ΠΡΑΓΜΑΤΕΙΑΙ ΤΗΣ ΑΚΑΔΗΜΙΑΣ ΑΘΗΝΩΝ
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BASIC PROPERTIES OF MERIDIONAL AND SAGITTAL CAUSTICS FROM CONIC REFLECTORS

BY
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ΑΘΗΝΑΙ
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SUMMARY

The caustic surfaces derived by illuminating conic reflectors by a point-light source lying along the axis of symmetry of the reflector were studied. All these caustics, consisting of highly illuminated surfaces are, classified into two categories, the meridional and the sagittal caustics. Meridional caustics form axisymmetric surfaces, which are defined as the envelopes of the light rays reflected from the mirror, while sagittal caustics form highly illuminated spikes along the axis of symmetry of the reflector formed by the intersections of the reflected light rays with the axis of symmetry of the reflector. A thorough study of the shape, evolution, position, properties and the dependence of meridional and sagittal caustics on the particular type of the conic reflector, as well as on the relative position of the point-light source and the reflector was undertaken. General laws for the characteristic properties of all these caustics were derived.

BASIC PROPERTIES OF MERIDIONAL AND SAGITTAL CAUSTICS FROM CONIC REFLECTORS

INTRODUCTION

Caustic surfaces formed by illuminating high stress-concentration regions in plane specimens loaded in their own plane have already been used by the author in a series of publications [**1-12**] for the study of the characteristic parameters of the stress field in such regions, that is the order of singularity and the stress concentration and intensity factors. These caustics can be considered as a means for the transformation of the existing stress singularities into optical singularities (caustics), which contain all the necessary information for the study of the corresponding stress singularities. A large number of interesting engineering problems has been solved by this method of caustics in a easy and accurate manner by the author and his co-workers. These problems included, among others, cracks in plates loaded in their elastic or plastic domain of deformation, discontinuities of any type, jointed dissimilar media and crack propagation phenomena. In all these problems the created surfaces had small slopes and curvatures.

The method of caustics was then extended to incorporate surfaces with large slopes and curvatures. First, the caustic surfaces formed in the simple case of conic reflectors were considered [**13**, **14**], and these caustics were used for surface topography purposes [**13**], that is for the formulation of a technique which enables the determination of the shape of a surface from its corresponding caustics. Nomograms have been also established for the determination of the deviation of a sphere-like surface from the ideal shape of a sphere. The caustic surfaces created by conic reflectors were afterwards used for distance measuring between two targets [**15**], one of which is materialized by a conic reflector and the other by the point-light source.

In the present paper the caustic surfaces created by illuminating conic reflectors by a point-light source lying along the principal axis of the reflector were thoroughly studied and important laws about the shape, evolution, position and properties of these caustics were disclosed.

MERIDIONAL CAUSTICS

a) Equations of Caustics.

Let us consider an axisymmetric reflector with equation of the form :

$$z = f(r) \tag{1}$$

and a point-light source placed along the z-axis and at a distance A from the Oxy plane. If a reference screen is placed at a distance z_0 from the Oxy plane (Fig. 1), the intersection with the screen of a light ray reflected from a generic point $P(r)$ of the reflector is given, according to Snell's law of reflection, by :

$$w = (z - z_0) \tan (2a + \varphi) \tag{2}$$

with :

$$\tan a = \frac{dz}{dr} \quad \text{and} \quad \tan \varphi = \frac{r}{A + z} \tag{3}$$

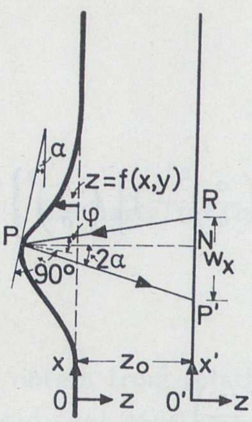


Fig. 1. Geometry of formation of a caustic by illuminating a surface by a parallel or divergent light beam.

If we refer the deviation w to the origin of the system $O'x'y'$, which is the parallel projection of the system Oxy on the screen, we obtain :

$$r' = r + (z - z_0) \tan (2a + \varphi) \tag{4}$$

Relation (4) maps each point $P(r)$ of the axisymmetric surface to a point $P'(r')$ on the screen. The necessary and sufficient condition for the points $P'(r')$ on the screen to belong to a curve is the zeroing of the derivative of r' with respect to r, that is :

$$\frac{dr'}{dr} = 0 \tag{5}$$

Relation (5) defines a curve on the surface $z = f(r)$, called the *initial curve*, while the system of equations (4) and (5) defines on the screen its corresponding *caustic*.

For the special case of an ellipsoid reflector with equation of the form :

$$\frac{z^2}{a^2} + \frac{r^2}{b^2} = 1 \tag{6}$$

relations (4) and (5) yield the following parametric equations of caustic in the form: $z_0 = z_0(r)$ and $r' = r'(r)$:

$$\frac{z_0}{b} = \frac{B_1 (1-(r/b)^2)^{1/2} + B_2}{\Delta_1 (1-(r/b)^2)^{1/2} + \Delta_2} \tag{7}$$

with:

$$B_1 = 2\left(\frac{a}{b}\right)^2 \left\{1 + \left[\left(\frac{a}{b}\right)^2 - 1\right]\left(\frac{r}{b}\right)^2 - \left(\frac{a}{b}\right)^2\right\} + \\ + \left\{1 - \left[\left(\frac{a}{b}\right)^2 + 1\right]\left(\frac{r}{b}\right)^2 - 2\left(\frac{a}{b}\right)^2\right\}\left(\frac{A}{b}\right)^2$$

$$B_2 = \left(\frac{A}{b}\right)\left(\frac{a}{b}\right)\left(3\left\{1 + \left[\left(\frac{a}{b}\right)^2 - 1\right]\left(\frac{r}{b}\right)^2\right\} - 4\left(\frac{a}{b}\right)^2\right)$$

$$\Delta_1 = \left(\frac{A}{b}\right)\left\{1 + \left[\left(\frac{a}{b}\right)^2 - 1\right]\left(\frac{r}{b}\right)^2 - 4\left(\frac{a}{b}\right)^2\right\}$$

$$\Delta_2 = \left(\frac{a}{b}\right)\left(\left\{1 + \left[\left(\frac{a}{b}\right)^2 - 1\right]\left(\frac{r}{b}\right)^2\right\} - 2\left\{\left(\frac{a}{b}\right)^2 - \left[\left(\frac{a}{b}\right)^2 - 1\right]\left(\frac{r}{b}\right)^2\right\} - \\ - 2\left(\frac{A}{b}\right)^2\right)$$

and:

$$\frac{r'}{b} = \frac{2\left(\frac{a}{b}\right)\left\{\left[\left(\frac{a}{b}\right)^2 - 1\right] - \left(\frac{A}{b}\right)^2\right\}\left(\frac{r}{b}\right)^3}{\Delta_1 (1-(r/b^2))^{1/2} + \Delta_2} \tag{8}$$

Equations (7) and (8) for the special case of a spherical reflector $(a = b = R)$ yield:

$$\frac{z_0}{R} = \frac{\left(\frac{A}{R}\right)^2\left[1 + 2\left(\frac{r}{R}\right)^2\right]\left(1 - \left(\frac{r}{R}\right)^2\right)^{1/2} + \left(\frac{A}{R}\right)}{3\left(\frac{A}{R}\right)\left(1 - \left(\frac{r}{R}\right)^2\right)^{1/2} + 2\left(\frac{A}{R}\right)^2 + 1}$$

$$\frac{r'}{R} = \frac{2\left(\frac{A}{R}\right)^2\left(\frac{r}{R}\right)^3}{3\left(\frac{A}{R}\right)\left(1 - \left(\frac{r}{R}\right)^2\right)^{1/2} + 2\left(\frac{A}{R}\right)^2 + 1} \tag{9}$$

The same equations (7) and (8) of the ellipsoid reflector for the case of a parallel light beam ($A \to \infty$) yield:

$$\frac{z_0}{b} = -\frac{\left[1 - \left[\left(\frac{a}{b}\right)^2 + 1\right]\left(\frac{r}{b}\right)^2 - 2\left(\frac{a}{b}\right)^2\right]}{2\left(\frac{a}{b}\right)}\left(1 - \left(\frac{r}{b}\right)^2\right)^{1/2}$$

$$\frac{r'}{b} = \left(\frac{r}{b}\right)^3 \tag{10}$$

For the particular case of a spherical reflector, illuminated by a parallel light beam, it can be obtained from equation (9) with $A \to \infty$, or from equation (10) with $a = b = R$, the following simple relations:

$$\frac{z_0}{R} = \left[\frac{1}{2} + \left(\frac{r}{R}\right)^2\right]\left(1 - \left(\frac{r}{R}\right)^2\right)^{1/2}$$

$$\frac{r'}{R} = \left(\frac{r}{R}\right)^3 \tag{11}$$

Similarly, for the case of a paraboloid reflector with equation of the form:

$$z_0 = \frac{r^2}{4b} \tag{12}$$

we obtain from relations (4) and (5) the following parametric equations of the meridional caustics:

$$\frac{z_0}{b} = \frac{\mathrm{E} + \mathrm{F}}{\mathrm{G}} \tag{13}$$

with:

$$\mathrm{E} = \left[\left[4 - \left(\frac{r}{b}\right)^2\right]\left[4\left(\frac{A}{b}\right) - \left(\frac{r}{b}\right)^2\right] + 16\left(\frac{r}{b}\right)^2\right]\left[8\left(\frac{A}{b}\right)\left(\frac{r}{b}\right)^2 + 16\left(\frac{A}{b}\right) + \left(\frac{r}{b}\right)^4\right]$$

$$\mathrm{F} = 16\left(\frac{r}{b}\right)^4\left[\left(\frac{A}{b}\right) - 1\right]\left[2\left(\frac{A}{b}\right) - \left(\frac{r}{b}\right)^2 - 6\right]$$

$$\mathrm{G} = 16\left[\left(\frac{A}{b}\right) - 1\right]\left[4 + \left(\frac{r}{b}\right)^2\right]\left[4\left(\frac{A}{b}\right) - 3\left(\frac{r}{b}\right)^2\right]$$

and:

$$\frac{r'}{b} = \frac{-4\left(\frac{r}{b}\right)^3}{4\left(\frac{A}{b}\right) - 3\left(\frac{r}{b}\right)^2} \tag{14}$$

Working as above for a hyperboloid reflector with equation of the form:

$$\frac{z^2}{a^2} - \frac{r^2}{b^2} = 1 \tag{15}$$

the following parametric equations of its caustic are derived:

$$\frac{z_0}{b} = \frac{B_1'\left[1 + \left(\frac{r}{b}\right)^2\right]^{1/2} + B_2'}{\Delta_1'\left[1 + \left(\frac{r}{b}\right)^2\right]^{1/2} + \Delta_2'} \tag{16}$$

with:

$$B_1' = 2\left(\frac{a}{b}\right)^2 \left[1 + \left[1 + \left(\frac{a}{b}\right)^2\right]\left(\frac{r}{b}\right)^2 + \left(\frac{a}{b}\right)^2\right] + $$

$$+ \left[1 + \left[1 - \left(\frac{a}{b}\right)^2\right]\left(\frac{r}{b}\right)^2 + 2\left(\frac{a}{b}\right)^2\right]\left(\frac{A}{b}\right)^2$$

$$B_2' = \left(\frac{A}{b}\right)\left(\frac{a}{b}\right)\left[3\left[1 + \left[1 + \left(\frac{a}{b}\right)^2\right]\left(\frac{r}{b}\right)^2\right] + 4\left(\frac{a}{b}\right)^2\right]$$

$$\Delta_1' = \left(\frac{A}{b}\right)\left[1 + \left[1 + \left(\frac{a}{b}\right)^2\right]\left(\frac{r}{b}\right)^2 + 4\left(\frac{a}{b}\right)^2\right]$$

$$\Delta_2' = \left(\frac{a}{b}\right)\left[1 + \left[1 + \left(\frac{a}{b}\right)^2\right]\left(\frac{r}{b}\right)^2 + 2\left[\left(\frac{a}{b}\right)^2 + \left[1 + \left(\frac{a}{b}\right)^2\right]\left(\frac{r}{b}\right)^2\right] + 2\left(\frac{A}{b}\right)^2\right]$$

and:

$$\frac{r'}{b} = \frac{2\left(\frac{a}{b}\right)\left[\left(\frac{A}{b}\right)^2 - \left[1 + \left(\frac{a}{b}\right)^2\right]\right]\left(\frac{r}{b}\right)^3}{\Delta_1'\left[1 + \left(\frac{r}{b}\right)^2\right]^{1/2} + \Delta_2'} \tag{17}$$

Equations (7) and (8), (13) and (14), and (16) and (17), defining the meridional caustics obtained by illuminating an ellipsoid, paraboloid and

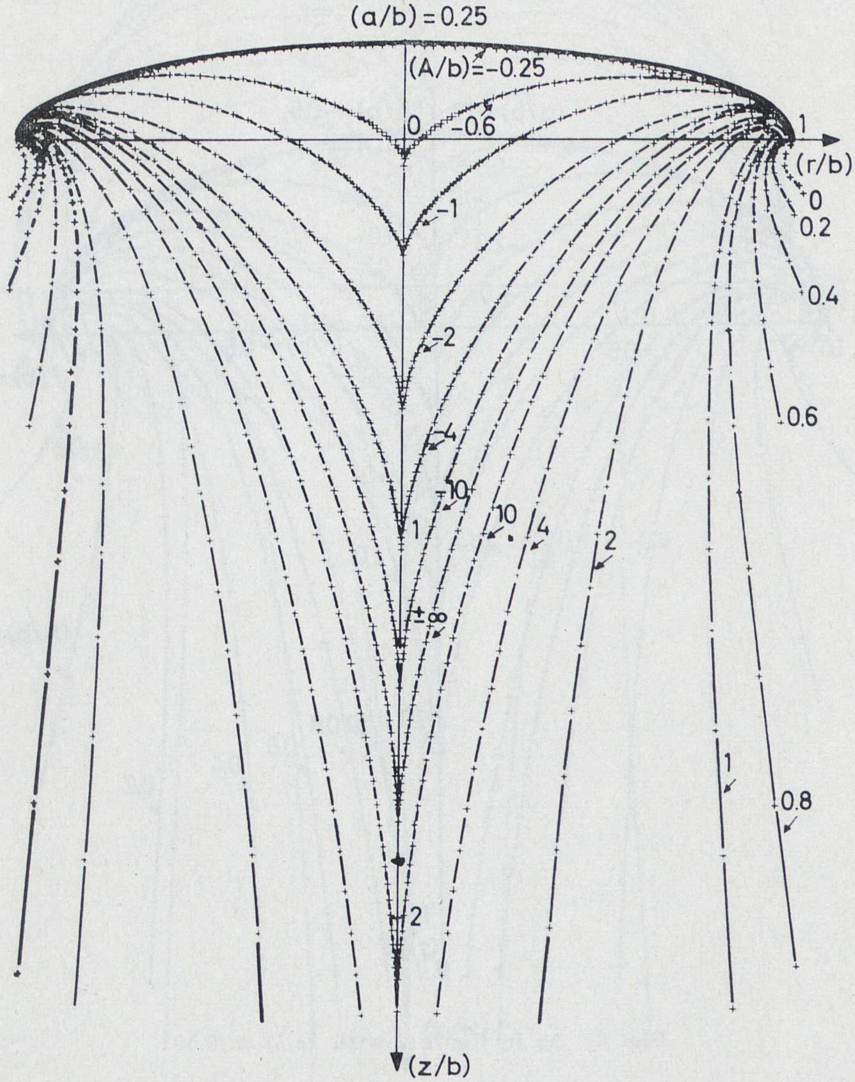


Fig. 2. Caustics obtained from a shallow ellipsoid reflector with $(a/b) = 0.25$ illuminated by a point-light source placed at various distances (A/b) from the equator of the reflector and along its principal axis. Positive values of A correspond to positions of the light source lying on the positive z-semi-axis.

hyperboloid reflector by a point-light source were programmed in a digital computer and the caustics corresponding to various types of these reflectors were plotted.

Figures 2 to 5 present the caustics formed by ellipsoid reflectors whose ratio (a/b) of their semi-axes takes the values $(a/b) = 0.25$, 0.50, 1.00 and 2.00. All these ellipsoid reflectors are illuminated by a point-light source

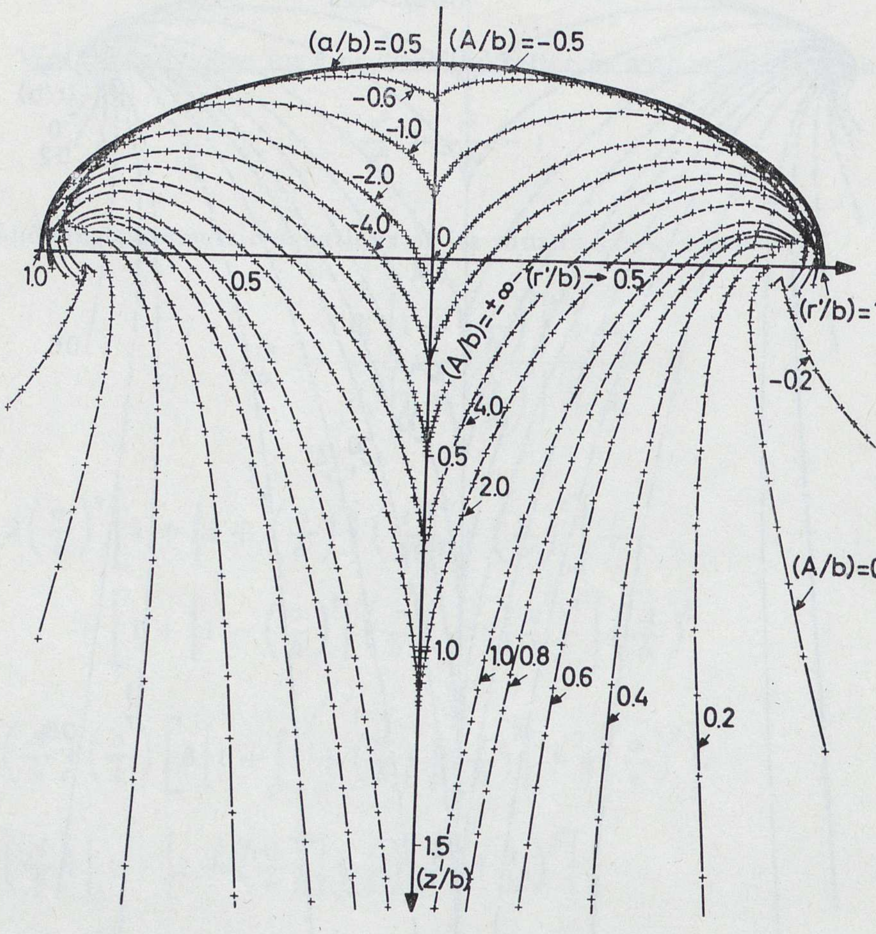


Fig. 3. As in figure 2 with $(a/b) = 0.50$.

placed at various distances (A/b) along the principal optical z-axis of the reflector. Positive values of A designate that the point-light source lies outside the part of the z-axis included between the reflecting surface of the mirror and its equatorial plane.

Similarly, figure 6 presents the caustics formed by illuminating a para-

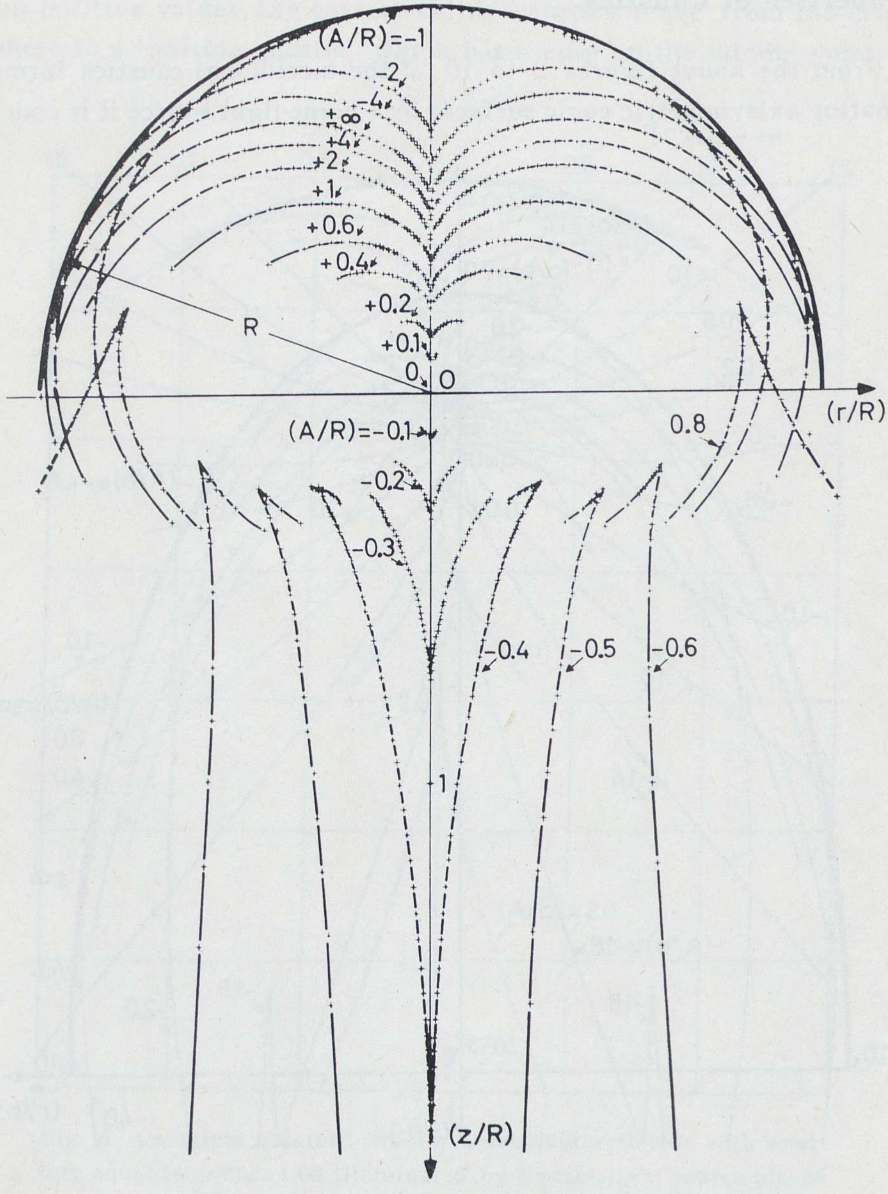


Fig. 4. As in figure 2 with $(a/b) = 1.00$.

boloid reflector, whose aperture is defined by $-b \leqq r \leqq b$, by a point-light source placed at various distances (A/b) from the reflector.

The caustics obtained by hyperboloid reflectors with $(a/b) = 0.25$, 0.50, 1.00 and 2.00 are presented in figures 7 to 10.

b) Properties of Caustics.

From the above figures 2 to 10 of the meridional caustics formed by illuminating axisymmetric conic surfaces by a point-light source it is concluded

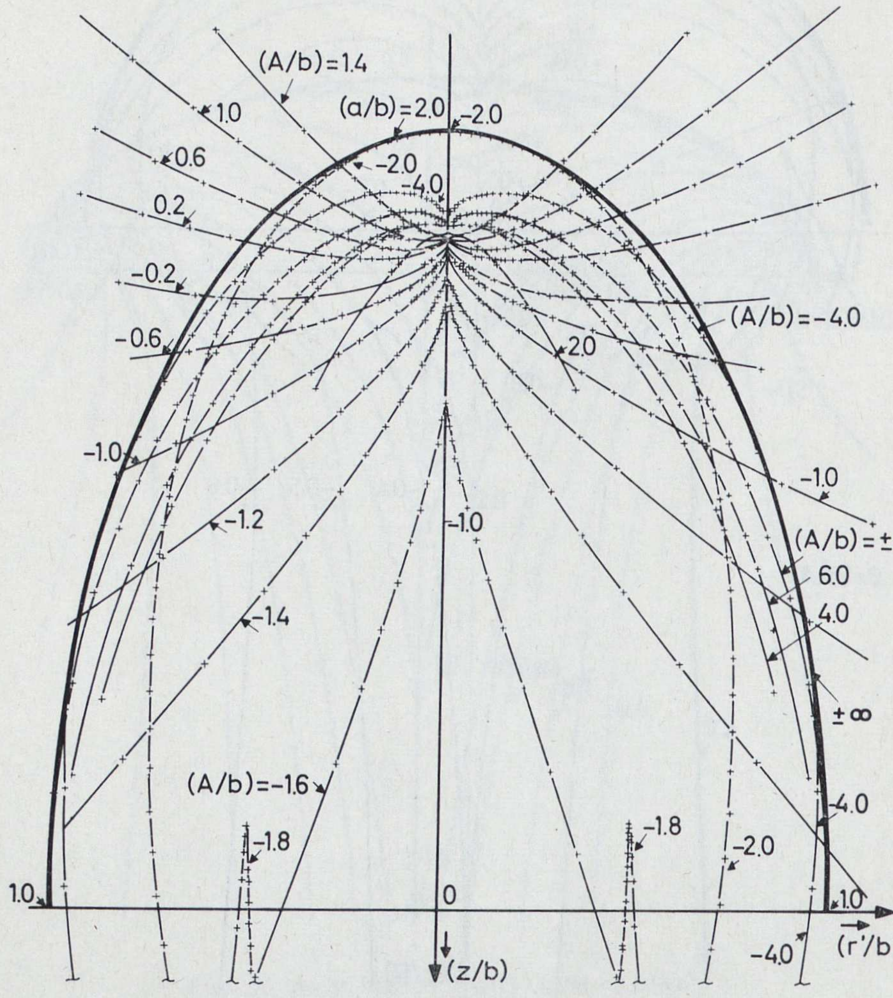


Fig. 5. As in figure 2 with $(a/b) = 2.00$.

that these caustics move progressively as the point source moves along the principal axis of the reflector and this movement depends on the particular type of the reflector. Thus, for the particular case of the spherical reflector, as the point-light source recedes from the center of the sphere to infinity, through positive values, the corresponding caustics move from the center of the sphere to a limiting caustic, which has a cusp on the middle point of the

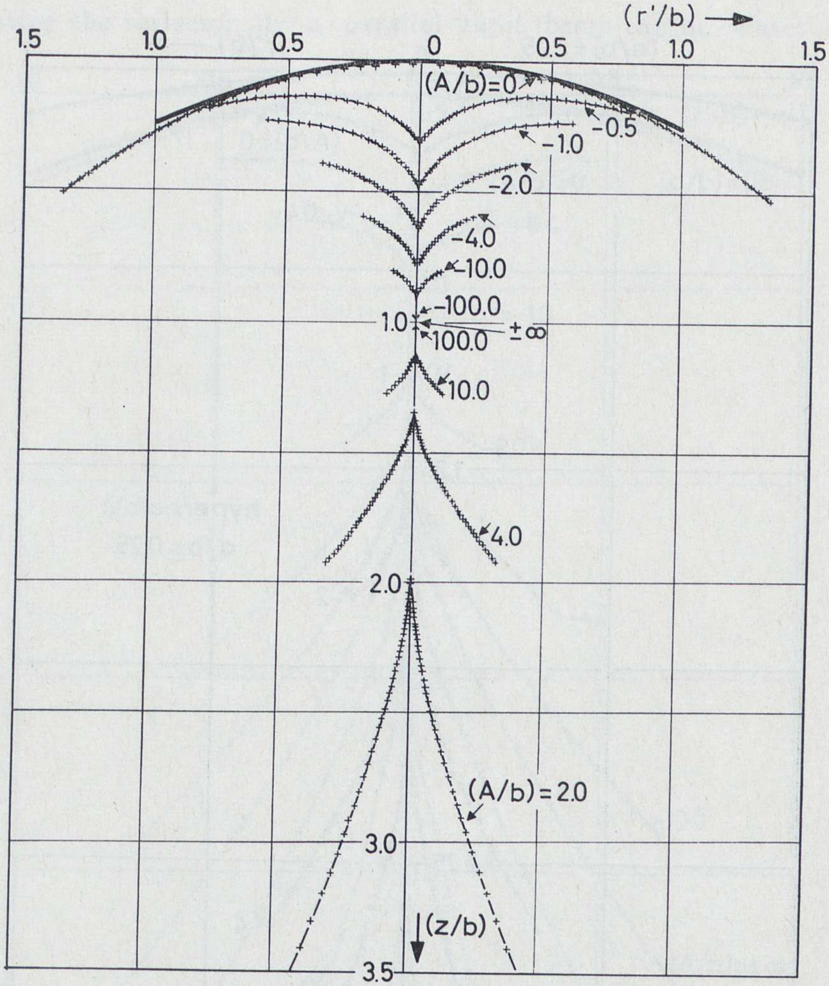


Fig. 6. Caustícts obtained from a paraboloid reflector with aperture equal to $(r/b) = 1.00$ illuminated by a point-light source placed at various distances (A/b) from the tangential plane of the reflector and along its principal axis. Positive values of A correspond to positions of the light source lying on the positive z-semi-axis.

principal radius between the center of the sphere and its surface, and terminates at the circumference of the sphere. For the shallow ellipsoid reflectors and as the ratio (a/b) of their axes diminishes $((a/b) < 1)$ this set of caustics recedes progressively from the reflector, beyond the equator of each reflector, whereas for deep ellipsoid reflectors ($(a/b) > 1$) and, as the ratio (a/b) increases, all these caustics are displaced, concentrating inside the

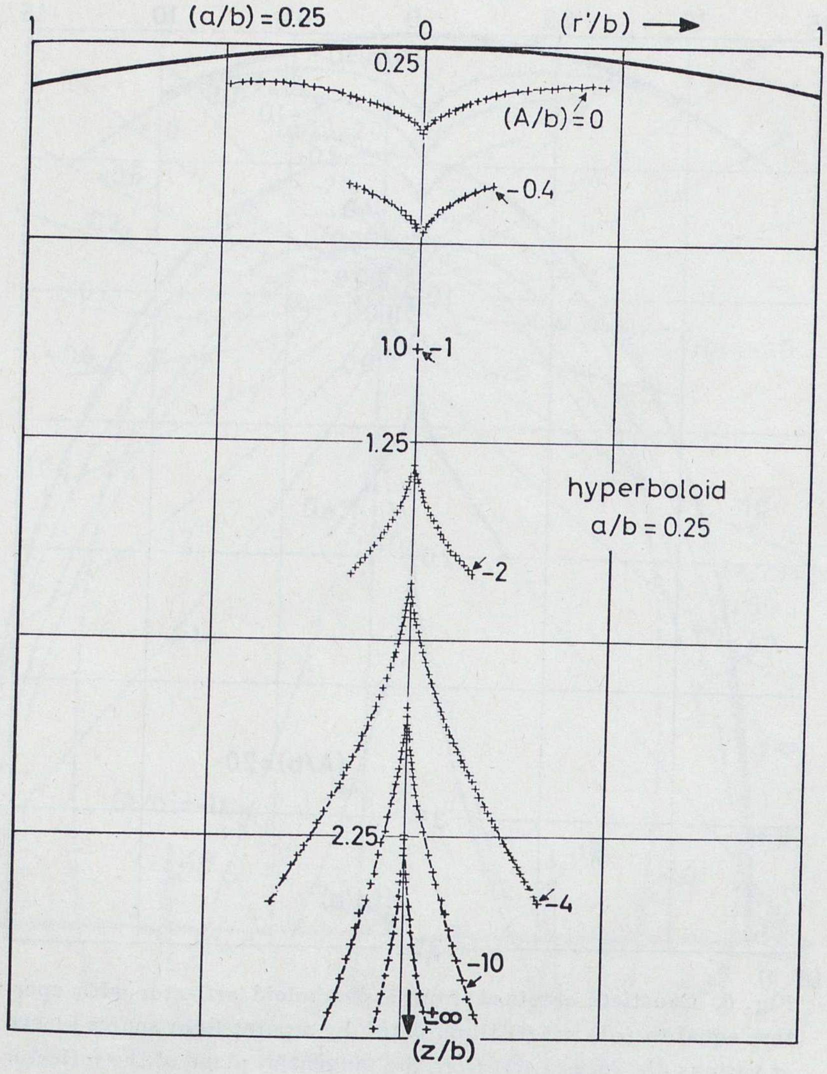


Fig. 7. Caustics obtained from a hyperboloid reflector with $(a/b) = 0.25$ and aperture equal to $(r/b) = 1.00$ illuminated by a point-light source placed at various distances (A/b) from the reference plane Orr of the reflector and along its principal axis. Positive values of A correspond to positions of the light source lying on the positive z-semi-axis.

reflector. Similar observations can be made for the case of the paraboloid and the hyperboloid reflectors.

On the other hand, the displacement of the typical caustics formed by illuminating the reflector by a parallel light beam can be materialized by

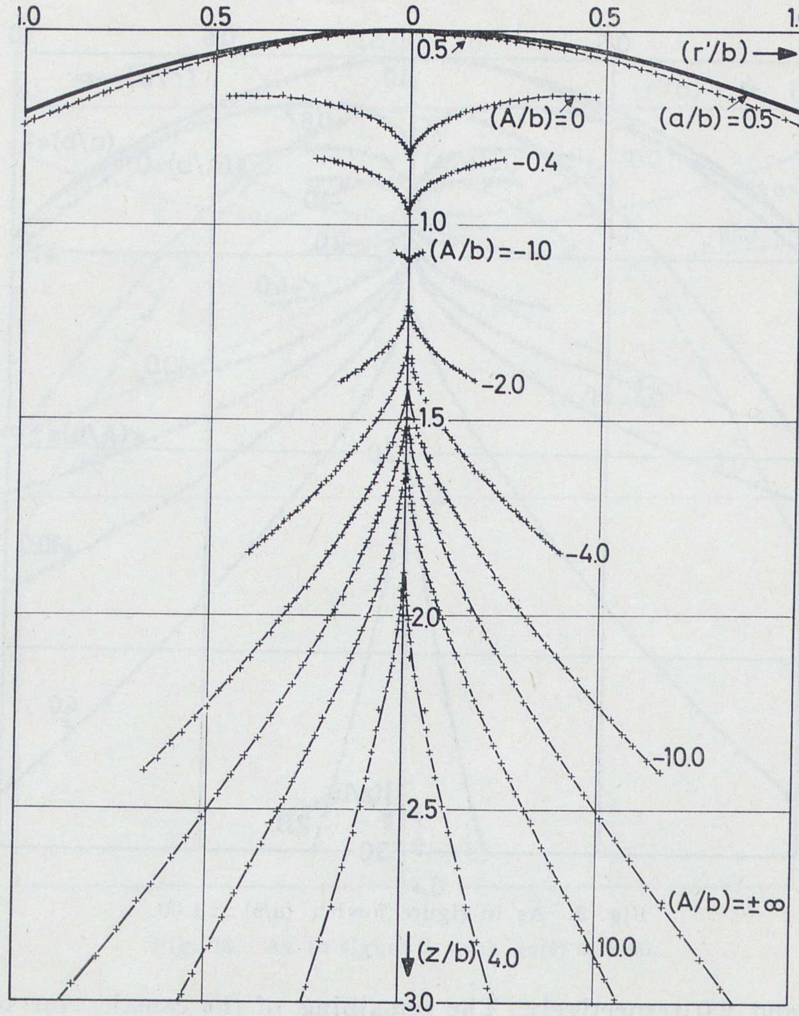


Fig. 8. As in figure 7 with $(a/b) = 0.50$.

studying the respective displacements of the cusp points along the axis of symmetry for reflectors of different shapes. For this reason the meridional caustics for ellipsoid reflectors with ratios of their respective axes $(a/b) = 0.25$, 0.50, 1.00, 2.00 and 4.00 were plotted in a computer by applying relations (10).

It was found that the cusp points of all these meridional caustics lie on the principal axes of the reflectors and at distances (z/b) from the origin equal to 1.75, 0.50, —0.50, 1.75 and —3.875 respectively. The positions of these cusp points were indicated in figures 2 to 5 for the caustics with $(a/b) = 0.25$,

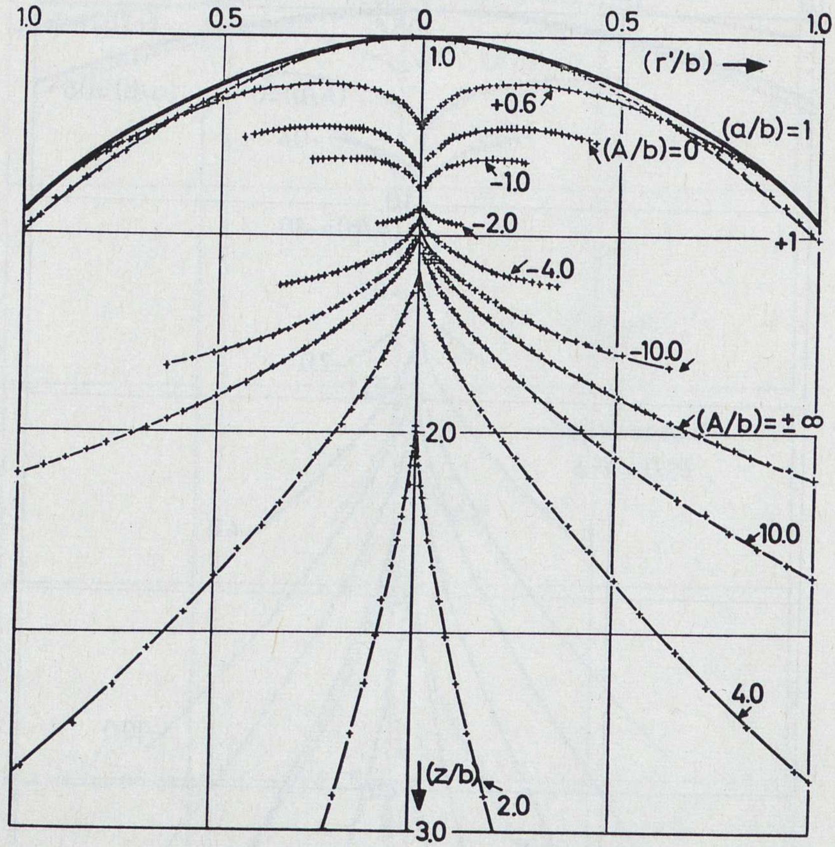


Fig. 9. As in figure 7 with $(a/b) = 1.00$.

0.5, 1.0 and 2.0 respectively. The remaining of the caustics for other geometries of reflectors were not plotted for saving of space.

It is also worthwhile noting from the plottings of caustics of figures 2 - 10 that, for all shapes of caustics for all conic reflectors, as the point-light source recedes along the positive or negative axis of symmetry of the reflector, the corresponding caustics approach more and more each other and tend to the caustic formed by the parallel light beam.

Furthermore, from the plottings of caustics of all conic reflectors the interesting result can be derived that, as the point-light source lies on the surface of the reflector ($A = -a$) the corresponding caustic is very close to the surface of the reflector and it touches it only at this point. This caustic

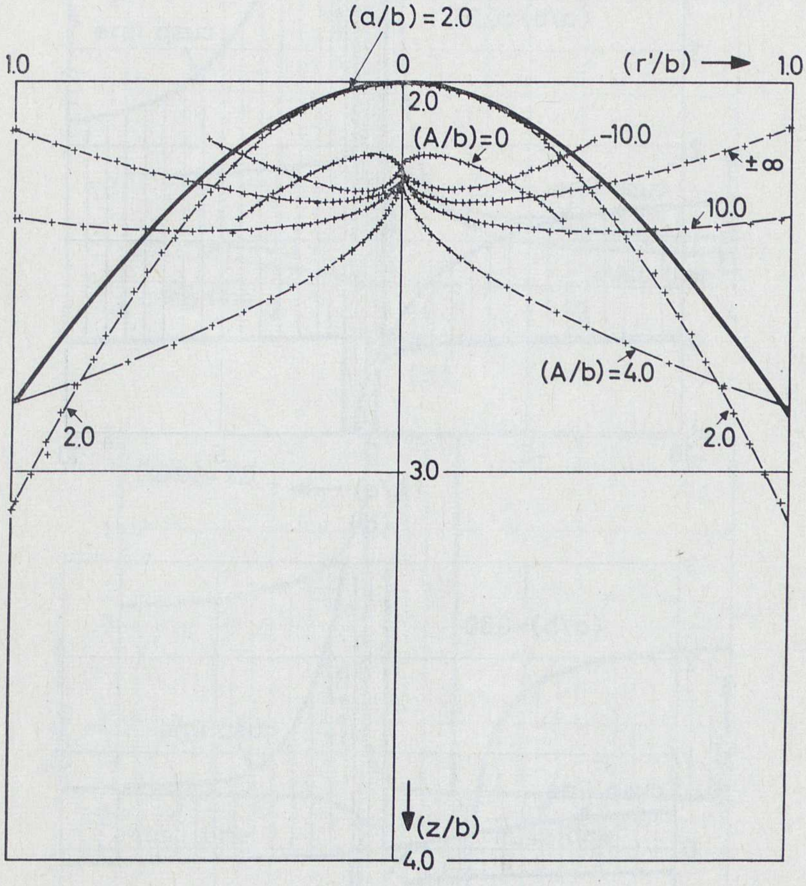


Fig. 10. As in figure 7 with $(a/b) = 2.00$.

is the only one, which does not present a cusp point at the axis of symmetry of the reflector.

The variation of the projections along the z-axis of the extremities of the caustic surfaces formed from ellipsoid reflectors with $(a/b) = 0.25$, 0.50, 1.00 and 2.00 are shown in figures 11 and 12 in terms of the relative position of the point-light source, while figures 13 and 14 present the same variations for the paraboloid and hyperboloid reflectors with $(a/b) = 0.50$, 1.00 and 2.00. In the same figures the positions of the respective cusp points are also indicated. From these figures it can be derived that, for each particular shape

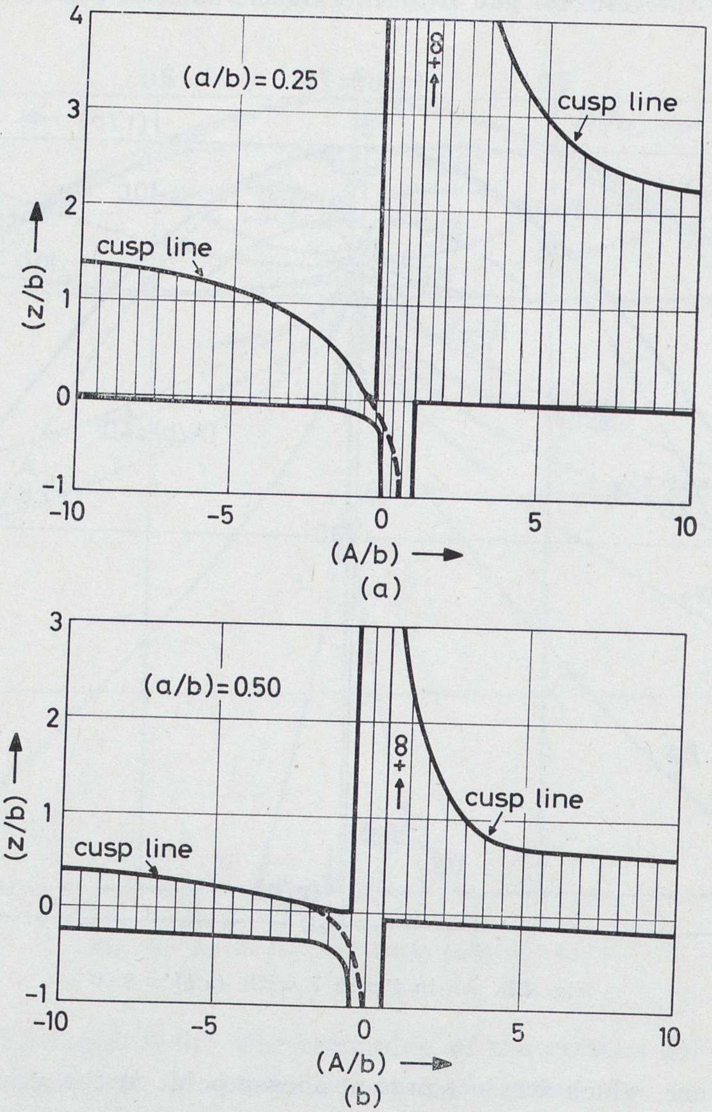


Fig. 11. Variation of the projections along the z-axis of the extremities of the caustics formed by ellipsoid reflectors with $(a/b) = 0.25$ (a) and 0.50 (b) versus the relative position of the light source. The position of the cusp point is also indicated.

of the reflector, there is a definite interval along its principal axis for which, when the point-light source lies within it, the corresponding caustics extend

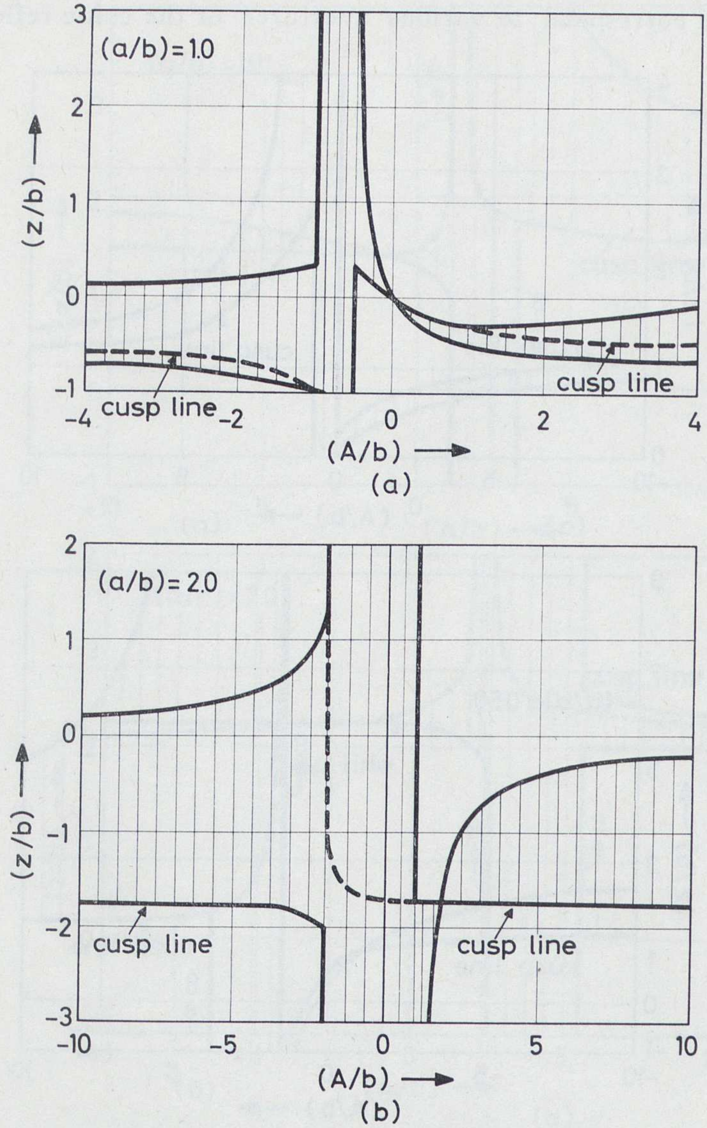


Fig. 12. As in figure 11 with $(a/b) = 1.00$ (a) and 2.00 (b).

to infinity. Furthermore, it can be derived from the plottings of the caustics of the various forms of the conic reflectors, that the major part of the light forming each caustic is concentrated around each cusp, surrounding the principal axis of the reflector, while the light density of each caustic is diminishing as we recede from the cusp point.

It is worthwhile defining the loci of the terminal points of each of the caustics, which correspond to various apertures of the conic reflectors. The

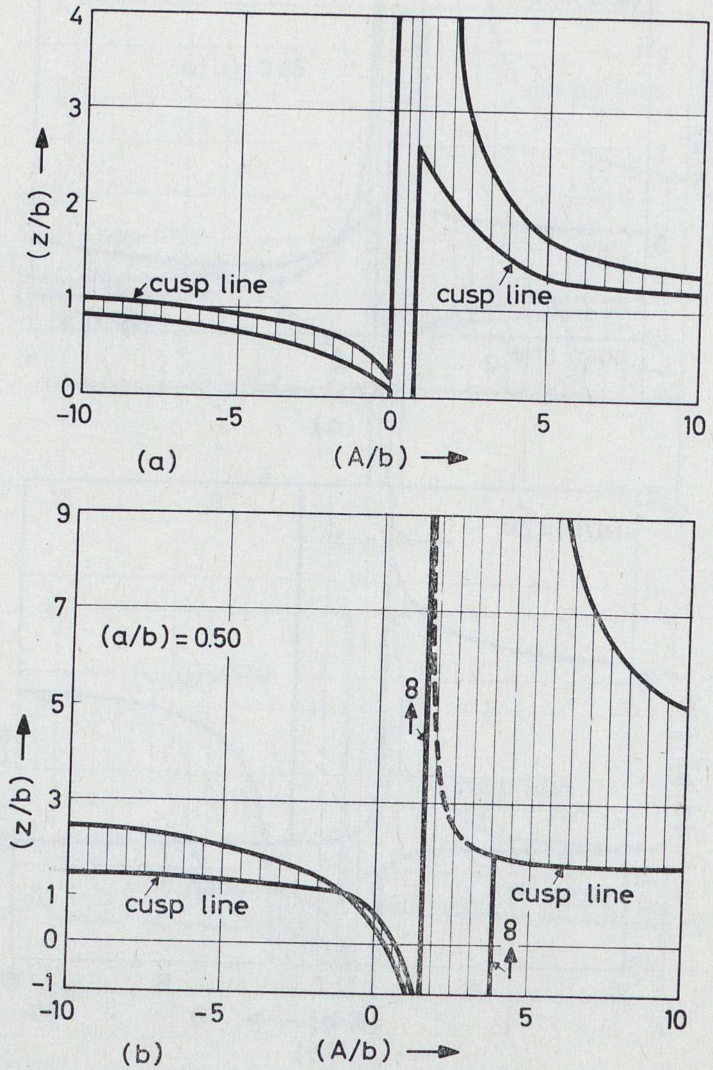


Fig. 13. Variation of the projections along the z-axis of the extremities of the caustics formed by a paraboloid (a) and a hyperboloid (b) reflector with $(a/b) = 0.50$ versus the relative position of the light source. The position of the cusp point is also indicated.

parametric equations of these loci, with the normalized distance (A/b) of the point-light source from the reflector as parameter, are defined by replacing

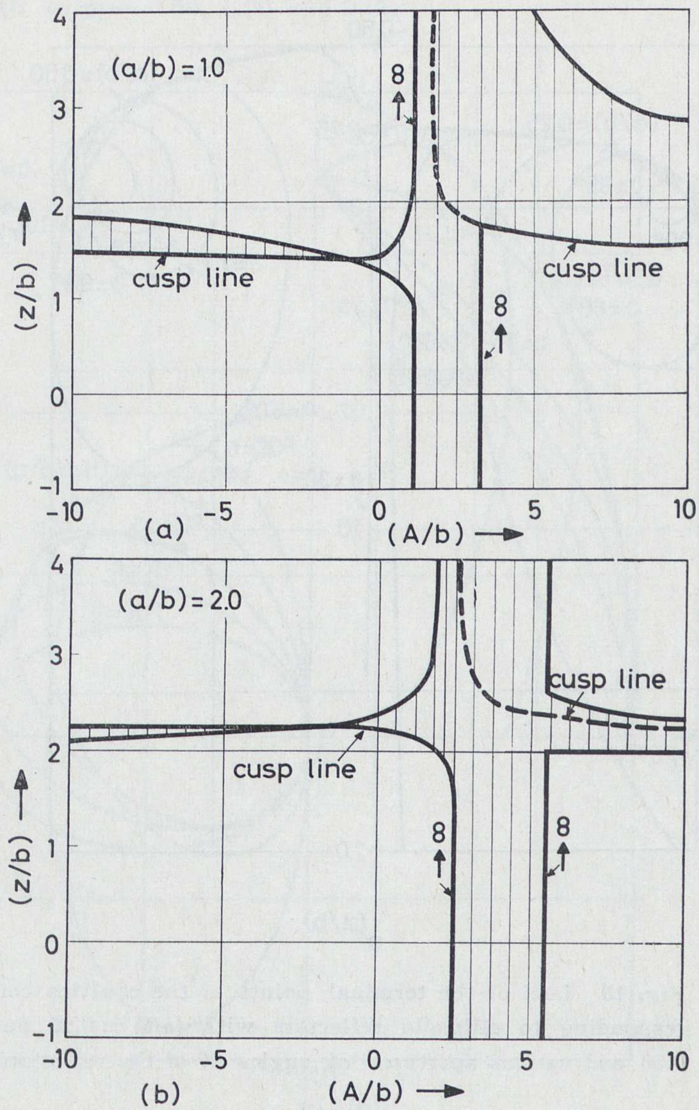


Fig. 14. As in figure 13 for hyperboloid reflectors with $(a/b) = 1.00$ (a) and 2.00 (b).

into equations (7) and (8) for the ellipsoid, (13) and (14) for the paraboloid and (16) and (17) for the hyperboloid reflectors the appropriate value of (r/b), corresponding to the aperture of the reflector. The kind of loci of the terminal points of caustics depends on the shape of each particular reflector, as well as on its aperture. Thus, for shallow ellipsoid reflectors ($(a/b) \leqslant 1$), these loci may be hyperbolae, parabolae, or ellipses, depending on the parti-

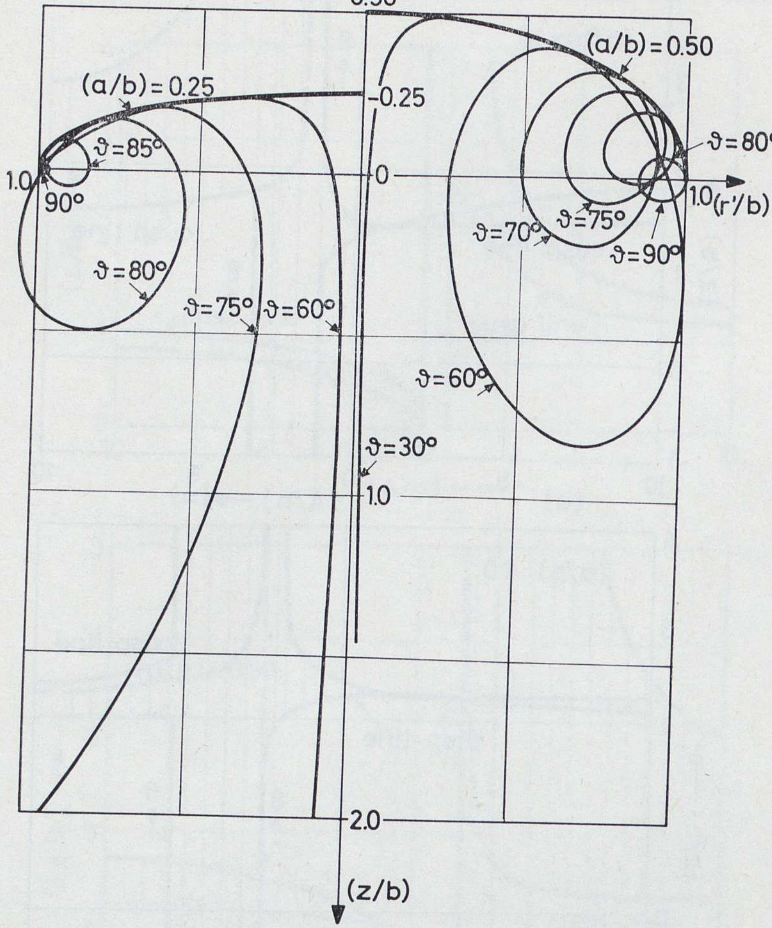


Fig. 15. Loci of the terminal points of the caustics corresponding to ellipsoid reflectors with $(a/b) = 0.25$ and 0.50 and various apertures of angles 2θ of the reflectors.

cular aperture of each reflector, whereas for deep reflectors ($(a/b) > 1$) these loci are always hyperbolae. For parabolic and hyperbolic reflectors, these loci are either parabolae or hyperbolae.

The loci of the terminal points of the caustics corresponding to the series of ellipsoid reflectors with $(a/b) = 0.25$, 0.50, 1.00 and 2.00 and to various apertures, of angles 2θ, are shown in figures 15 and 16, while figures 17 and 18 present the corresponding loci for the paraboloid and hyperboloid reflectors with $(a/b) = 0.50$, 1.00 and 2.00.

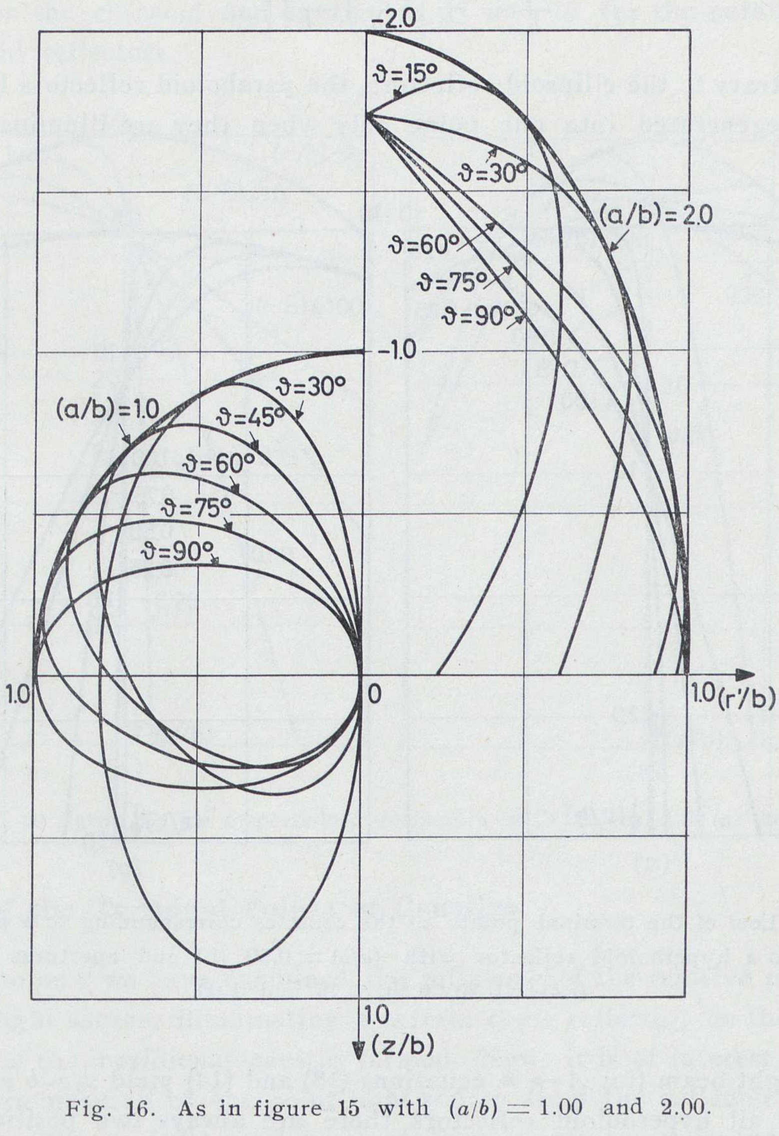


Fig. 16. As in figure 15 with $(a/b) = 1.00$ and 2.00.

From relation (8) it can be readily derived that for each deep ellipsoid reflector $((a/b) > 1)$ there are two particular positions of the point-light source, along the z-axis, for which the radius r' of the corresponding caustic degenerates to a point, for every aperture (defined by (r/b)) of the reflector. These positions are symmetric with respect to the equator of the reflector and they are given by:

$$\frac{A}{b} = \pm \left[\left(\frac{a}{b} \right)^2 - 1 \right]^{1/2} \tag{18}$$

Contrary to the ellipsoid reflectors, the paraboloid reflectors have their caustics degenerated into one point only when they are illuminated by a

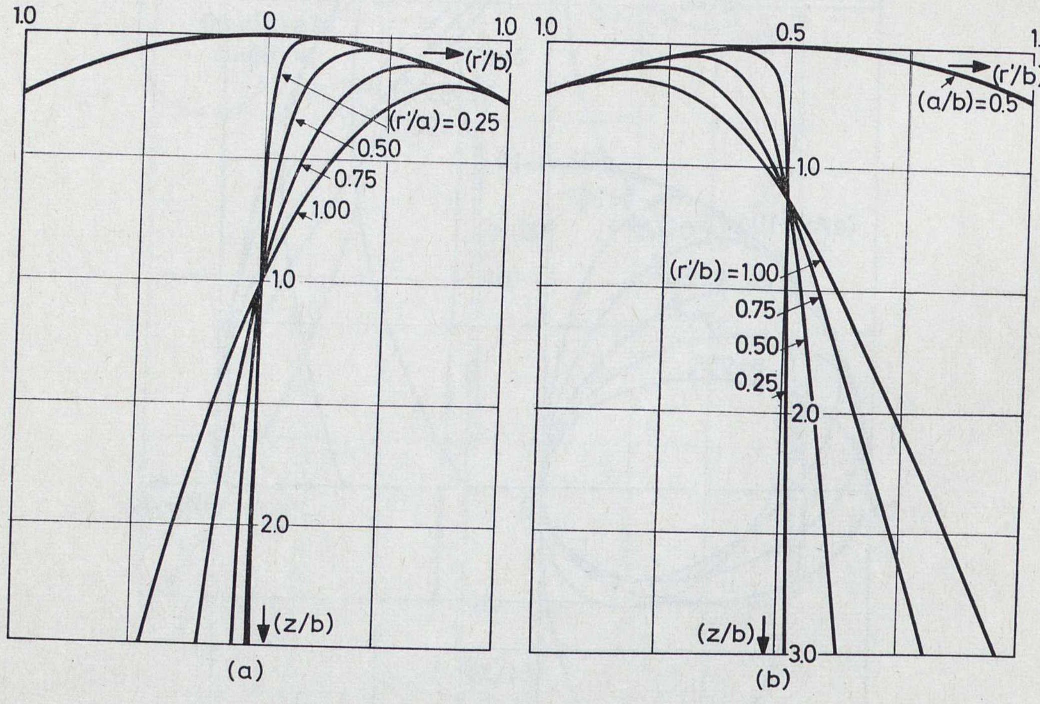
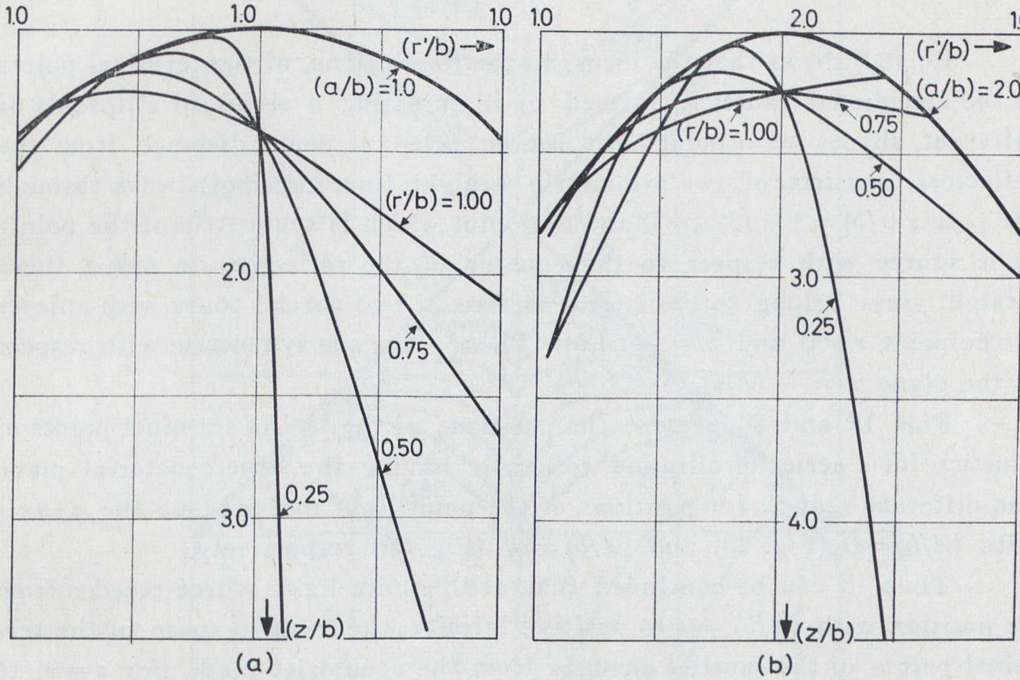


Fig. 17. Loci of the terminal points of the caustics corresponding to a paraboloid (a) and to a hyperboloid reflector with $(a/b) = 0.50$ (b) and apertures equal to $(r/b) = 0.25$, 0.50, 0.75 and 1.00.

parallel light beam (for $A \rightarrow \infty$ equations (13) and (14) yield $z_0 = b$ and $r' = 0$), while for all hyperboloid reflectors there are always two positions of the point-light source, for which their caustic vanishes into a single point. These positions A are given by:

$$\frac{A}{b} = \pm \left[\left(\frac{a}{b} \right)^2 + 1 \right]^{1/2} \tag{19}$$

The existence of a particular no-caustic position of the point-light source along the z-axis for the deep ellipsoid, the paraboloid and the hyperboloid reflectors may be also verified by the loci of terminal points of caustics corresponding to different apertures of the reflectors shown in figures 15 and 16 for the ellipsoid and in figures 17 and 18 for the paraboloid and hyperboloid reflectors.



Ffg. 18. As in figure 17 for hyperboloid reflectors with $(a/b) = 1.00$ (a) and 2.00 (b).

c) Loci of the Terminal Points of Caustics.

Up to now we have examined the influence of the relative position of the point-light source, illuminating a certain conic reflector, on the size and geometry of the meridional caustic formed. Now, it is of interest to study, for a certain position of the point-light source from the reflector, how the shape and position of the meridional caustics is influenced by the shape of the reflector.

Let now consider a point-light source fixed at a certain distance along the principal optical z-axis of a series of ellipsoid reflectors of different shapes. The equations of the loci of the terminal points of the caustics formed by such a series of reflectors for various positions of the light source can be obtained by putting the value $(r/b) = 1$ to the parametric equations expressing the shape of each particular caustic. Introducing this value into Eqs. (7) and (8) and eliminating between them the ratio (a/b), we obtain:

$$\frac{r'}{b} = \left[1 - \frac{(z_0/b)}{(A/b)}\right] \tag{20}$$

Eq. (20) shows that the locus, on the (Orz) - plane, of the terminal points of the meridional caustics, formed by illuminating a series of ellipsoids of different shapes by a point-light source fixed at some distance from the reflectors, consists of two symmetric straight lines passing always through the points $(r/b) = 1$ and $z_0 = 0$ and the point which is symmetric of the point-light source with respect to the equator of the reflector. In space these straight lines belong to the lateral surface of two normal cones with apieces at the point $r = 0$ and $z = -(A/b)$. These cones are symmetric with respect to the plane $z = -(A/b)$.

Figs 19 and 20 present the positions of the loci of terminal points of caustics for a series of ellipsoid reflectors having the same equatorial plane and different shapes, for positions of the point-light source along the z-axis with $(A/b) = 0$ (Fig. *19*) and $(A/b) = \infty$ (Fig. *20*) respectively.

Thus, it can be concluded that as the point-light source recedes from its position with $(A/b) = 0$ to positive infinity, the locus in space of the terminal points of the caustics changes from the equatorial plane (for $z = 0$) to two inversed cones with apices at the points $r = 0$, $z = -A$ and it tends to the circumference of the common equator of the ellipsoid reflectors.

For the case of hyperboloid reflectors of different shapes extending in $0 < (a/b) < \infty$ we consider parts of the reflectors corresponding always to $(r/b) = 1$ and having coinciding their planes of symmetry $(z = 0)$ of the two parts of the hyperboloids. For various values of the ratio (A/b), Eqs. (16) and (17) may be used for various parametric values of the ratio (a/b) and for $(r/b) = 1$ to define the loci of the terminal points of the corresponding meridional caustics for the hyperboloid reflectors.

Figure 21 presents the caustics corresponding to a series of hyperboloid reflectors illuminated by a point-light source placed at the position $(A/b) = 0$. The locus of the terminal points of caustics is also indicated. From this figure and similar ones, not presented here, it can be concluded that the loci of terminal points of caustics corresponding to fixed positions of the point-light source lying along the positive z-semi-axis constitute multi-

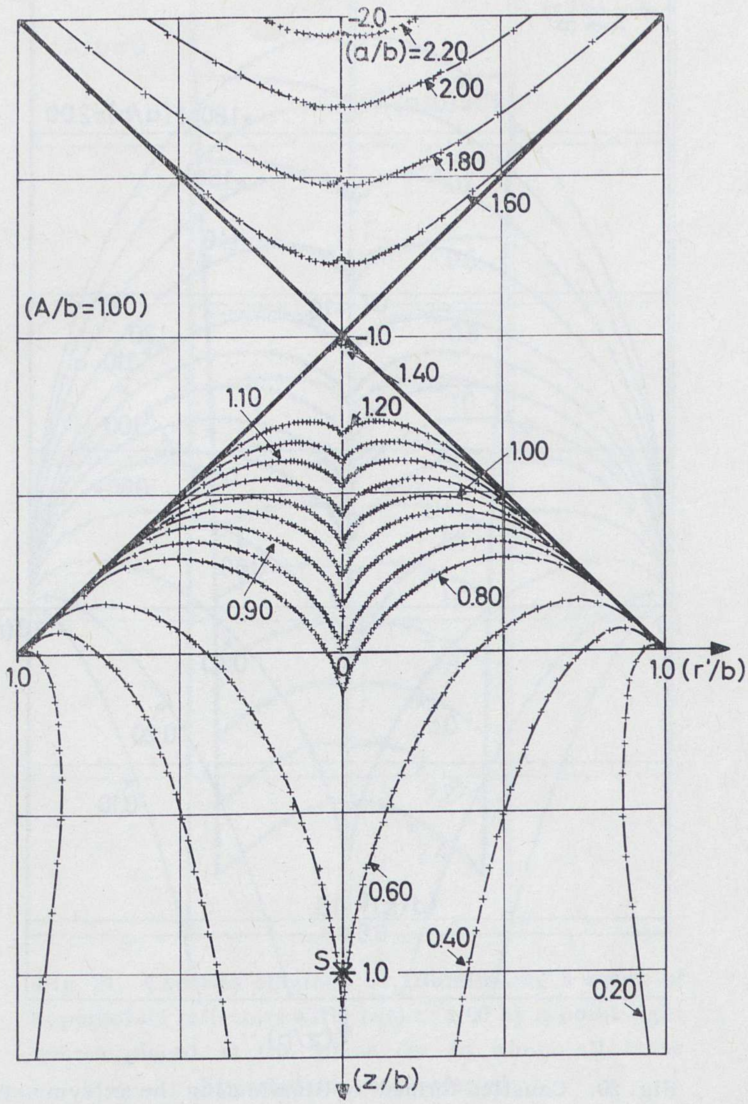


Fig. 19. Caustics formed by a point-light source placed at a position with $(A/b)=0$ and illuminating axisymmetric reflectors with elliptical cross-sections with the following values of the ratio (a/b) of their semi-axes : $(a/b)=0.80,\ 0.85,\ 0.90,\ 0.95,\ 1.00,\ 1.05,\ 1.10,\ 1.15,\ 1.20$.

branch curves with one branch lying in the region with positive values of z. These curves, beyond a small curved part near the positive z-semi-axis,

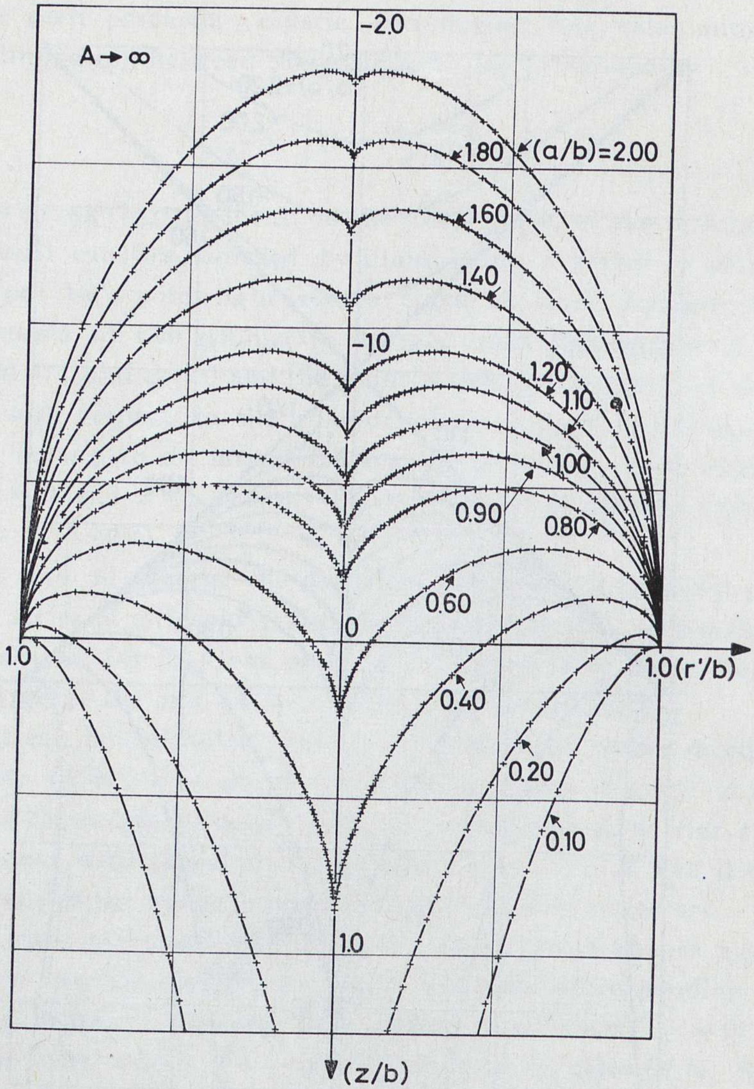


Fig. 20. Caustics formed by illuminating the axisymmetric mirrors of Fig. 19 with a parallel light beam.

become progressively almost parallel to the z-semi-axis and they extend up to infinity for the hyperboloid reflector with $(a/b) \rightarrow \infty$. On the other side, for positions of the point-light source lying along the negative z-semi-axis these loci constitute curves of hyperbolic type having two branches and presenting points at infinity.

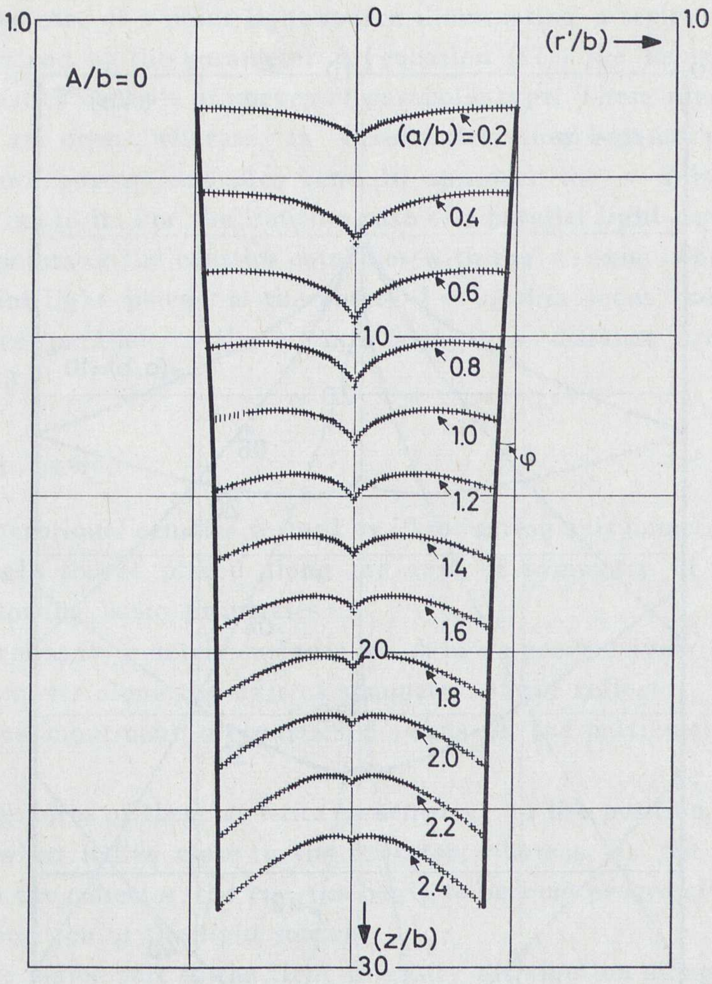


Fig. 21. Caustics obtained by illuminating a series of hyperboloid reflectors with $(r/b) = 1.00$ by a point-light source placed at the plane *Orr* to whom all these hyperboloids are referred.

For the particular case of a parallel light beam illuminating a series of hyperboloid reflectors, the locus of the terminal points of the caustics is defined by the simple relation :

$$(r'/b) = (r/b)^3 \tag{21}$$

as this can be derived from equations (16) and (17) by putting $A \rightarrow \infty$ and $(r/b) = 1$. This relation indicates that the terminal points of caustics lie on the lateral surface of a normal cylinder, whose radius is equal to the cube of the

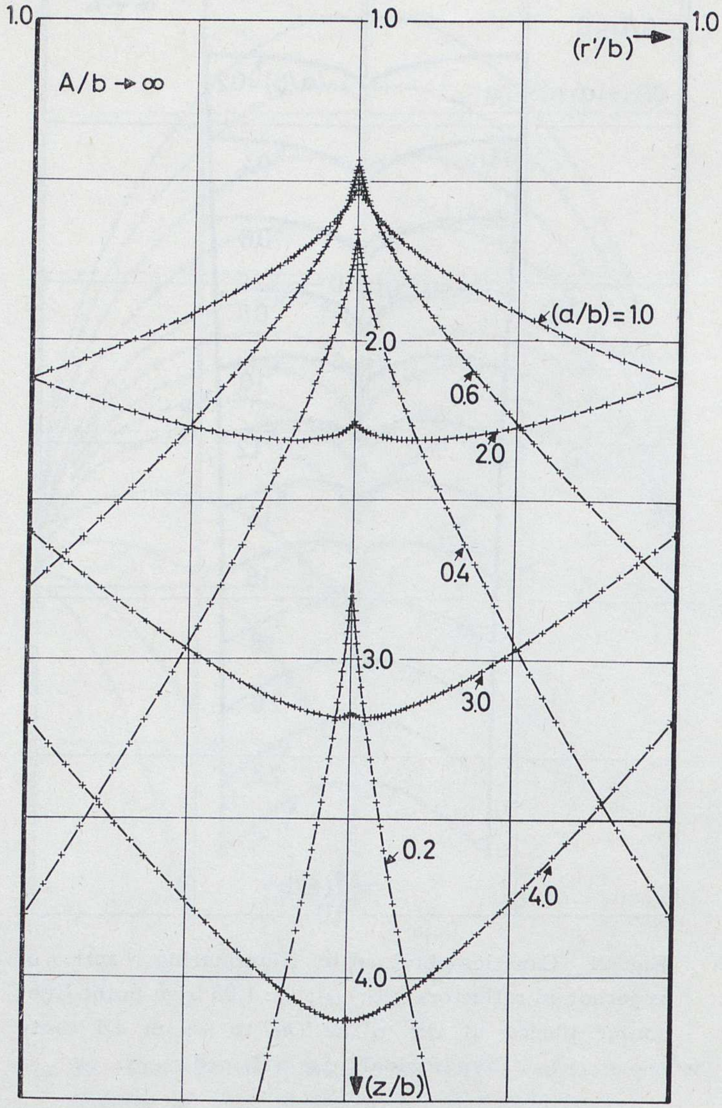


Fig. 22. Caustics obtained by illuminating a series of hyperboloid reflectors with $(r/b) = 1.00$ by a parallel light beam. The terminal points of all these caustics lie on a cylinder with radius $(r/b) = 1.0$.

aperture of the mirror. Figure 22 presents the caustics obtained by illuminating a series of hyperboloid reflectors by a parallel light beam. The terminal points of all these caustics lie on the lines $(r/b) = \pm 1$.

For the case of a point-light source illuminating a series of paraboloid reflectors defined by the parameter b (equation (17)) the locus of terminal points of caustics consists of curves of parabolic type. These curves for small values of A are open, whereas, as A increases, they become progressively more and more narrow and they bend to approach the z-axis, tending to become parallel to it. For the limiting case of a parallel light beam, the locus of terminal points of the caustics coincides with the z-axis, whereas for the case of a point-light placed at the point $A = 0$, this locus consists of two straight lines parallel to the z-axis and at a distance from it equal to $r = 1.3333$.

d) General Laws.

The meridional caustics formed by illuminating axisymmetric reflectors by a point-light source placed along the axis of symmetry of the reflector have the following basic properties :

1. Form axisymmetric surfaces which move progressively as the point-light source moves along the axis of symmetry of the reflector. The law of this successive movement of caustics depends on the particular type of the reflector.

2. The form of these caustics is sensitive to the position of the light source only when it lies close to the reflector, whereas, as the light source recedes from the reflector, the caustics begin to become progressively insensitive to the position of the light source.

3. The major part of the light intensity distribution along the caustic is concentrated near the principal axis of the reflector, where the caustic presents a cusp point. As we recede from the cusp point, the light intensity diminishes gradually.

4. The only caustic which does not present a cusp point along the axis of symmetry of the reflector is that formed by the light source lying on the surface of the reflector. This caustic is very close to the surface of the reflector and touches it at the axis of symmetry of the reflector. Along this caustic the light intensity is equally distributed and does not present a maximum value near the principal axis.

5. For each reflector there is a definite interval along its principal axis for which, when the light source lies within it, the corresponding caustics present points at infinity. All other caustics are finite.

i) Special Properties of Ellipsoid Reflectors.

1. The loci of the terminal points of caustics corresponding to various apertures of the reflector as the point source recedes along the axis of the reflector are for shallow ellipsoid reflectors hyperbolae, parabolae or ellipses depending on the particular aperture of the reflector, whereas for deep reflectors these loci are always hyperbolae. When the whole aperture of the ellipsoid reflector is illuminated by the light flux, these loci are ellipses for the values of the ratio (a/b) of the semi-axes a and b of the ellipsoid reflector lower than $2^{1/2}$, $((a/b) < 2^{1/2})$, hyperbolae for $(a/b) > 2^{1/2}$ and a parabola for $(a/b) = 2^{1/2}$.

2. For each deep ellipsoid reflector, these are two distinct positions of the light source for which no caustic is formed and all the reflected light rays pass through the conjugate point of the point-light source with respect to the equator of the ellipsoid. On both sides of these no-caustic images of the point-light source the respective caustics have oriented their cusp points aiming toward these characteristic points. For shallow ellipsoid reflectors there are no such positions of the point-light source, for which no caustic is formed, and all their caustics have their cusp points aiming outwards to their respective reflectors.

3. The locus of the terminal points of the meridional caustics formed by illuminating a series of ellipsoid reflectors of different shapes by a point-light source fixed at some distance from the reflectors consist of a conic surface of revolution with its apex coinciding with the symmetric point of the point-light source with respect to the equator of the reflector. This locus changes from the equatorial plane of the reflector to the circumference of the common equator of the reflectors, as the light source recedes from the equatorial plane to infinity.

ii) Special Properties of Hyperboloid Reflectors.

1. The loci of the terminal points of caustics corresponding to various apertures of the reflector, as the point-light source recedes along the axis of the reflector, are always hyperbolae for all shapes of the reflectors.

2. For each reflector there is a particular position of the point-light source for which no caustic is formed and all the reflected light rays pass through the conjugate point of the light source with respect to the reflector.

3. The loci of the terminal points of caustics corresponding to fixed positions of the point-light source lying along the positive z-semi-axis and illuminating a series of hyperboloid reflectors constitute multibranch curves with one branch lying in the region with positive values of z. These curves, beyond a small curved part near the positive z-semi-axis become progressively almost parallel to the z-semi-axis and they extend up to infinity for hyperboloid reflectors with $(a/b) \rightarrow \infty$. For positions of the point-light source lying along the negative z-semi-axis these loci constitute curves of hyperbolic type having two branches and presenting points at infinity.

iii) Special Properties of Paraboloid Reflectors.

1. The loci of the terminal points of caustics for any aperture of the paraboloid reflector, as the point-light source recedes along the axis of the reflector are parabolae.

2. Each paraboloid reflector gives a zero-caustic, formed at its focus, only when it is illuminated by a parallel light beam. The cusp point of all caustics, lying between the reflector and its focus, aims outwards the reflector, whereas this rule is reversed for caustics lying outside the above region.

3. The loci of terminal points of caustics corresponding to a fixed position of the point-light source, which illuminates a series of parabolic reflectors are always parabolae.

SAGITTAL CAUSTICS

a) Equations of Caustics.

Besides the meridional caustics studied above, which are defined as the envelopes of the light rays reflected from the mirror, another kind of caustics is also formed by illuminating a conic reflector by a point-light source.

Let us consider in Fig. 23 a bundle of parallel light rays, which impinge on a meridional section of a spherical reflector. The envelope of the reflected light rays forms the meridional caustic studied previously. This caustic is an axisymmetric surface formed by rotating about the axis of symmetry of the reflector the inplane caustic of Fig 23. However, it can be observed from

Fig. 23 that each light ray reflected from the reflector intersects the principal axis of it at a definite point. Thus, if we consider the portion of the reflector between points O and A all the reflected light rays from the aperture OA lie between the cusp C of the meridional caustic and the point A', at which the reflected ray from the rim A of the reflector intersects the principal axis. Since between C and A' lie all the intersection points of the reflected light rays from the corresponding portions OA of all the meridional sections of the

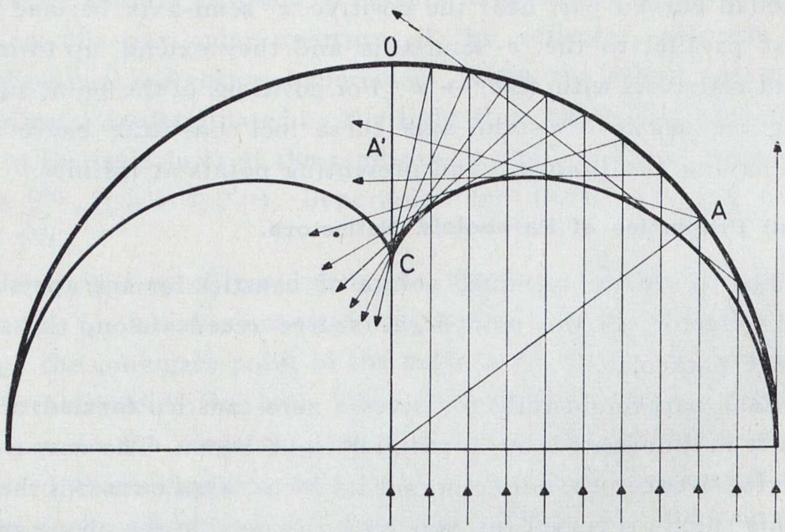


Fig. 23. Reflected light rays according to Snell's law of reflection from a spherical mirror illuminated by a parallel light beam. Meridional caustic is defined as the envelope of the reflected light rays, while sagittal caustic lies along the principal axis of the reflector and is defined by the cusp point of the meridional caustic and the reflection from the rim of the mirror.

spherical reflector, the portion $A'C$ of the principal axis will be highly illuminated. This portion $A'C$ forms another kind of caustic, which due to its shape is called *sagittal caustic.*

Thus, when a conic reflector is illuminated by a point-light source lying along the principal axis of the reflector two caustics are formed: the meridional caustic as the envelope of the reflected light rays and the sagittal caustic consisting of a highly illuminated portion along the principal axis of the reflector, formed by the intersections of the reflected rays and the principal axis of the reflector. Both caustics have a common point coinciding with the cusp of the meridional caustic.

The z coordinate of point A' along the principal axis of the reflector, derived from relation (4) by putting $r' = 0$, is given by:

$$z_0 = z + \frac{r}{\tan(2a + \varphi)} \tag{22}$$

This equation defines the one extremity of the sagittal caustic, while the other extremity is always coinciding with the cusp point of the meridional caustic.

From equations (3), (6), (12) and (15) we obtain the following relations defining the z coordinate of point A':

$$\frac{z_0}{b} = \frac{K\left(1 - (r/b)^2\right)^{1/2} + L}{M\left(1 - (r/b)^2\right)^{1/2} + N} \tag{23}$$

with:

$$K = 2\left(\frac{a}{b}\right)\left(\frac{r}{b}\right)\left[\left(\frac{a}{b}\right)^2 - 1\right]$$

$$L = 2\left(\frac{a}{b}\right)^2\left(\frac{A}{b}\right)\left(\frac{r}{b}\right)\left[1 - \left(\frac{r}{b}\right)^2\right] + \left(\frac{A}{b}\right)\left(\frac{r}{b}\right) - \left(\frac{A}{b}\right)\left(\frac{r}{b}\right)^3\left[1 + \left(\frac{a}{b}\right)^2\right]$$

$$M = -2\left(\frac{a}{b}\right)\left(\frac{A}{b}\right)\left(\frac{r}{b}\right) \quad N = \left(\frac{r}{b}\right)\left[\left(\frac{a}{b}\right)^2 - 1\right]\left[\left(\frac{r}{b}\right)^2 - 1\right] - \left(\frac{a}{b}\right)^2\left(\frac{r}{b}\right)$$

for the ellipsoid reflector,

$$\frac{z_0}{b} = -\frac{K'\left(1 + (r/b)^2\right)^{1/2} + L'}{M'\left(1 + (r/b)^2\right)^{1/2} + N'} \tag{24}$$

with:

$$K' = 2\left(\frac{a}{b}\right)\left(\frac{r}{b}\right)\left[\left(\frac{a}{b}\right)^2 + 1\right]$$

$$L' = 2\left(\frac{a}{b}\right)^2\left(\frac{A}{b}\right)\left(\frac{r}{b}\right)\left[1 + \left(\frac{r}{b}\right)^2\right] - \left(\frac{A}{b}\right)\left(\frac{r}{b}\right) - \left(\frac{A}{b}\right)\left(\frac{r}{b}\right)^3\left[1 - \left(\frac{a}{b}\right)^2\right]$$

$$M' = 2\left(\frac{a}{b}\right)\left(\frac{A}{b}\right)\left(\frac{r}{b}\right) \quad N' = \left(\frac{r}{b}\right)\left[\left(\frac{a}{b}\right)^2 + 1\right]\left[\left(\frac{r}{b}\right)^2 + 1\right] + \left(\frac{a}{b}\right)^2\left(\frac{r}{b}\right)$$

for the hyperboloid reflector and,

$$\frac{z_0}{b} = \frac{\left(\frac{r}{b}\right)^4 + 8\left(\frac{r}{b}\right)^2 + 16\left(\frac{A}{b}\right)}{16\left[\left(\frac{A}{b}\right) - 1\right]} \tag{25}$$

for the parapoloid reflector, respectively.

The z coordinate of the other extremity of the sagittal caustics, coinciding with the cusp point of the corresponding meridional caustic, is defined from relations (7), (16) and (13) for the ellipsoid, the hyperboloid and the paraboloid reflector respectively and it is given by :

$$\frac{z_0}{b} = \frac{2\left(\frac{a}{b}\right)^2\left[1-\left(\frac{a}{b}\right)^2\right]+\left(\frac{A}{b}\right)^2\left[1-2\left(\frac{a}{b}\right)^2\right]+\left(\frac{A}{b}\right)\left(\frac{a}{b}\right)\left[3-4\left(\frac{a}{b}\right)^2\right]}{\left(\frac{A}{b}\right)\left[1-4\left(\frac{a}{b}\right)^2\right]+\left(\frac{a}{b}\right)\left[1-2\left(\frac{a}{b}\right)^2-2\left(\frac{A}{b}\right)^2\right]} \tag{26}$$

for the ellipsoid reflector,

$$\frac{z_0}{b} = \frac{2\left(\frac{a}{b}\right)^2\left[1+\left(\frac{a}{b}\right)^2\right]+\left(\frac{A}{b}\right)^2\left[1+2\left(\frac{a}{b}\right)^2\right]+\left(\frac{A}{b}\right)\left(\frac{a}{b}\right)\left[3+4\left(\frac{a}{b}\right)^2\right]}{\left(\frac{A}{b}\right)\left[1+4\left(\frac{a}{b}\right)^2\right]+\left(\frac{a}{b}\right)\left[1+2\left(\frac{a}{b}\right)^2+2\left(\frac{A}{b}\right)^2\right]} \tag{27}$$

for the hyperboloid reflector, and

$$\frac{z}{b} = \frac{(A/b)}{(A/b) - 1} \tag{28}$$

for the paraboloid reflector.

b) Properties of Sagittal Caustics.

The curves with continuous lines in figures 24 to 32 present the variation of the (z/b) - coordinate of the one extremity of the sagittal caustics and with dashed lines the (z/b) - coordinate of the cusp point of the corresponding meridional caustic, which defines the other extremity of the respective sagittal caustic versus the aperture (r/b) of the mirror for various positions of the point light source. Figures 24 to 27 correspond to the case of an ellipsoid

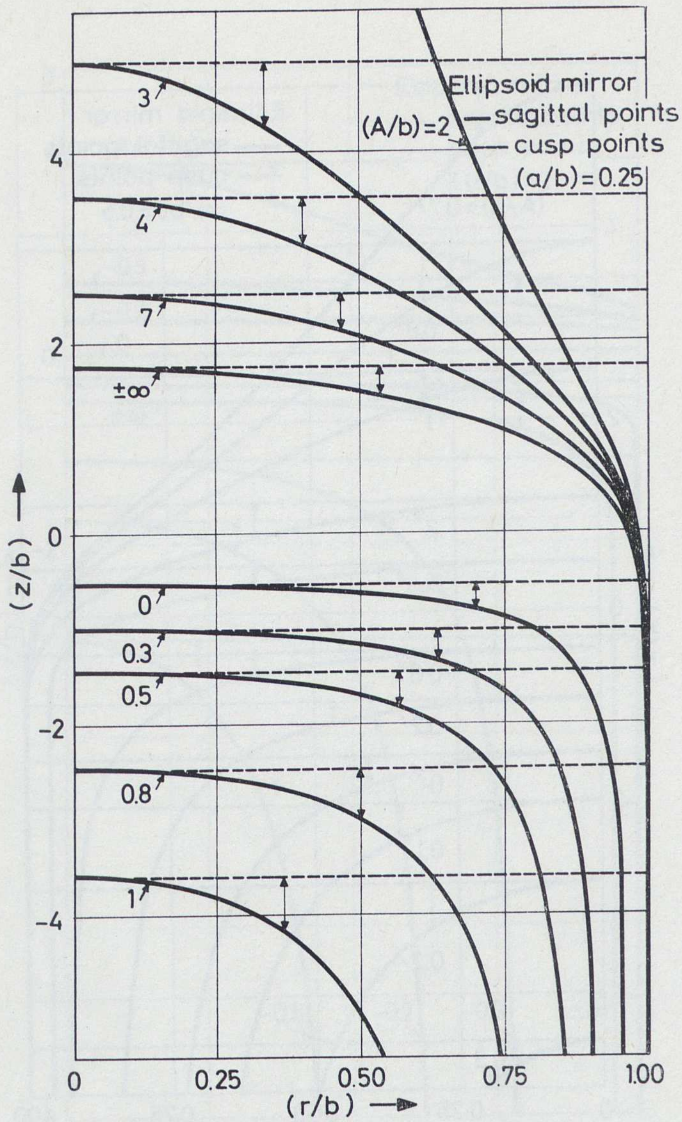


Fig. 24. Variation of the cusp point of the meridional caustics constituting also the one extremity of the sagittal caustics (dashed lines) and the other extremity of the sagittal caustics (continuous lines) versus the aperture (r/b) of an ellipsoid reflector with $(a/b) = 0.25$ for various positions (A/b) of the point-light source illuminating the reflector.

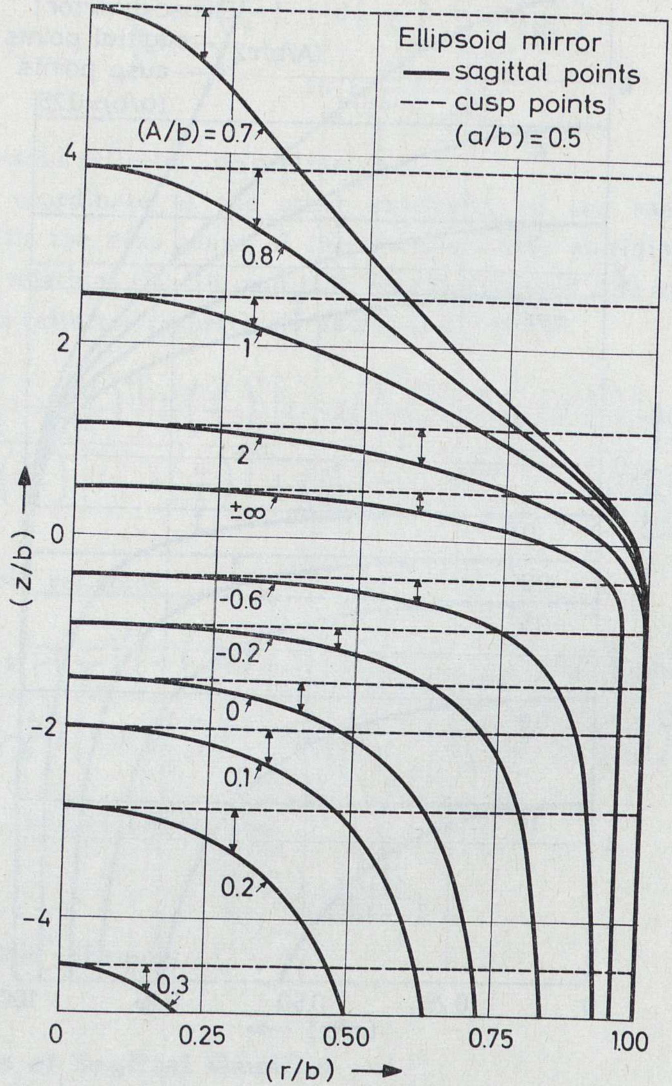


Fig. 25. As in figure 24 with $(a/b) = 0.50$.

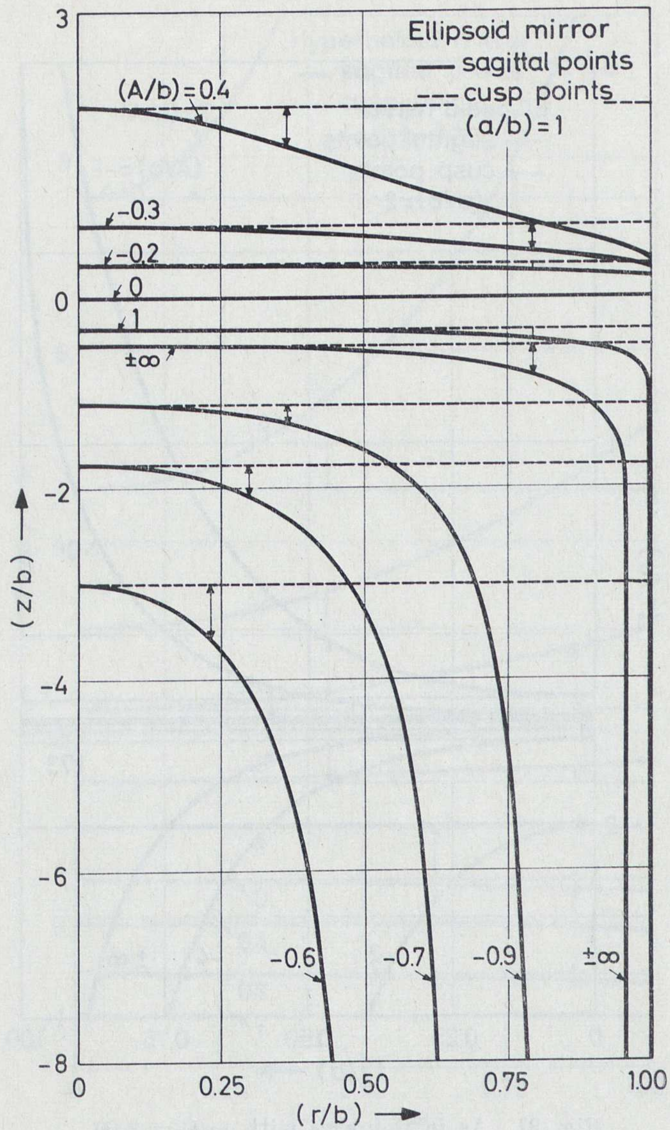


Fig. 26. As in figure 24 with $(a/b) = 1.00$.

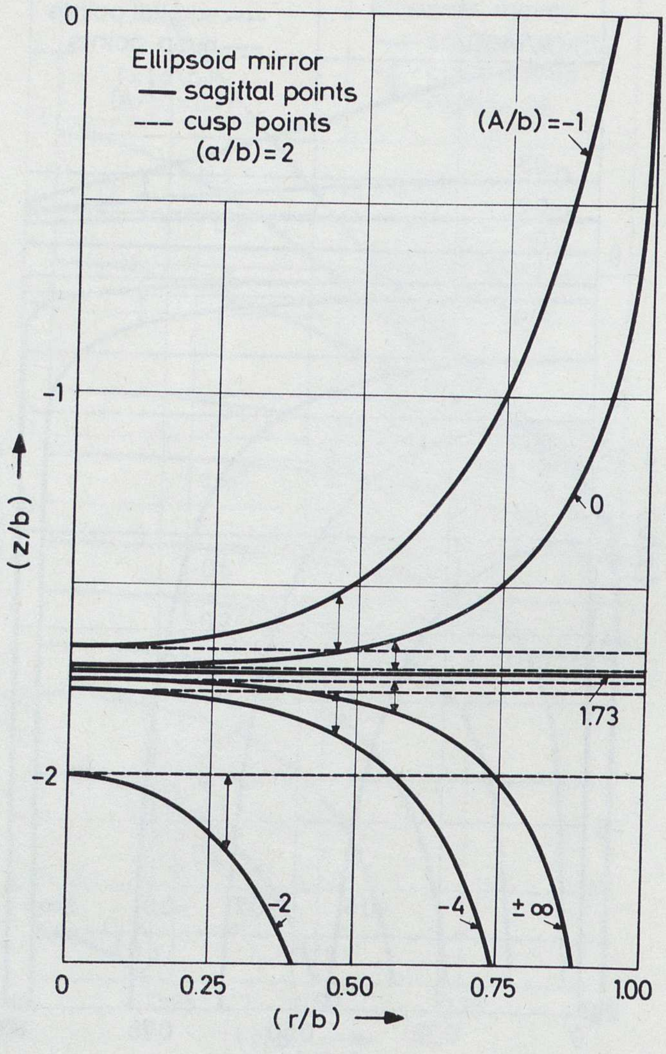


Fig. 27. As in figure 24 with $(a/b) = 2.00$.

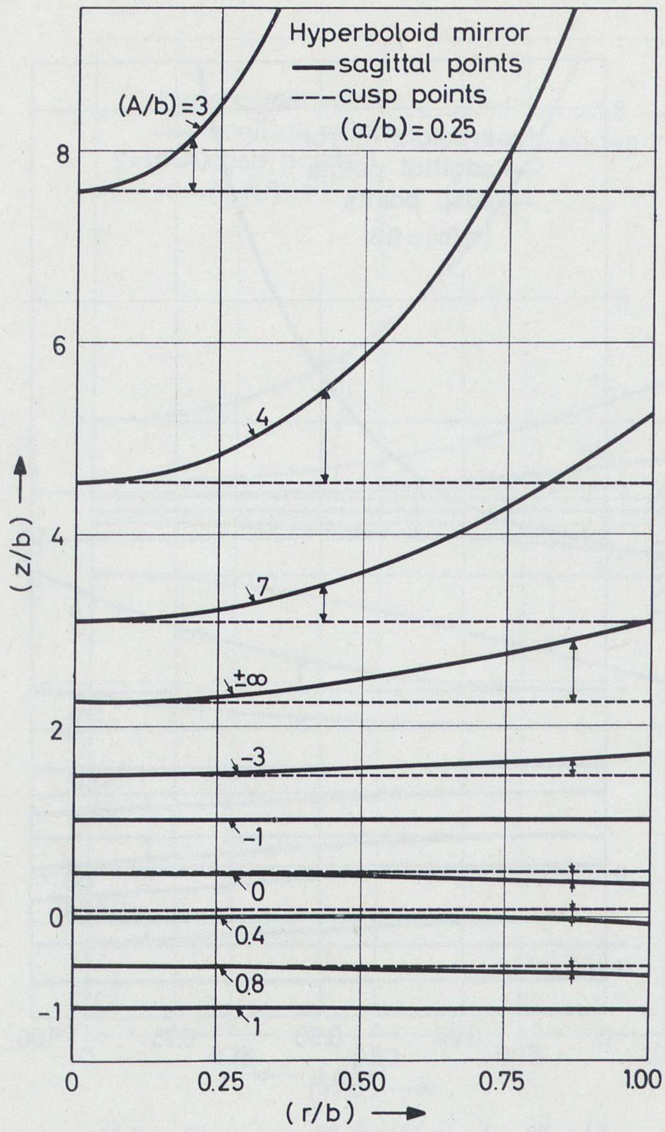


Fig. 28. As in figure 24 for a hyperboloid reflector with $(a/b) = 0.25$.

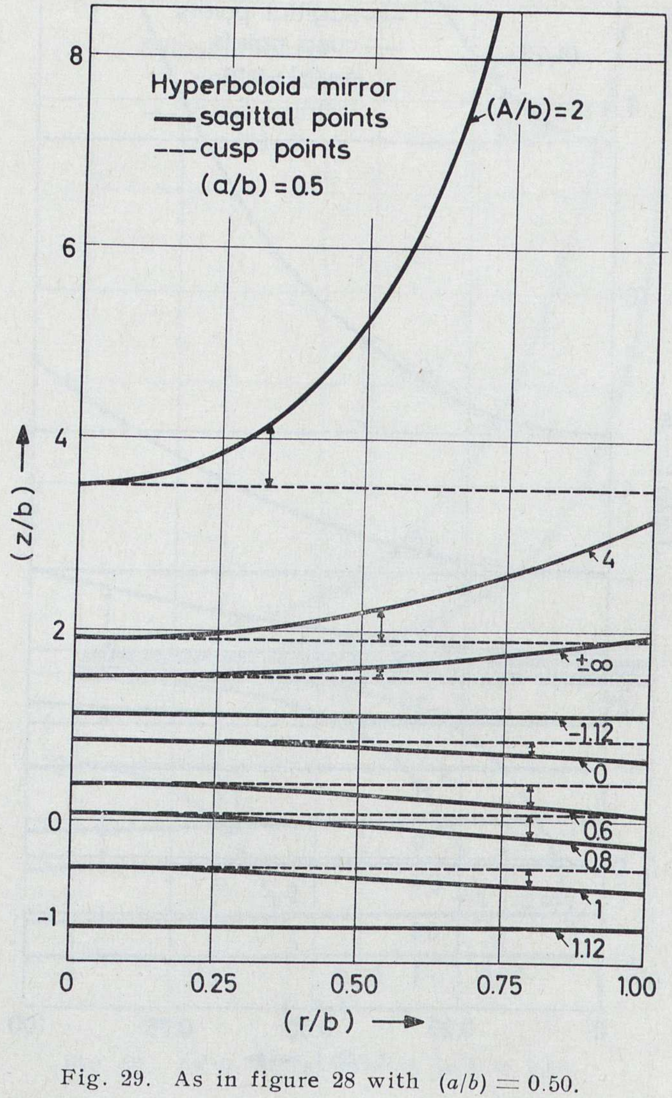


Fig. 29. As in figure 28 with $(a/b) = 0.50$.

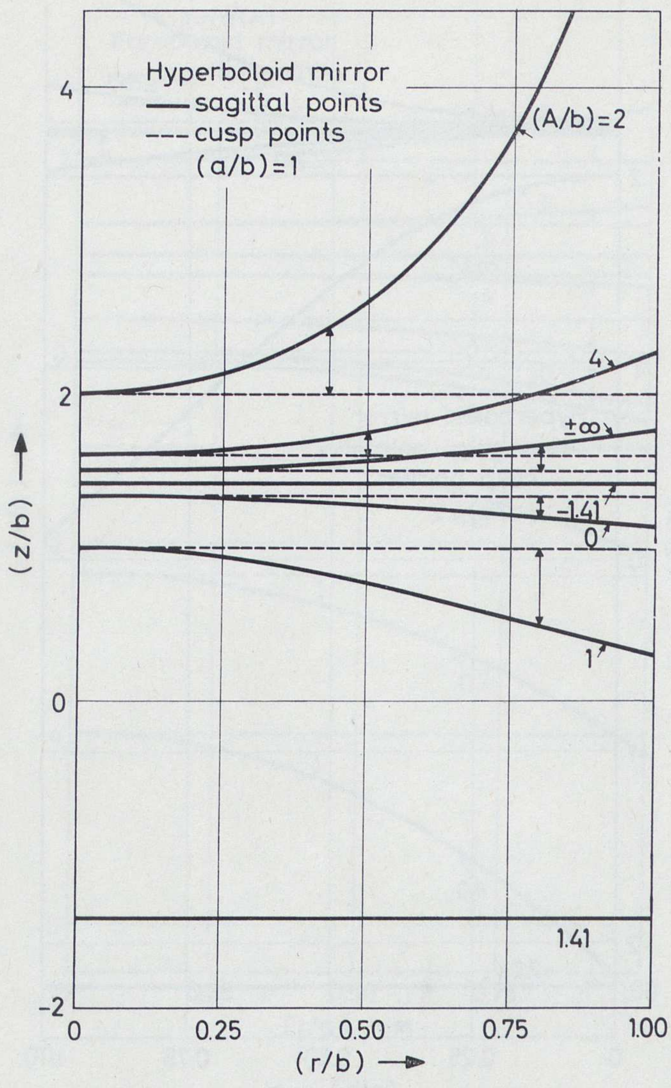


Fig. 30. As in figure 28 with $(a/b) = 1.00$.

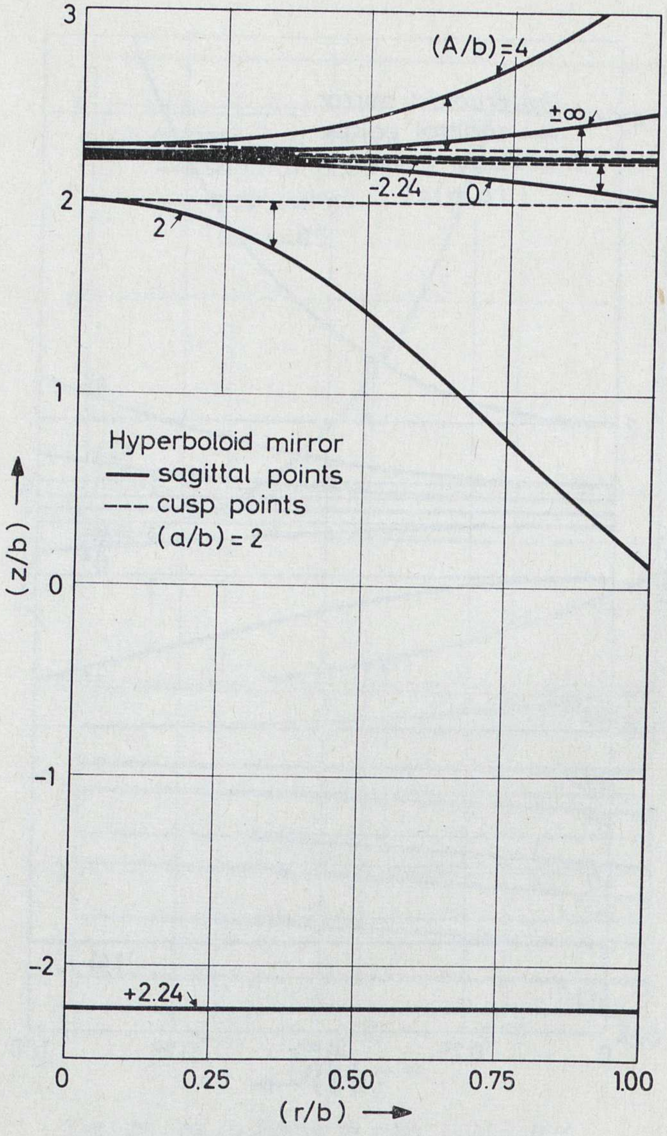


Fig. 31. As in figure 28 with $(a/b) = 2.00$.

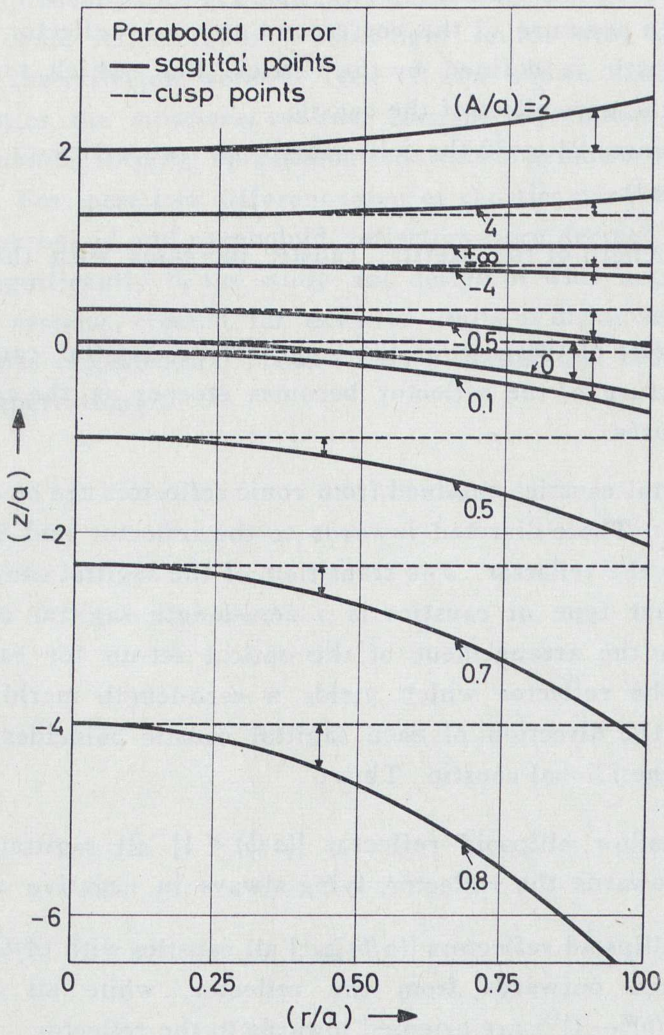


Fig. 32. As in figure 24 for a paraboloid reflector.

reflector with $(a/b) = 0.25$, 0.50, 1.00 and 2.00, figures 28 to 31 to the case of a hyperboloid reflector with $(a/b) = 0.25$, 0.50,. 1.00 and 2.00, while figure 32 to a paraboloid reflector. For each position of the point-light source and for a given aperture of the corresponding conic reflector the length of the sagittal caustic is defined by the vertical line, which touches the two curves giving the exremities of the caustic.

From figures 24 to 32 the following basic properties of these caustics can be concluded:

1. The length of the sagittal caustic increases with the aperture of the reflector.

2. The rate of increase of the length of the sagittal caustic with the respective aperture of the reflector becomes steeper as the aperture of the reflector increases.

3. Sagittal caustics obtained from conic reflectors can be classified into two categories: Those directed inwards to the reflector and those directed outwards from the reflector. The transition of the sagittal caustics from the one to the other type of caustics is a zero-length sagittal caustic, which coincides with the arrangement of the optical set-up for each particular geometry of the reflector which yields a zero-length meridional caustic. Furthermore, the direction of each sagittal caustic coincides with that of its respective meridional caustic. Thus:

i) In the shallow ellipsoid reflector $[(a/b) < 1]$ all sagittal caustics are oriented towards the reflector, lying always in negative z-coordinates.

ii) For deep ellipsoid reflectors $[(a/b) \geqq 1]$ all caustics with $|A/b| > [(a/b)^2 - 1]^{1/2}$ are oriented outwards from the reflector, while all caustics with $|A/b| < [(a/b)^2 - 1]^{1/2}$ are oriented towards to the reflector.

iii) For hyperboloid reflectors all caustics with $|A/b| > [(a/b)^2 + 1]^{1/2}$ are directed outwards from the reflector, while all caustics with $|A/b| <$ $< [(a/b)^2 + 1]^{1/2}$ are directed inwards to the reflector.

iv) For paraboloid reflectors all caustics with $(A/b) < 0$ are directed inwards to the reflector, while all caustics with $(A/b) > 0$ are directed outwards from the reflector.

CONCLUSIONS

In the present paper general laws regarding the caustics obtained by illuminating conic reflectors by a point-light source lying along the axis of symmetry of the reflector were derived. It was shown that two different types of caustics, the *meridional caustics,* forming axisymmetric surfaces, and the *sagittal caustics,* forming spikes along the axis of symmetry of the reflector, are obtained. For these two different types of caustics particular laws for the ellipsoid, hyperboloid and paraboloid reflectors were derived. All these laws contribute significantly to the study and design of wide angle all-reflective multi-mirror systems, created for definite limits of distances and which, by successive pairs of reflectors, present only a limited and reduced amount of third order aberrations.

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ΒΑΣΙΚΑΙ ΙΔΙΟΤΗΤΕΣ ΤΩΝ ΜΕΣΗΜΒΡΙΝΩΝ ΚΑΙ ΤΟΞΟΕΙΔΩΝ ΚΑΥΣΤΙΚΩΝ ΔΗΜΙΟΥΡΓΟΥΜΕΝΩΝ ΥΠΟ ΚΩΝΙΚΩΝ ΚΑΤΟΠΤΡΩΝ

Εἰς τὴν παροῦσαν ἐργασίαν μελετῶνται αἱ καυστικαὶ ἐπιφάνειαι αἱ δημιουργούμεναι κατὰ τὸν φωτισμὸν κατόπτρων κωνικῶν τομῶν ἐκ περιστροφῆς ὑπὸ σημειακῆς φωτεινῆς πηγῆς εὑρισκομένης ἐπὶ τοῦ κυρίου ἄξονος συμμετρίας τοῦ κατόπτρου. Αἱ οὕτω δημιουργούμεναι καυστικαί, αἱ ὁποῖαι ἀποτελοῦν ἐπιφανείας μεγάλης φωτεινῆς ἐντάσεως, διακρίνονται εἰς δύο κατηγορίας, τὰς μεσημβρινὰς καὶ τὰς τοξοειδεῖς. Αἱ μεσημβριναὶ καυστικαὶ ἀποτελοῦν ἀξονοσυμμετρικὰς ἐπιφανείας, αἱ ὁποῖαι ὁρίζονται ὡς περιβάλλουσαι τῶν ἀνακλωμένων ὑπὸ τοῦ κατόπτρου ἀκτίνων, ἐνῷ αἱ τοξοειδεῖς καυστικαὶ ἀποτελοῦν ἰσχυρῶς φωτιζόμενα εὐθύγραμμα τμήματα ἐπὶ τοῦ ἄξονος συμμετρίας τοῦ κατόπτρου, σχηματιζόμενα ὑπὸ τῶν τομῶν τῶν ἀνακλωμένων φωτεινῶν ἀκτίνων μετὰ τοῦ ἄξονος τούτου.

Αἱ δημιουργούμεναι μεσημβριναὶ καὶ τοξοειδεῖς καυστικαὶ μελετῶνται ἐνδελεχῶς καὶ συνάγονται βασικαὶ ἰδιότητες διὰ τὸ σχῆμα, τὴν θέσιν καὶ τὴν ἐξέλιξιν τῶν καυστικῶν τούτων ἐκ τοῦ τύπου τοῦ κωνικοῦ κατόπτρου ὡς καὶ τῆς θέσεως τῆς σημειακῆς φωτεινῆς πηγῆς ἐπὶ τοῦ ἄξονος συμμετρίας τοῦ κατόπτρου.

Οὕτω, διὰ τὰς μεσημβρινὰς καυστικὰς συνάγονται αἱ ἀκόλουθοι βασικαὶ ἰδιότητες, αἱ ὁποῖαι ἰσχύουν δι᾽ ὅλους τοὺς τύπους κωνικῶν κατόπτρων :

i) Ἡ μορφὴ τῆς καυστικῆς ἐξαρτᾶται ἐκ τῆς θέσεως τῆς φωτεινῆς πηγῆς. Διὰ θέσεις ταύτης πλησίον τοῦ κατόπτρου ἡ μορφὴ τῆς καυστικῆς εἶναι εὐαίσθητος ἐκ τῆς θέσεως τῆς πηγῆς, ἐνῷ, ὅσον ἡ φωτεινὴ πηγὴ ἀπομακρύνεται ἐκ τοῦ κατόπτρου, τόσον αἱ καυστικαὶ καθίστανται ὀλιγώτερον εὐαίσθητοι ἐκ τῆς θέσεως τῆς φωτεινῆς πηγῆς.

ii) Τὸ σχῆμα τῶν μεσημβρινῶν καυστικῶν ἐπιφανειῶν διὰ φωτεινὴν πηγὴν ἐπὶ τοῦ κυρίου ἄξονος τοῦ κατόπτρου εἶναι συμμετρικὸν ὀμβρελλοειδὲς μετὰ κέρατος εἰς τὸ κέντρον τῆς ἐπιφανείας.

iii) Τὸ μεγαλύτερον μέρος τῆς φωτεινῆς ἐνεργείας κατὰ μῆκος τῆς καυστικῆς εἶναι συγκεντρωμένον πλησίον τοῦ ἄξονος συμμετρίας τοῦ κατόπτρου περὶ τὸ κέρας τῆς ἐπιφανείας καὶ ὅσον ἀπομακρυνόμεθα τοῦ ἄξονος τούτου ἡ φωτεινὴ ἐνέργεια ἐλαττοῦται βαθμιαίως. Ἡ μόνη καυστικὴ εἰς τὴν ὁποίαν ὁ κανὼν οὗτος δὲν ἐφαρμόζεται εἶναι ἡ σχηματιζομένη ὑπὸ φωτεινῆς πηγῆς κειμένης ἐπὶ τοῦ κατόπτρου. Ἡ καυστικὴ αὕτη εἶναι σχεδὸν ὁμοιομόρφως φωτιζομένη καὶ εὑρίσκεται πολὺ πλησίον τοῦ κατόπτρου.

iv) Δι' ἕκαστον τύπον κωνικοῦ κατόπτρου ὑφίσταται ὡρισμένη περιοχὴ ἐπὶ τοῦ ἄξονος συμμετρίας τούτου διὰ τὴν ὁποίαν ὅταν ἡ φωτεινὴ πηγὴ εὑρίσκεται ἐντὸς ταύτης, αἱ ἀντίστοιχοι καυστικαὶ ἐκτείνονται μέχρι τοῦ ἀπείρου. Δι' ὅλας τὰς ἄλλας θέσεις τῆς φωτεινῆς πηγῆς αἱ δημιουργούμεναι καυστικαὶ εἶναι πεπερασμέναι.

v) Δι' ἕκαστον κωνικὸν κάτοπτρον ὑφίστανται δύο θέσεις τῆς φωτεινῆς πηγῆς ἐπὶ τοῦ ἄξονος συμμετρίας τούτου, συμμετρικαὶ ὡς πρὸς τὸ ἰσημερινὸν ἐπίπεδον τοῦ κατόπτρου, διὰ τὰς ὁποίας ἡ ἀντίστοιχος καυστικὴ ἐκφυλίζεται εἰς σημεῖον. Ὁ προσανατολισμὸς τῶν καυστικῶν ἑκατέρωθεν τῆς μηδενικῆς ταύτης καυστικῆς εἶναι διάφορος, οὕτως ὥστε αἱ κείμεναι πρὸς τὴν μίαν πλευρὰν καυστικαὶ νὰ κατευθύνωνται πρὸς τὸ κάτοπτρον, ἐνῷ αἱ ἄλλαι νὰ κατευθύνωνται ἐκτὸς τοῦ κατόπτρου. Τοιαῦται θέσεις τῆς φωτεινῆς πηγῆς εἶναι φανταστικαὶ μὲν διὰ τὰ ἀβαθῆ ἐλλειψοειδῆ κάτοπτρα, ἐνῷ συμπίπτουν μὲ τὰ ἐπ' ἄπειρον σημεῖα τοῦ ἐπιπέδου διὰ τὰ παραβολοειδῆ κάτοπτρα.

vi) Τὰ ἀκραῖα σημεῖα τῶν καυστικῶν τῶν δημιουργουμένων κατὰ τὴν κίνησιν τῆς φωτεινῆς πηγῆς ἐπὶ τοῦ ἄξονος τοῦ κατόπτρου καὶ ἀντιστοιχουσῶν εἰς διάφορα ἀνοίγματα τοῦ κατόπτρου κεῖνται ἐπὶ ἐπιφανειῶν ἐκ περιστροφῆς τῶν ὁποίων αἱ μεσημβριναὶ τομαὶ εἶναι καμπύλαι δευτέρου βαθμοῦ.

vii) Πλὴν τῶν γενικῶν τούτων ἰδιοτήτων αἱ ὁποῖαι ἰσχύουν δι' ὅλους ἀνεξαιρέτως τοὺς τύπους τῶν κωνικῶν κατόπτρων, ἐνδιαφέρουσαι ἰδιότητες συνάγονται διὰ τὰς καυστικὰς τὰς δημιουργουμένας ὑφ' ἑκάστου τύπου κατόπτρων (ἐλλειψοειδῶν, ὑπερβολοειδῶν, παραβολοειδῶν).

viii) Ὁ γεωμετρικὸς τόπος τῶν ἀκραίων σημείων τῶν καυστικῶν αἱ ὁποῖαι δημιουργοῦνται διὰ φωτισμοῦ ὅλων τῶν μορφῶν τῶν ἐλλειψοειδῶν κατόπτρων ὑπὸ σταθερᾶς φωτεινῆς πηγῆς εἶναι κωνικὴ ἐπιφάνεια ἔχουσα κορυφὴν τὸ συμμετρικὸν σημεῖον τῆς φωτεινῆς πηγῆς ὡς πρὸς τὸ κάτοπτρον καὶ διερχομένη διὰ τοῦ χείλους τοῦ κατόπτρου. Ὁ ἀντίστοιχος γεωμετρικὸς τόπος διὰ τὰ ὑπερβολοειδῆ καὶ τὰ παραβολοειδῆ κάτοπτρα ἀποτελεῖται ὑπὸ ὑπερβολοειδῶν καὶ παραβολοειδῶν ἐπιφανειῶν ἐκ περιστροφῆς.

Ἐξ ἄλλου διὰ τὰς τοξοειδεῖς καυστικὰς συνάγεται ὅτι :

i) Τὸ μῆκος τούτων αὐξάνει μετὰ τοῦ ἀνοίγματος τοῦ κατόπτρου.

ii) Ἡ ταχύτης αὐξήσεως τοῦ μήκους τῶν καυστικῶν γίνεται ἐντονωτέρα ὅσον τὸ ἄνοιγμα τοῦ κατόπτρου αὐξάνει.

iii) Τὸ ἓν σημεῖον τῶν τοξοειδῶν καυστικῶν συμπίπτει μὲ τὴν κορυφὴν τοῦ κέρατος τῆς ἀντιστοίχου μεσημβρινῆς καυστικῆς κειμένης ἐπὶ τοῦ ἄξονος συμμετρίας τοῦ κατόπτρου. Τὸ δὲ ἕτερον σημεῖον τούτων κεῖται εἴτε πρὸς τὴν πλευρὰν εἴτε πρὸς τὴν ἀντίθετον κατεύθυνσιν τῆς πλευρᾶς τοῦ κατόπτρου, ἀκολουθοῦν τὴν ἀντίστοιχον θέσιν τῆς μεσημβρινῆς καυστικῆς. Αἱ θέσεις μηδενισμοῦ τῆς τοξοειδοῦς καυστικῆς συμπίπτουν μὲ τὰς θέσεις μηδενισμοῦ τῆς ἀντιστοίχου μεσημβρινῆς καυστικῆς.

Κύριε Πρόεδρε,

Τελειώνοντας, ἐπιτρέψατέ μου νὰ ἀναμνησθῶ τοῦ πρώτου Ἕλληνος συγχρόνου ἐπιστήμονος, τοῦ Φραγκίσκου Μαυρόλυκου.

Ὁ Φραγκίσκος Μαυρόλυκος ἐγεννήθη εἰς τὴν Μεσσήνην τῆς Σικελίας τὸ ἔτος 1494 καὶ ἀπέθανε ἐκεῖ τὸ ἔτος 1575 ἢ 1577. Ἦτο ὁ υἱὸς τοῦ Ἀντωνίου Μαυρόλυκου, εὐπατρίδου τῆς Κωνσταντινουπόλεως, ὁ ὁποῖος ἐγκατέλειψε τὴν Βασιλεύουσαν κατὰ τὴν πτῶσιν της εἰς τοὺς Τούρκους, ζητήσας ἄσυλον εἰς τὴν Σικελίαν.

Περὶ τῆς οἰκογενείας τοῦ Μαυρόλυκου, ὡς γνωρίζετε, συνέγραψε σειρὰν ὅλην μυθιστορημάτων ὁ συγγραφεὺς Θανάσης Πετσάλης - Διομήδης, μεταξὺ τῶν ὁποίων ἐξέχουσαν θέσιν ἔχει τὸ τρίτομον χρονικὸν τῆς Τουρκοκρατίας μὲ τὸν τίτλον «Μαυρόλυκοι».

Αἱ ἐξαιρετικαὶ διὰ τὴν ἐποχὴν ἐκείνην γνώσεις τοῦ ἐπιστήμονος τούτου καὶ ἡ μεγάλη φήμη του τοῦ ἐπεδαψίλευσαν τὴν εὔνοιαν καὶ μεγάλας τιμὰς τῶν ἰσχυρῶν ἀνδρῶν τῆς ἐποχῆς του. Οὕτω, ὁ αὐτοκράτωρ Κάρολος ὁ πέμπτος τὸν ἐπαρασημοφόρησε μετὰ τὴν ἐπιστροφήν του ἀπὸ τὴν ἐκστρατείαν εἰς τὴν Ἀφρικήν. Τὸ ἀξίωμα τοῦ ἡγουμένου εἰς τὴν μονὴν Santa Maria de Partu εἰς τὸ Castelnuovo τὸ ὤφειλεν εἰς τὰς φροντίδας τοῦ διασήμου Ἀλεξάνδρου Φαρνέζε. Προεῖπε τὴν νίκην τοῦ Don Juan τῆς Αὐστρίας ἐπὶ τῶν Τούρκων εἰς τὴν Ναύπακτον τῷ 1571 καὶ λόγῳ αὐτῆς του τῆς προφητείας ἔχαιρε μεγάλου σεβασμοῦ.

Συνέγραψε πολλὰ βιβλία, μέγα μέρος τῶν ὁποίων ἀφοροῦν εἰς τὴν ἑρμηνείαν τῶν μαθηματικῶν τῶν ἀρχαίων Ἑλλήνων καὶ τοῦ Ἀριστοτέλους. Ἀνεκάλυψε σειρὰν νόμων τῆς φυσικῆς ἀσχοληθεὶς ἰδιαιτέρως μὲ τὴν ὀπτικήν. Εἰς τὸ βιβλίον του: «Photismi de Lumine et umbra ad prospectivam radiorum incidentium facientes», τὸ ὁποῖον μετὰ τὸν θάνατόν του ἐδημοσιεύθη εἰς τὴν Βενετίαν, διακρίνεται ὁ μέγας ἐπιστήμων, διότι κατορθώνει εἰς ὀλίγας μόνον σελίδας νὰ διατυπώνῃ σπουδαίους νόμους τῆς Φυσικῆς κατὰ τρόπον αὐστηρόν, περιεκτικόν, ἄνευ ἀπεραντολογιῶν καὶ μὲ πλήρη σαφήνειαν, ἐνθυμίζων σοφὸν τοῦ 20οῦ αἰῶνος μᾶλλον, παρὰ ἐργάτην τῆς ἐπιστήμης εἰς τὸ λυκαυγές της.

Ἄξιος ἀναφορᾶς τυγχάνει ὁ τρόπος μὲ τὸν ὁποῖον ἀναπτύσσει διὰ τῶν ἀπολύτως ἀπαραιτήτων συλλογισμῶν καὶ μετὰ περισσῆς σαφηνείας τοὺς ἀκριβεῖς νόμους σχηματισμοῦ τῆς σκιᾶς. Μὲ τὸ αὐτὸ θέμα ἀσχολεῖται πολλὰ ἔτη ἀργότερον ὁ μέγας Γερμανὸς σοφὸς Ἰωάννης Kepler (1571 - 1631). Διὰ τὴν σύγκρισιν τῶν σκέψεων τῶν δύο σοφῶν ἀναφέρομεν ἐπὶ λέξει τὰς κρίσεις Γερμανοῦ ἱστορικοῦ τοῦ παρελθόντος αἰῶνος. Οὗτος λέγει : «Ἐπίσης ὁ Κέπλερος ἠσχολήθη μὲ τὸ αὐτὸ θέμα χωρὶς νὰ γνωρίζῃ τὰ γραφόμενα τοῦ Μαυρόλυκου. Οὗτος ἔθεσε μὲ ἄχρηστον πολυλογίαν, πλῆθος γεωμετρικῶν προτάσεων καὶ δὲν εὗρε ἄλλο ἀποτέλεσμα, εἰμὴ ἐκεῖνο τὸ ὁποῖον εἶχεν εὕρει ὁ Μαυρόλυκος μὲ τόσον ἁπλὸ τρόπο».

Μεταξὺ τῶν μεγάλων ἀνακαλύψεων τοῦ Μαυρόλυκου εἶναι ἡ ἀκριβὴς ἑρμηνεία τοῦ σχηματισμοῦ τοῦ οὐρανίου τόξου, πολὺ πρὸ ἀπὸ τὴν ἀντίστοιχον συμβολὴν τοῦ Νεύτωνος (1642 - 1726) καὶ ἡ διατύπωσις τοῦ νόμου τῶν καυστικῶν. Ὁ Μαυρόλυκος εἶναι ὁ πρῶτος ὀπτικὸς ἐπιστήμων ποὺ διετύπωσε τὸν νόμον τῆς ὑπάρξεως διπλῶν καυστικῶν, τῶν μεσημβρινῶν καὶ τῶν τοξοειδῶν, ὅταν φωτεινὴ δέσμη προσπίπτει εἴτε ἐπὶ κατόπτρου εἴτε ἐπὶ φακοῦ. Ἡ ἀκρίβεια καὶ ἡ σαφήνεια τῆς διατυπώσεως καὶ τοῦ νόμου αὐτοῦ εἶναι καταπλήσσουσα.

Ἡ ἡμετέρα ἐργασία, ἥτις παρουσιάσθη σήμερον εἰς τὴν Ἀκαδημίαν Ἀθηνῶν βασίζεται οὕτως εἰπεῖν εἰς τὰς σκέψεις τοῦ πρώτου τούτου μεγάλου συγχρόνου Ἕλληνος ἐπιστήμονος. Ἡ ἀναφορὰ αὐτὴ σκοπὸν ἔχει νὰ τιμήσῃ τὸν Ἕλληνα σοφὸν ἐπὶ τῇ ἐπετείῳ τῶν 400 χρόνων ἀπὸ τοῦ θανάτου του.