίας τοῦ τόπου τῶν ὀρθοτρόπων ὑλικῶν καὶ τοῦ κυρίου διαγωνίου ἐπιπέδου, ὅταν τὸ ὀρθότροπον ὑλικὸν θεωρηθῇ ὡς ἐγκαρσίως ἰσότροπον μὲ $\sigma_2=\sigma_3$, ὁρίζεται ἀπὸ τὴν γωνίαν τομῆς τοῦ σ_3-ἄξονος καὶ τοῦ μείζονος ἄξονος τῆς ἐλλειπτικῆς τομῆς τοῦ παραβολοειδοῦς ὑπὸ τοῦ ἀποκλίνοντος ἐπιπέδου, δεδομένου ὅτι ἀμφότερα τὰ παραβολοειδῆ, τοῦ ὀρθοτρόπου καὶ τοῦ ἀντιστοίχου ἐγκαρσίως ἰσοτρόπου μέσου, διατηροῦν τοὺς ἄξονας συμμετρίας των παραλλήλους πρὸς τὸν ὑδροστατικὸν ἄξονα τοῦ χώρου τῶν κυρίων τάσεων $\sigma_1=\sigma_2=\sigma_3$.

Αἱ τομαὶ τοῦ παραβολοειδοῦς τοῦ ὀρθοτρόπου ὑλικοῦ ἀπὸ τὰ κύρια ἐπίπεδα τῶν τάσεων $(\sigma_1\sigma_2)$, $(\sigma_2\sigma_3)$ καὶ $(\sigma_3\sigma_1)$ καθὼς καὶ ἀπὸ τὸ ἀποκλίνον ἐπίπεδον ἀποδεικνύεται ὅτι εἶναι πάλιν ἐλλείψεις τῶν ὁποίων ἡ θέσις, ὁ προσανατολισμὸς καὶ αἱ διαστάσεις τῶν κυρίων ἀξόνων των ὁρίζονται δι' ἁπλῶν σχετικῶς τύπων.

Ἀποδεικνύεται ἐκ τῶν ἐφαρμογῶν ὅτι, διὰ μεταβολῆς τῆς ἐνδιαμέσου ἀντοχῆς τοῦ ὀρθοτρόπου ὑλικοῦ μεταξὺ τῶν ἀκραίων ὁρίων τῶν ἀντοχῶν ἀπὸ τὰς ἀσθενεστέρας μέχρι τὰς ἰσχυροτέρας, αἱ θέσεις τῶν ἐλλειπτικῶν τομῶν, ἰδίᾳ ὑπὸ τῶν κυρίων ἐπιπέδων τῶν τάσεων, δὲν μετακινοῦνται σημαντικῶς ἀπὸ τὰς ἀντιστοίχους θέσεις διὰ τὸ ἰσοδύναμον ἐγκαρσίως ἰσότροπον ὑλικόν, γεγονὸς ποὺ ἐπιτρέπει τὴν χρῆσιν τῶν ἁπλουστέρων τύπων τοῦ ἐγκαρσίως ἰσοτρόπου μέσου διὰ περιπτώσεις ὀρθοτρόπων ὑλικῶν, τῶν ὁποίων ἡ ἐνδιάμεσος ἀντοχὴ πλησιάζει ἑκατέραν τῶν ἀκραίων ἀντοχῶν τοῦ μέσου.

Ὅλαι αἱ εὑρεθεῖσαι συνθῆκαι, αἱ ἰσχύουσαι διὰ τὰ ἐλλειπτικὰ παραβολοειδῆ διὰ τὰ ὀρθότροπα καὶ τὰ ἐγκαρσίως ἰσότροπα μέσα χρησιμεύουν διὰ τὴν καλυτέραν χρησιμοποίησιν τῶν ὑλικῶν αὐτῶν εἰς τὰς κατασκευάς.