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ΜΑΘΗΜΑΤΙΚΑ.— **Antiplane shear stress intensity factor evaluated by caustics,** *by P. S. Theocaris* *.

ABSTRACT

The optical method of reflected caustics, which up to now was applied for the evaluation of stress intensity factors in deformed cracked plates under mode I and II, was extended in this paper for the evaluation of the same factor in cracked plates subjected to mode III deformation. It was shown that the method of reflected caustics was capable in detecting and evaluating this factor, where all other experimental methods, i.e. photoelasticity, holographic interferometry and especially the method of transmitted cauctics are invalid to yield this quantity. Based on the first-order approximation of the elastic theory around the crack tip and Sneddon's formulas the theory of formation of the reflected caustics was developed and the characteristic geometric properties of this envelope curve were defined. It was shown that this envelope is again a generalized epicycloid, whose characteristic dimensions are directly related to $K_{III}$. Experimental evidence with specimens made of optically isotropic materials (plexiglas) and elastically loaded corroborated the theoretical results.

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* ΠΕΡΙΚΛΗ ΘΕΟΧΑΡΗ, **Ἡ μέθοδος τῶν ἀνακλωμένων καυστικῶν διὰ τὸν ὑπολογισμὸν τοῦ $K_{III}$ — συντελεστοῦ ἐντάσεως τάσεων εἰς ρωγμὰς καταπονουμένας εἰς ἐγκαρσίαν διάτμησιν.**

## INTRODUCTION

The optical method of reflected caustics [1] and its complement of transmitted caustics [2] were used for evaluating the orders of singularities at the crack tips and other elastic cases when the cracked plates were submitted to an arbitraty in-plane loading creating either opening mode (mode I) or edge-sliding mode (mode II) of deformation. This was because with these modes of deformation the lateral $\varepsilon_z$ - strain, developed because of the Poisson's ratio effect, created a deformation of the lateral faces of the specimen and deviations of the impinging light rays at the vicinity of the crack tip when these rays were either reflected or transmitted through the specimen.

Mode III deformations in anti-plane shear were not encountered up to now because it was thought that, since for this mode the components of stresses $\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$, and only the shear stresses $\tau_{xy}$ and $\tau_{yz}$ are different than zero, there was no lateral deformation of the specimen contributing mainly to the creation of caustics. Since the variation of refractive index at the vicinity of the crack was not expected to create significant deviations of the light rays there, and since the thickness variations of the specimen at the vicinity of the crack tip are annuled, no caustics were expected for this type of deformation.

Meanwhile, the method of reflected caustics was extended to the study of the stress fields in non-cracked plate submitted to bending [3, 4]. Based on the same principle as for the cases of generalized plane stress problems the method was also extended for the study of the distribution of curvatures on plates and shells [5]. Finally, the method was extended to the study of the stress intensity factors in symmetrically cracked plates under symmetric bending [6].

Since the anti-plane shear mode of deformation of a cracked plate constitutes one of the simplest cases of transverse loading of cracked plates, it is evident that the method of reflected caustics may solve the problem of evaluating $K_{III}$ in such cracked plates. On the contrary, the method of transmitted caustics is incapable of yielding $K_{III}$, since the light rays impinging at the vicinity of the crack tip are only parallelly displaced due to refraction phenomena.

In this paper the theoretical background of the method was presented and experimental evidence with cracked plexiglas plates corroborated the results of theory.

THE STRESS FIELD AT THE CRACK - TIP FOR MODE-III DEFORMATION

Consider a thin elastic plate containing a single edge crack and submitted to anti-plane shear so that a mode-III deformation is created at the crack tip. A direct evaluation of the stresses around the crack tip may be derived from Westergaard's solution, if we are interested for the components of stresses and displacements at the close vicinity of the crack tip and we truncate the powers solution of the problem only to the first and singular term.

If a system of Cartesian coordinates Oxy is related to the crack with its origin O coinciding with the crack tip and the Ox-axis tangent to the crack axis at its tip, it has been shown that the only non-zero stresses existing at the crack tip are the shear components [**7**]:

$$\tau_{xz} = -\frac{K_{III}}{(2\pi r)^{1/2}} \sin \vartheta/2 \quad \text{and} \quad \tau_{yz} = \frac{K_{III}}{(2\pi r)^{1/2}} \cos \vartheta/2 \tag{1}$$

while the only non-vanishing displacement w is along the Oz-axis normal to the mid-plane of the specimen and equal to :

$$w = \frac{2r}{G} \frac{K_{III}}{(2\pi r)^{1/2}} \sin \vartheta/2 . \tag{2}$$

Whereas there is no thickness variation of the plate at the crack tip because $\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$, there is this deflection w, which makes the impinging rays of a light beam and reflected either from the front or from the rear face of the specimen to deviate and to create a caustic, while the transmitted through the thickness rays are only parallelly displaced to themselves by small amounts according to the theory of refraction.

## THE EQUATION OF THE REFLECTED CAUSTIC FOR $K_{III}$ - DEFORMATION

For obtaining a reflected caustic from the laterally deformed lips of a crack under mode III of deformation a parallel (convergent or divergent) light beam impinges on the reflecting faces of the two legs of the crack, which under mode-III deformation become cylindrical surfaces of opposite direction having a curvature indicated in Fig. 1.

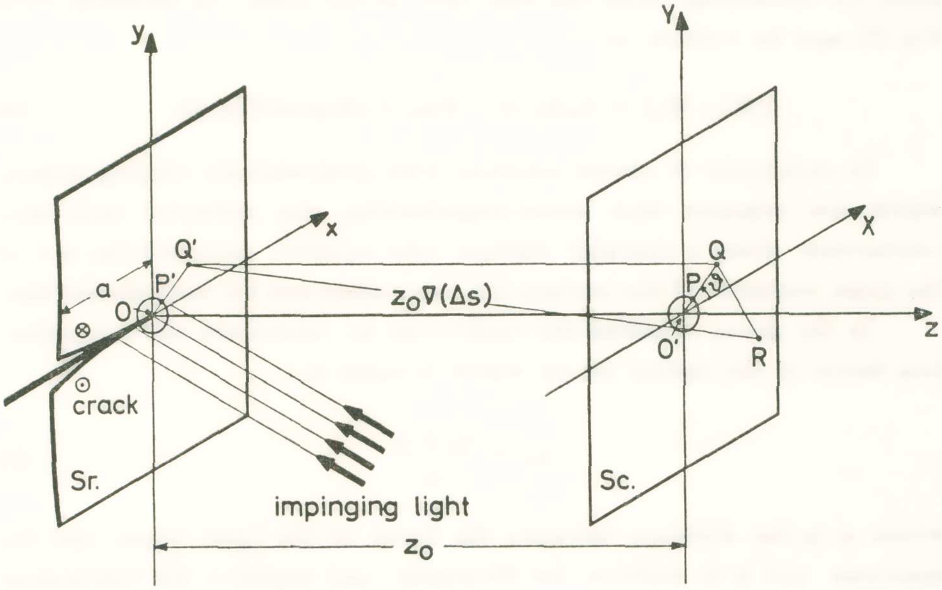


Fig. 1.

The slope of these cylindrical surfaces increases as x is approaching the origin of the Cartesian coordinate system Oxy, which coincides with the crack tip, while the Ox-axis coincides with the crack axis and the Oz-axis is normal to the middle plane of the specimen having also the direction of the light rays.

The reflected light rays from both the front and rear faces of the specimen expressed by the relation $z = f(x, y)$, obey Snell's reflection law and when they are received on a reference screen (Sc), placed at a distance $z_0$ from the specimen, deviate and their deviations are expressed in parametric form by the simplified relations :

$$W_x = \lambda_m x - 2(z_0 + \varepsilon d)\frac{\partial f(x, y)}{\partial x}, \quad W_y = \lambda_m y - 2(z_0 + \varepsilon d)\frac{\partial f(x, y)}{\partial y} \qquad (3)$$

where $(W_x, W_y)$ are the coordinates of a point E′ of the screen Sc, which corresponds to a point E(x, y) of the deformed surface Sr $(z = f(x, y))$ of the specimen. In these relations d is the thickness of the plate and $\varepsilon$ is either equal to zero for reflections from the front face or equal to unity for reflections from the rear face of the plate. In vectorial form Eqs (3) may be written as :

$$(W_x, W_y) = \lambda_m (x, y) - 2(z_0 + \varepsilon d)\,\mathrm{grad}\, f(x, y). \qquad (4)$$

In such types of lateral surfaces with progressively varying slopes, which are common near stress-singularities, the reflected rays may concentrate along a singular surface (the caustic), provided the law of the slope variation of the surface is a convenient one for such phenomena.

In the above relations the coefficient $\lambda_m$ represents the magnification factor of the optical set-up which is equal to :

$$\lambda_m = \frac{z_0 + z_i}{z_i} \qquad (5)$$

where $z_i$ is the distance between the focus of the light beam and the specimen and it is positive for divergent and negative for convergent light. In the cases studied in this paper where only reflected rays are considered the quantity $z_0$ is always positive. Furthermore, since for large magnifications of the caustic it is customary to consider optical set-ups with $z_0 > z_i$ the magnification ratio $\lambda_m$ is always positive.

In the case of mode-III deformation the initially flat and plane surface around the crack tip becomes during deformation cylindrical and its form is expressed by the deflection w given by Eq. (2). Then the parametric relations (3) become :

$$W_x = \lambda_m x - 2(z_0 + \varepsilon d)\frac{\partial w(x, y)}{\partial x}, \quad W_y = \lambda_m y - 2(z_0 + \varepsilon d)\frac{\partial w(x, y)}{\partial y}. \qquad (6)$$

Differentiating relation (2) with respect to r and $\vartheta$ and using the transformation for the coordinate system Oxy we can readily find that :

$$\frac{\partial w}{\partial x} = - \frac{K_{III}}{(2\pi r)^{1/2} G} \sin \frac{\vartheta}{2} \quad \text{and} \quad \frac{\partial w}{\partial y} = \frac{K_{III}}{(2\pi r)^{1/2} G} \cos \frac{\vartheta}{2} . \tag{7}$$

Then, the parametric equations of the caustic are derived from relations (6) and given by :

$$\begin{aligned} W_x &= \lambda_m r \cos \vartheta + 2 (z_0 + \varepsilon d) \frac{K_{III}}{(2\pi r)^{1/2} G} \sin \vartheta/2 \\ W_y &= \lambda_m r \sin \vartheta - 2 (z_0 + \varepsilon d) \frac{K_{III}}{(2\pi r)^{1/2} G} \cos \vartheta/2 \end{aligned} \tag{8}$$

where r are the coordinates (for $-\pi < \vartheta < \pi$) of convenient curve on the specimen whose points (x, y) correspond to points $(W_x , W_y)$ of the caustic. This curve is called *initial* or *generatrix curve* of the caustic. This curve can be determined by the fact that, for the caustic to be formed, the coordinates $W_x (W_y)$ must take maximum or minimum values for $W_y = \text{const}$ $(W_x = \text{const})$. These conditions are satisfied if the *Jacobian determinant* $J = \dfrac{\partial (W_x , W_y)}{\partial (r, \vartheta)}$ vanishes, then :

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \vartheta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \vartheta} \end{vmatrix} = r \lambda_m^2 - r^{-2} (z_0 + \varepsilon d)^2 \frac{K_{III}^2}{2\pi G^2} = 0$$

which yields:

$$r_0 = \left( \frac{(z_0 + \varepsilon d) K_{III}}{(2\pi)^{1/2} \lambda_m G} \right)^{2/3} = \frac{1}{(2\pi)^{1/3}} (|C K_{III}|)^{2/3} \tag{9}$$

where C is an overall constant given by $C = \left[ \dfrac{(z_0 + \varepsilon d)}{\lambda_m G} \right]$.

In practically all cases the fulfilment of Eq. (9) for a set of pairs (x, y) on the specimen means the formation of a caustic on the screen Sc by their corresponding points $(W_x , W_y)$, which are determined from the parametric equations (8).

Eq. (9) indicates that the initial curve of the caustic on the specimen depends only on the absolute value of the stress intensity factor $K_{III}$ and on the constant C depending on the experimental arrangement and the mechanical properties of the material. This constant C for mode III deformations, where reflected caustics are only encountered, is always positive.

It can immediately be derived from relation (9) that this curve for reflections from the front or rear faces of the specimen is a circumference of a circle surrounding the crack tip 0.

It is evident that Eq. (9) is valid only up to the extent of validity of truncating to the first and singular term the power expansion of the complex stress function $\Phi(z)$ of Muskhelishvili (or the $Z(z)$ function of Westergaard).

Otherwise, $r_0$ should result from Eq. (9) sufficiently small, so that the initial curve of the caustic lies inside the near vicinity of the crack tip.

In reality, this initial curve is a circumference cut from the crack lips. Then, it starts from $\vartheta = -\pi$ and it terminates at $\vartheta = +\pi$.

Furthermore, Eqs. (8) for the caustic, because of Eq. (9) take the form:

$$\mathbf{W} = W_x + iW_y = \lambda_m r_0 \left\{ \exp(i\vartheta) + 2e \exp\left[ i\left(\frac{\vartheta}{2} - \frac{\pi}{2}\right)\right]\right\} \tag{10}$$

where e is given by $e = K_{III}/|K_{III}|$.

Relation (10) yields the following set of parametric equations for the caustic:

$$\left.\begin{aligned} W_x &= \lambda_m r_0 (\cos\vartheta + 2\sin\vartheta/2) \\ W_y &= \lambda_m r_0 (\sin\vartheta - 2\cos\vartheta/2) \end{aligned}\right\} -\pi \leqq \vartheta \leqq \pi \tag{11}$$

It is first worthwhile indicating the simplicity of expressions (11) for the caustic, which is again a generalized epicycloid curve. On the basis of the above equations we can draw the form of a typical caustic for reflections from the front face where $\varepsilon = 0$. The caustic from the rear face is similar to the caustic from the front face and slightly displaced from it, if d is significant relatively to $z_0$. But for our experiments where $z_0 \gg d$ the two caustics practically coincide.

Fig. 2 presents the vectorial diagram for tracing the caustic ABCDEF for mode-III deformation. For a generic point P of the initial curve projected on the screen (reference frame O'XY) PQ is the vector $\lambda_m r_0 (\cos\vartheta, \sin\vartheta)$ and QR is the vector $2\lambda_m r_0 (\sin\vartheta/2, -\cos\vartheta/2)$.

From the parametric relations of the caustic (Eqs. (11)) it can be deduced that this curve does not present any kind of symmetry with

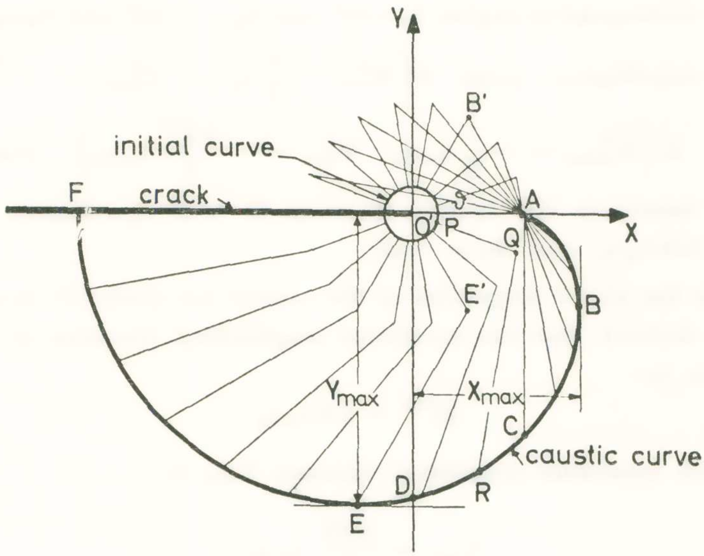


Fig. 2.

respect to the O'X- and O'Y-axes. This was expected from the type of anti-plane shear loading. Furthermore, it can be readily derived that for the angle $\vartheta$ lying between $\pi \geqq \vartheta \geqq 0$ the part ABC of the caustic is traced, starting from the intersection A of the caustic with the positive half of the O'X-axis and ending at point C, the intersection of the caustic with the normal AC at A to the O'X-axis. The tale of the caustic, that is the part CDEF, is formed from angles $\vartheta$ in the interval $-\pi \leqq \vartheta \leqq 0$.

Points A and F, where the caustic intersects the O'X-axis, have coordinates: point $A(\lambda_m r_0, 0)$ and point $F(-3\lambda_m r_0, 0)$, so that the distance AF along the O′X-axis is:

$$(AF) = 4\lambda_m r_0 . \tag{12}$$

It can be further shown, by taking the limits of the derivatives yielding the slope of the curve, that the caustic is tangent to O′X-axis at point A and perpendicular to the same axis at point F.

The extreme points B and E of the caustic along the O′X- and O′Y- axes correspond to angles $\vartheta_B = 60°$ and $\vartheta_E = -60°$ and these maxima have the coordinates: point $B\left(X^B_{max} = \frac{3}{2}\lambda_m r_0,\ Y^B_{(max)} = -\frac{\sqrt{3}}{2}\lambda_m r_0\right)$ and point $E\left(X^E_{(max)} = -\frac{1}{2}\lambda_m r_0,\ Y^E_{max} = -\frac{3\sqrt{3}}{2}\lambda_m r_0\right)$. Finally, the O′Y-axis intersects the caustic at point D with coordinates $X_D = 0$, $Y_D = -2.543\,\lambda_m r_0$ and $\vartheta_D = -43°$.

From the above properties of the caustic for mode-III deformation it may be derived that the maximum longitudinal diameter of the caustic is given by:

$$D_l^{max} = 4.5\,\lambda_m r_0 \tag{13}$$

whereas the maximum transverse distance $Y_{max}$ is:

$$Y_{max} = \frac{3\sqrt{3}}{2}\lambda_m r_0 . \tag{14}$$

On the other hand, the position of the crack tip may by defined by the coordinates $X_0$, $Y_0$ measured from points B and E of the caustic corresponding to the maximum points of this curve parallel and normal to the crack axis. These coordinates are given by:

$$X_0 = -1.5\,\lambda_m r_0 , \quad Y_0 = \frac{3\sqrt{3}}{2}\lambda_m r_0 . \tag{15}$$

In this way not only the instantaneous position of the crack tip may be accurately defined, but also the corresponding value of the stress intensity factor may be given by:

$$K_{III} = 0.263\frac{G}{z_0\lambda_m^{1/2}}\left(D_e^{max}\right)^{3/2} \quad \text{or} \quad K_{III} = 0.599\frac{G}{z_0\lambda_m^{1/2}}\left(Y_{max}\right)^{3/2} \tag{16}$$

Finally, the angle $\varphi$ subtended between the line FB connecting the maximum point B of the caustic along the direction of the crack axis with the tail point F of the caustic and the crack axis is equal to:

$$|\tan\varphi| = \frac{1}{3\sqrt{3}}$$

and $$|\hat{\varphi}| = 10° 53' 36''. \tag{17}$$

Figure 3 presents the traces on a plane placed at distance $z_0$ from the specimen of the light rays reflected from the front surface of the

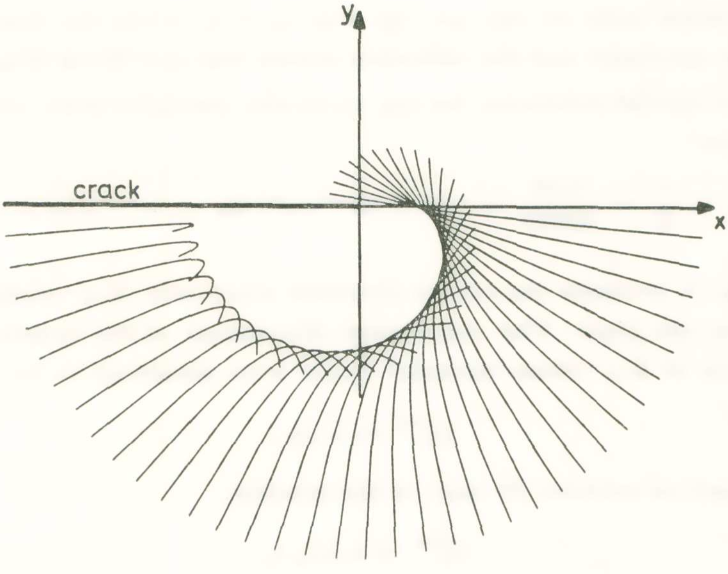


Fig. 3.

specimen, at the close vicinity of the crack tip. Each of these curves contains rays lying along the same radius from the crack tip. These curves have been traced by the computer as parametric families of angles $\vartheta = \text{const.}$ when the cracked plate is submitted to a mode-III deformation. It is clear from this figure that all rays bend back as they contact the envelope surface which constitutes the caustic. Furthermore, as the parameter $\vartheta$ is decreasing from $+\pi$ to zero and to $-\pi$ the bending back of the rays, as they approach the caustic, becomes more and more sharp so that at $\vartheta = -\pi$ the rays return back on the same straight path.

## EXPERIMENTAL EVIDENCE

In order to check the results derived from the caustic obtained at the tip of a crack subjected to antiplane shear a test was undertaken where an edge cracked plexiglas plate was submitted to a couple of forces normal to the lateral faces of the plate as indicated in Fig. 1.

The length of the crack was $a = 2\,\text{cm}$, whereas the dimensions of the plate were: width, $b = 12.5\,\text{cm}$, length $l = 15\,\text{cm}$ and thickness $d = 1.0\,\text{cm}$. The experimental arrangement was simple. A coherent light (divergent) was impinging on the faces of the plate at the vicinity of the crack. The magnification ratio of the set-up was $\lambda_m = 4$, while the distance between the specimen and the reference screen was $z_0 = 60\,\text{cm}$ (Fig. 1).

The optical constants for the particular plexiglas plate used in the tests were:

$$c_f = \frac{\nu}{E} = \frac{0.33}{32000} = 1.03 \times 10^{-5}\,\text{cm}^2/\text{Kg} \quad c_r = -3.24 \times c_f .$$

Fig. 4 presents the caustic obtained when only $K_{III}$-mode is operative on the plate. The convenient dimensions of the caustic for the evaluation of $K_{III}$-stress intensity factor were measured to be:

$$D_e^{max} = 5\,\text{cm}.$$

Based on relation (9) and on the relation,

$$D_e^{max} = 4.5\,\lambda_m\, r_0 ,$$

an experimental value $K_{III} = 9{,}31\,\text{Kp/cm}^{3/2}$ of S. I. F. was found.

This value compared well with its theoretical value $K_{III} = 10\,\text{Kp/cm}^{3/2}$ and gives a relative error of 6,9% only. This error is considered as acceptable for practical purposes.

## ΠΕΡΙΛΗΨΙΣ

Ἡ ὀπτικὴ μέθοδος τῶν ἀνακλωμένων καυστικῶν ὡς αὐτὴ ἔχει εἰσαχθῆ ὑπὸ τοῦ συγγραφέως [1] καθὼς καὶ ἡ ἀντίστοιχος μέθοδος τῶν διερχομένων καυστικῶν [2] ἔχουν εὐρέως χρησιμοποιηθῆ κατὰ τὴν τελευταίαν δεκαετίαν διὰ τὸν καθορισμὸν τοῦ συντελεστοῦ ἐντάσεως τῶν τάσεων εἰς τὰς αἰχμὰς ρωγμῶν πλακῶν καταπονουμένων, εἴτε εἰς καθαρὸν ἐφελκυσμόν, ὁπότε ἔχομεν τὸν συντελεστὴν $K_I$, εἴτε εἰς συνεπίπεδον διάτμησιν, ὁπότε ἔχομεν τὸν συντελεστὴν $K_{II}$.

Ὁ περιορισμὸς τῆς ἐφαρμογῆς τῆς μεθόδου τῶν καυστικῶν διὰ τοὺς δύο αὐτοὺς γενικοὺς τρόπους φορτίσεως ρηγματωμένων πλακῶν εἰς τὸ ἐπίπεδόν των ὀφείλεται εἰς τὸ γεγονὸς ὅτι κατὰ τὴν παραμόρφωσιν τῆς πλακὸς εἰς τὴν γειτονίαν τῆς ρηγματώσεως αἱ τάσεις καθίστανται ἰδιόμορφοι, τείνουσαι εἰς τὸ ἄπειρον ὅσον πλησιάζομεν εἰς τὴν αἰχμὴν τῆς ρωγμῆς. Βάσει τῆς θεωρίας τῆς ἐλαστικότητος καὶ τοῦ φαινομένου Poisson αἱ παράπλευροι ἕδραι τῆς πλακὸς παραμορφοῦνται σημαντικῶς, λαμβάνουσαι μορφὴν ἐπιφανειῶν μὲ αὔξουσαν κλίσιν εἰς τὴν γειτονίαν τῆς αἰχμῆς τῆς ρωγμῆς.

Ἑπομένως αἱ ὀπτικαὶ ἀκτῖνες αἱ προσπίπτουσαι παραλλήλως (ἢ συγκλίνουσαι ἢ ἀποκλίνουσαι) εἰς τὴν γειτονίαν τῆς αἰχμῆς τῆς ρωγμῆς, ἀνακλώμεναι ἢ διαθλώμεναι ἐπὶ τῶν παραπλεύρων ἐπιφανειῶν τοῦ δοκιμίου, διασκορπίζονται εἰς τὸν χῶρον καὶ ἐὰν ἡ μορφὴ τῶν παραπλεύρων ἐπιφανειῶν τοῦ δοκιμίου εἶναι κατάλληλος, περιορίζονται εἰς τὸν χῶρον ὑπὸ φωτεινῆς ἐπιφανείας, ἡ ὁποία ἀποτελεῖ τὴν περιβάλλουσαν τῶν δεσμῶν αὐτῶν φωτός, ἐπὶ τῆς ὁποίας τὸ πλεῖστον τῶν φωτεινῶν ἀκτίνων ἐφάπτεται. Ἐὰν ἡ ἐπιφάνεια αὐτὴ τμηθῇ ὑπὸ ἐπιπέδου παραλλήλου πρὸς τὸ δοκίμιον, ἡ καυστικὴ αὐτὴ ἐπιφάνεια δίδει τομὴν ἡ ὁποία συνήθως εἶναι γ ε ν ι κ ε υ μ έ ν η ἐ π ι κ υ κ λ ο ε ι δ ὴ ς κ α μ π ύ λ η.

Εἰς τὴν περίπτωσιν ὅπου ἡ ρηγματωμένη πλὰξ καταπονεῖται εἰς ἐγκαρσίαν διάτμησιν ἡ ὁποία ἔχει ὡς ἀποτέλεσμα τὴν κάμψιν τῶν χειλέων τῆς ρωγμῆς κατ' ἀντιθέτους κατευθύνσεις (σχῆμα 1), ἡ λύσις ἡ βασιζομένη εἰς τὴν θεωρίαν τῆς ἐλαστικότητος δίδει τὰς κάτωθι συνιστώσας τῶν τάσεων :

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \qquad (\alpha)$$

καὶ

$$\tau_{xz} \neq 0 \qquad \tau_{yz} \neq 0 . \qquad (\beta)$$

Ἐκ τῶν σχέσεων (α) συνάγεται ὅτι κατὰ τὴν παραμόρφωσιν τῆς ρηγματωμένης πλακὸς οὐδεμία μεταβολὴ τοῦ πάχους της λαμβάνει χώραν εἰς τὴν πλάκα. Ἑπομένως δὲν ἀναμένεται νὰ σχηματισθῇ καυστικὴ ἐπιφάνεια εἰς τὸν χῶρον τοῦ τύπου τοῦ δημιουργουμένου ἐκ τῆς μεταβολῆς τοῦ πάχους τοῦ δοκιμίου, ἤτοι τοῦ τύπου τοῦ ἀντιστοιχοῦντος εἰς τὰς παραμορφώσεις τὰς δημιουργούσας συντελεστὰς ἐντάσεως τῶν τάσεων τοῦ τύπου I καὶ II.

Ἐν τούτοις ὅμως κατὰ τὴν ἐγκαρσίαν παραμόρφωσιν τοῦ ρηγματωμένου δοκιμίου (σχῆμα 1) ἀμφότερα τὰ χείλη τῆς ρωγμῆς κάμπτονται ἀντικλαστικῶς καὶ ἑπομένως κατὰ τὴν πρόσπτωσιν τῶν ὀπτικῶν ἀκτίνων παρουσιάζουν μεταβαλλομένην κλίσιν τῶν ἀνακλωσῶν ἐπιφανειῶν, αἱ ὁποῖαι συντελοῦν εἰς τὴν ἀπόκλισιν τῶν ἀνακλωμένων ἀκτίνων εἴτε ἐκ τῆς προσθίας ἐπιφανείας τοῦ δοκιμίου (ἐὰν τοῦτο εἶναι ἀδιαφανὲς) εἴτε ἐξ ἀμφοτέρων τῶν παραπλεύρων ἐπιφανειῶν του (ἐὰν τοῦτο εἶναι διαφανές).

Ἡ παραμόρφωσις τῶν παραπλεύρων ἐπιφανειῶν τοῦ δοκιμίου δίδεται ἀπὸ τὴν σχέσιν τὴν ἐκφράζουσαν τὸ βέλος κάμψεως w συναρτήσει τοῦ συντελεστοῦ ἐντάσεως τῶν τάσεων $K_{III}$ εἰς ἐγκαρσίαν διάτμησιν ἐκφραζομένην ὑπὸ τῆς σχέσεως :

$$w = \frac{2r}{G} \frac{K_{III}}{(2\pi r)^{1/2}} \sin\frac{\vartheta}{2} \qquad (\gamma)$$

ὅπου r εἶναι ἡ πολικὴ ἀπόστασις ἀπὸ τῆς αἰχμῆς τῆς ρωγμῆς καὶ ϑ ἡ γωνία τῆς ἐπιβατικῆς ἀκτῖνος r τῶν ὑπ' ὄψιν σημείων καὶ τοῦ ἄξονος Ox τῆς ρωγμῆς καὶ G τὸ ἐλαστικὸν μέτρον διατμήσεως.

Πρὸς καθορισμὸν τῆς καυστικῆς ἐπιφανείας εἰς τὸν χῶρον ἐφαρμόζομεν τὸν νόμον τῆς ἀνακλάσεως τῶν ἀκτίνων καὶ ζητοῦμεν τὴν συνθήκην μηδενισμοῦ τῆς Ἰακωβιανῆς ὁριζούσης τῆς ἐκφραζούσης τὸν νόμον αὐτὸν διὰ τὸν καθορισμὸν τῆς ἰδιομόρφου ἐπιφανείας τῆς καυστικῆς εἰς τὸν χῶρον.

Ἡ συνθήκη αὐτή, ἐκφραζομένη ὑπὸ τῆς σχέσεως (9) τοῦ κειμένου, δίδει τὴν ἐξίσωσιν τῆς γενετείρας καμπύλης ἐπὶ τοῦ δοκιμίου τῆς καλουμένης καὶ ἀ ρ χ ι κ ῆ ς  κ α μ π ύ λ η ς.

Ἀποδεικνύεται ὅτι διὰ τὴν περίπτωσιν ἐγκαρσίας διατμήσεως ἡ καμπύλη αὐτὴ εἶναι κύκλος περιβάλλων τὴν αἰχμὴν τῆς ρωγμῆς, ἡ ἀκτὶς τοῦ ὁποίου δίδεται ὑπὸ τῆς σχέσεως :

$$r_0 = \frac{1}{(2\pi)^{1/3}} \left(|CK_{III}|\right)^{2/3} \qquad (\delta)$$

ὅπου ἡ γενικὴ σταθερὰ C δίδεται ἀπὸ τὴν σχέσιν :

$$C = \left| \frac{z_0 + \varepsilon d}{\lambda_m G} \right| . \qquad (\varepsilon)$$

Εἰς τὴν σχέσιν αὐτὴν $z_0$ εἶναι ἡ ἀπόστασις μεταξὺ δοκιμίου καὶ πετάσματος, d εἶναι τὸ πάχος τοῦ δοκιμίου, ε ἀκέραιος ἀριθμὸς ἴσος πρὸς τὴν μονάδα δι' ἀνάκλασιν εἰς τὴν προσθίαν ἐπιφάνειαν καὶ ε = 2 δι' ἀνάκλασιν ἐκ τῆς ὀπισθίας ἐπιφανείας τοῦ δοκιμίου καὶ $\lambda_m$ ὁ συντελεστὴς τῆς ὀπτικῆς μεγενθύνσεως τῆς διατάξεως.

Αἱ ἀνακλώμεναι ἀκτῖνες ἐκ τῆς γειτονίας τῆς ἀρχικῆς καμπύλης σχηματίζουν καυστικὴν ἐπιφάνειαν εἰς τὸν χῶρον, ἡ ὁποία, ἀποτεμνομένη ὑπὸ ἐπιπέδου ἀναφορᾶς εἰς ἀπόστασιν $z_0$ ἀπὸ τὸ δοκίμιον, δίδει καυστικὴν καμπύλην ἐκφραζομένην ὑπὸ τῶν παραμετρικῶν ἐξισώσεων :

$$\left.\begin{aligned} W_x &= \lambda_m r_0 (\cos\vartheta + 2\sin\vartheta/2) \\ W_y &= \lambda_m r_0 (\sin\vartheta + 2\cos\vartheta/2) \end{aligned}\right\} \quad -\pi \leqq \vartheta \leqq \pi . \qquad (\zeta)$$

Ἀποδεικνύεται ὅτι ἡ καυστικὴ αὐτὴ καμπύλη εἶναι καὶ αὐτὴ γ ε ν ι κ ε υ μ έ ν η ἐ π ι κ υ κ λ ο ε ι δ ή ς, τῆς ὁποίας ἡ μορφὴ δίδεται εἰς τὸ σχῆμα 3. Ἡ καμπύλη αὐτὴ ἔχει ἐφαπτόμενον τὸν ἄξονα τῆς ρωγμῆς εἰς τὸ ἐμπρόσθιόν της σημεῖον A καὶ εἶναι κάθετος ἐπὶ τὸν αὐτὸν ἄξονα εἰς τὴν οὐράν της F. Τὸ μῆκος (AF) εἶναι τετραπλάσιον τοῦ ἀνηγμένου μήκους τῆς ἀκτῖνος τῆς ἀρχικῆς καμπύλης $(\lambda_m r_0)$.

Τὰ ἀκρότατα σημεῖα τῆς καυστικῆς B ἔχουν συντεταγμένας :

$$B\left(X^B_{max} = \frac{3}{2}\lambda_m r_0 , \quad Y^B_{(max)} = -\frac{\sqrt{3}}{2}\lambda_m r_0\right)$$

καὶ

$$E\left(X^E_{(max)} = -\frac{1}{2}\lambda_m r_0 , \quad Y^E_{max} = -\frac{3\sqrt{3}}{2}\lambda_m r_0\right),$$

ἐνῷ ἡ μεγίστη ἐπιμήκης κατὰ τοῦ ἄξονα τῆς ρωγμῆς διάμετρος τῆς καυστικῆς εἶναι $D^{max}_e = 4.5\,\lambda_m r_0$ καὶ τὸ μέγιστον ἐγκάρσιον ἄνοιγμά της $Y_{max} = \frac{3\sqrt{3}}{2}\lambda_m r_0$.

Διὰ μετρήσεως οἱασδήποτε ἐκ τῶν ἀνωτέρω χαρακτηριστικῶν διαστάσεων τῆς καυστικῆς τῆς λαμβανομένης πειραματικῶς ὑπολογίζεται ἡ ἀκτὶς $r_0$ τῆς ἀρχικῆς της καμπύλης καὶ ἐξ αὐτῆς ὑπολογίζεται ὁ συντελεστὴς ἐντάσεως τῶν τάσεων $K_{III}$ δι' ἐφαρμογῆς τῆς σχέσεως (δ).

Τὸ σχῆμα 4 δεικνύει πειραματικῶς ληφθεῖσαν καυστικὴν διὰ $K_{III} \neq 0$. Διὰ περίπτωσιν τῆς πειραματικῆς διατάξεως τοιαύτην ὥστε νὰ δώσῃ $K_{III} = 10\,Kp/cm^{3/2}$ ὑπελογίσθη πειραματικῶς ἐκ τῶν διαστάσεων τῆς καυστικῆς συντελεστὴς $K^e_{III} = 9{,}31\ Kp/cm^{3/2}$. Ἄρα ὁ εὑρεθεὶς ἐκ τοῦ πειράματος συντελεστὴς $K^e_{III}$ διαφέρει ἐκ τῆς πραγματικῆς του τιμῆς μόνον κατὰ 6,9%. Ἡ ἀκρίβεια αὐτὴ θεωρεῖται ἱκανοποιητική.

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