ΜΑΘΗΜΑΤΙΚΑ.— **Closed-form solutions for the general three-dimensional problem of the theory of elasticity,** *by P. S. Theocaris**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Π. Θεοχάρη.

A B S T R A C T

A large number of three-dimensional boundary-value problems are investigated by considering the numerical solutions of certain systems of singular integral equations. It is of particular importance to know the families of the abovereferred equations, which possess a closed-form solution.

Several tables are constructed according to the problem, which is concerned (i. e. static, dynamic, thermoelastic boundary-value problems of the classical elasticity). Each one of these Tables contains the analytical solutions of a great number of boundary-value problems of the $I^{\pm}$, $II^{\pm}$, $III^{\pm}$ or $IV^{\pm}$ type, defined over canonical regions (i. e. half-space, a layer, a sphere, a round cylinder, an ellipsoid, a paraboloid, a hyperboloid etc.).

On the other hand, along with the analytical solutions of the general 3-D inclusion problem we can investigate certain 3-D crack problems of a special geometry (spherical cracks, cylindrical cracks etc.).

An extended bibliography, which covers the field of the analytical solutions of certain categories of 3-D boundary-value problems of elasticity and thermoelasticity, is included at the end of the paper.

1. INTRODUCTION

Several investigations in the field of three-dimensional theory of elasticity started in the last fifteen years with the works of a number of scientists concerning the branch of Mechanics of a solid deformable body: Aleksandrov and Solovév [**1**]; Grinchenko [**2**]; Kupradze et al. [**3**]; Guz' and Golovchan [**4**]; Kosmodamianskii and Shaldyrvan [**5**]; Lur'e [**6**]; Podil'chuk [**7**]; Ulitko [**8**]; and others. In all important books of the theory of elasticity the methods of solution of three-dimensional boundary-value problems are restricted to bodies of special shape (a half-space, a sphere, some cases of axially symmetrical bodies etc.). Furthermore, a great attention has been given to static problems, less

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attention to oscillation problems, and still less to problems of general dynamics. Such a situation might be well expected because of the fact that, during the entire preceding period, the historical background of the theory of elasticity was inadequate for developing a rigorous and sufficiently complete theory of 3 - D boundary-value problems. The situation is currently changing. The theory of three - dimensional problems may now be worked out by a variety of methods. Among them we shall mention the following possibilities:

i) The modern theory of generalized solutions of differential equations (the method of Hilbert spaces, variational methods, etc.) which is characterized by a great generality, involving the case of variable coefficients and boundary manifolds of the general type [**9**, **10**].

ii) The method of singular integrals and integral equations, which allows us to investigate in detail cases of particular interest for the theory and application, retaining the efficiency of the methods of the classical mechanics of continua. The works of Kupradze et al. [**3**], Vaindiner and Moskvitin [**11**], Parton and Perlin [**12**] are devoted to the construction of a system of singular integral equations, which solves the 3 - D boundary-value problems of the theory of elasticity.

iii) The Fourier-transform method which yields analytically exact solutions of the three-dimensional problems of the theory of elasticity for canonical regions (see Table III) (half-space, a layer, a sphere, a round cylinder, an ellipsoid, a paraboloid, a hyperboloid etc.). This method was preferentially applied to isotropic linearly-elastic media. A large amount of work, based on this approach, has presented by the Soviet scientists: Abramyan [**13**], Galerkin [**14**], Galin [**15**], Gromov [**16**], Uflyand [**17**], Savin [**18**], Podil'chuk [**19**], Rvachev [**20**], Ulitko [**21**], Shapiro [**22**], Guz' [**23**], Lur'e [**24**], Obolashvili [**25**], Solkyanik - Krassa [**26**] and others, who have investigated many 3 - D static, dynamic and thermoelastic boundary-value problems of the theory of elasticity in several curvilinear orthogonal systems of coordinates, allowing the separation of variables in the 3 - D Laplace equation (problem of statics) or the Helmholz equation (problem of dynamics).

iv) The theory of the generalized analytic functions (p-analytic and p, q-analytic) was successfully applied to the solution of axisymmetrical 3-D problems of the theory of elasticity (Polozhii [**27**]). A similar method was developed by Aleksandrov [29] for the investigation of certain axisymmetrical or non-axisymmetrical boundary-value problems of the theory of elasticity and thermoelasticity of isotropic and transversally isotropic bodies. The solutions of these problems are expressed in the form of linear operators of analytic functions of a complex variable (Table III) (see also the work by Vol'pert and Solov'ev on the same subject [28]).

An extensive analysis of individual efforts for the development of the above-referred methods was presented in the review-papers of Abramyan and Aleksandrov [13], Vorovich and Prokopov [30], Guz' and co-workers [31], Kil'chevskii [32], Miyamoto [33], Neiber and Hahn [34], Prokopov [35], Rvachev and Protsenko [36], Sternberg [37], Guz' [38], Ustinov and Shlevev [39], Theocaris and Kazantzakis [40], and others.

v) A number of theoretical methods, related to a certain degree with the methods of perturbations, expansions with respect to a parameter, successive approximations, etc. have been included in the monographs of Vorovich, et al. [41], Van Dyke [42], A. and O. Guz' [43 - 46], Ivlev and Ershov [47], Il'yushin [48], Kauderer [49], Cowle [50], Kantorovich and Krylov [51], Lomakin [52], Savin [53], Naiffe [54], Khusu et al. [55], Tsurpal [56], and others.

On the other hand, along with analytic solutions, various numerical techniques (finite differences, finite elements, variational differences, discrete orthogonalization and others) have been developed to become efficient tools in solving axisymmetrical problems of the theory of elasticity, particularly during the last decade. On the other hand, it should be noticed that for the case of 3-D regions with a complex surface geometry (especially when the form of the surface changes rapidly) the effectiveness of the numerical methods is considerably lowered. The difficulties arising in this case are systematically examined dy Marchuk [57].

In another wording, in the present work we try to give a complete spectrum of the closed-form analytical solutions of the systems of the singular integral equations defined over Lyapunov surfaces of special shape (i. e. spherical surface, ellipsoid of revolution, cylindrical, semi-infinite nonround cylinder, whose direction is an epitrochoid, etc.) [1] (see Tables I to III).

It was worth mentioning that ever since its introduction by Laplace in the eighteenth century, the integral equation has lagged behind the differential equation as a tool of the applied mathematician for studying physical phenomena. In many areas of engineering and physical sciences, however, a formulation of mathematical problems in terms of integral equations is more direct and more easily visualized than the corresponding formulation in terms of differential equations.

For the treatment of boundary-value problems, for instance, the solutions were mainly based upon differential equations which, combined together with the appropriate boundary conditions, yielded the solution of the particular physical problem. However, in many cases these boundary-value problems may be represented compactly in the form of integral equations, which include the boundary conditions.

But these are not the only reasons for studying the closed-form solvability of the aforementioned integral equations. There are many situations which depend upon hereditary influences, i. e. where the future of the system depends upon former states and, therefore, cannot be represented in terms of differential equations.

This case appears in the study of diffusion and transport phenomena (see Morse and Feshbach [**58**]) and in problems of growth (see Nemish [**59**]). These situations usually lead to integral equations, or more generally, to integro-differential equations.

From the above considerations one can see that integral equations should be playing a greater role in solving physical problems than they presently do. The cause of this disparity is mainly due to the fact that mathematicians, dealing with integral equations, have been primarily interested in existence proofs, rather than in practical solutions. Consequently, general methods of solving these equations have not been developed except in the simplest linear cases.

Much of nature's physical processes, however, are nonlinear in character. The contemporary theories of mechanics, elasticity and hydrodynamics, for example, are all areas abundant of such systems, but attempts to linearize such systems have been successful only in the more restrictive situations. While great advances are being made today in the design and development of high-speed computing devices, in particular, the electronic digital computers, the development of general computer oriented methods for solving non-linear problems is still in its rudimentary stages.

2. THREE-DIMENSIONAL BOUNDARY-VALUE PROBLEMS FOR ORTHOGONAL REGIONS

For convenience in the subsequent exposition the Lyapunov surface S is defined in such a manner that for any unit vector $\mathbf{n}$:

$$\mathbf{n}(x, y, z) = n_1(x, y, z)\,\mathbf{i} + n_2(x, y, z)\,\mathbf{j} + n_3(x, y, z)\,\mathbf{k}, \tag{1}$$

with

$$M(x, y, z) \in S,$$

the contitions of orthogonality are satisfied, (i. e. $\mathbf{n}_i \mathbf{n}_j \equiv 0 \forall (j \neq i)$ and $\mathbf{n}_i \mathbf{n}_i \equiv 1$). Surface S will be arbitrarily called orthogonal. Consequently, the boundary surfaces of the bodies under consideration coincide with the coordinate surfaces of the corresponding curvilinear system of coordinates.

Let D^+ be a domain bounded by such a surface S. The singular integral equation, which solves the $(I^{\pm})$ boundary-value problem has the form **[40]**:

$$2u(\mathbf{X}) = \int_S K(\mathbf{X} - \mathbf{Y})\, q(\mathbf{Y})\, d_y S - Q(\mathbf{X}), \tag{2}$$

$$\forall \mathbf{X} \in D^+$$

$$\int_S K(\mathbf{X} - \mathbf{Y})\, q(\mathbf{Y})\, d_y S = Q(\mathbf{X}), \tag{3}$$

$$\forall \mathbf{X} \in D^-$$

where D^- is the complement of $D^+ \cup S$ to E_3 (Three-Dimensional Euclidean space), and:

$$q(\mathbf{Y}) = [\, T(\partial_y, n)\, u(\mathbf{Y})\,]^+. \tag{4}$$

Moreover,

$$Q(\mathbf{X}) = \int_S [\, T(\partial_y, n)\, K(\mathbf{X} - \mathbf{Y})\,]\, f(\mathbf{Y})\, d_y S - \int_{D^+} K(\mathbf{X} - \mathbf{Y})\, H(\mathbf{X})\, dX, \tag{5}$$

where $K(\mathbf{X}-\mathbf{Y})$ denotes Kelvin's tensor [**40**], $f(\mathbf{Y})$ the known density of the external forces normally distributed on S, $H(\mathbf{X})$ the continuous function of the total forces defined for every $\mathbf{X} \in D^+$, $T(\partial_y, n)$ the stress tensor defined in [**40**], $\mathbf{X}$ the point of application of the force, which equals to twice the unity, $\mathbf{X} \in D^+$ and $\mathbf{Y} \in S$ is the field point.

Let S^* be an arbitrary smooth surface enclosing S and having no common point with it. Let us consider now a countable set of points $\{y^k\}_{k=1}^{\infty}$ belonging to the surface S^* densely distributed everywhere. According to the developments of Kupradze, [**60** to **70**], Basheleishvili, [**71** to **80**] and Burchuladze, [**81** to **90**] the following cubature formulas are valid:

$$u^n(\mathbf{X}) = \frac{1}{2} \int_S K(\mathbf{X}-\mathbf{Y}) \sum_{k=1}^{n} S_k P^{(k)}(\mathbf{Y})\, d_y S - \frac{1}{2} Q(\mathbf{X}), \tag{6}$$

where n is an arbitrary natural number,

$$S_m = \int_S \left(\sum_{n=1}^{m} a_{mn} \varphi^{(n)}(\mathbf{Y}) \right) q(\mathbf{Y})\, dS = \sum_{L=1}^{s} A^{mi} Q(\mathbf{X}^i), \tag{7}$$

with $m = 1, 2, \ldots$ and $s = [(m+2)/3]$. These relations allow the evaluation of the Fourier coefficients.

Let us consider now the vector:

$$2u_N(\mathbf{X}) \equiv \int_S K(\mathbf{X}-\mathbf{Y}) \sum_{n=1}^{m} S_n \varphi^n(\mathbf{Y})\, dS - H(\mathbf{X}), \tag{8}$$

where N is an arbitrary natural number. It can be readily seen that this vector represents an approximation of the exact solution in the sense of uniform convergence as $N \to \infty$ in an arbitrary closed domain, lying in D^+. That is because of the fact:

$$\lim_{N \to \infty} | u(\mathbf{X}) - u_N(\mathbf{X}) | \equiv 0. \tag{9}$$

In considering the boundary-value problems $(II^{\pm})$ $(III^{\pm})$ and $(IV^{\pm})$ we construct the components of Table I.

On the other hand, in the following we deal with domains bounded by several surfaces and we form problems, which are called mixed boundary-value problems. As an example let us suppose that it is required to find in the domain D^+ the regular vector $u(\mathbf{X})$, which solves the system:

$$A(\partial_x)\, u(\mathbf{X}) + H(\mathbf{X}) = 0\,, \quad \mathbf{X} \in D^+, \tag{10}$$

and satisfies the boundary conditions:

$$\left.\begin{array}{l} u^+(\mathbf{Z}) = f^{(k)}(\mathbf{Z})\,, \quad \mathbf{Z} \in S_k\,, \quad k = 0, 1, \ldots, r, \\ \left[\tau(\partial_z, n(\mathbf{Z}))\right]^+ + \sigma(\mathbf{Z})\, u^+(\mathbf{Z}) = f^{(k)}(\mathbf{Z})\,, \quad z \in S_k\,, \\ k = r+1, \ldots, m \quad \text{with} \quad 0 \leqq r < m\,, \end{array}\right\} \tag{11}$$

where $\sigma(z)$ is a non-negative definite matrix of class $C^{0,a}(S)$.

Let us now consider a domain D_k^* lying strictly inside D_k, $k = 1, \ldots, m$, S_k^* denotes the boundary of D_k, $r(S, S^*) > 0$, while $\{\mathbf{X}^k\}_{k=1}^{\infty}$ is a countable set of points, dense everywhere on S^*. We introduce the matrix:

$$Q(\mathbf{X} - \mathbf{Y}) = \{Q^{(1)}, Q^{(2)}, Q^{(3)}\}, \tag{12}$$

defined as:

$$Q(\mathbf{X} - \mathbf{Y}) = \begin{cases} K(\mathbf{X} - \mathbf{Y})\,, \quad \mathbf{Y} \in \bigcup_{k=0}^{r} S_k\,, \quad \mathbf{X} \in E_3\,, \\ [\tau(\partial_y, n) + \sigma(\mathbf{Y})]\, K(\mathbf{X} - \mathbf{Y}), \quad \mathbf{Y} \in \bigcup_{k=r+1}^{m} S_k\,, \quad \mathbf{X} \in E_3\,. \end{cases} \tag{13}$$

Then, the set of vectors $\left[Q^{(i)}(\mathbf{X}^k - \mathbf{Y})\right]_{k=1}^{\infty}$, $i = 1, 2, 3$, $\mathbf{Y} \in S = \bigcup_{k=0}^{m} S_k$, is linearly independent and complete in the space $L_2(S)$. Therefore, by introducing the notation:

$$\psi^{(k)}(\mathbf{Y}) = Q^{(l_k)}(\mathbf{X}^{[(k+2)/3]} - \mathbf{Y})\,, \quad \mathbf{Y} \in S\,, \quad k = 1, 2, 3, \ldots \tag{14}$$

where $l_k = k - 3\,[(k-1)/3]$ it is not difficult to see (Nemish et al. [**91**]) that the orthonormalization function of the vector $\psi(\mathbf{Y})$ on S has the form:

$$\varphi^{(k)}(\mathbf{Y}) = \begin{cases} \sum_{j=1}^{k} f_{k,j}\, K^{(l_j)}(\mathbf{X}^{[(j+2)/3]} - \mathbf{Y})\,, \quad \mathbf{Y} \in \bigcup_{k=0}^{r} S_k\,, \\ \sum_{j=1}^{k} f_{k,j}\, [\tau(\partial_y, n) + \sigma(\mathbf{Y})]\, K^{(l_j)}(\mathbf{X}^{[(j+2)-3]} - \mathbf{Y})\,, \quad \mathbf{Y} \in \bigcup_{k=r+1}^{m} S_k\,, \end{cases} \tag{15}$$

where $f_{k,j}$ are the coefficients of orthonormalization of the functions ψ on S. We introduce the notation [**92, 93**]:

$$p(\mathbf{X}) \equiv u(\mathbf{X}) - \frac{1}{2}\int_{D^+} K(\mathbf{X}-\mathbf{Y})\, H(\mathbf{Y})\, d\mathbf{Y}, \tag{16}$$

where $p(\mathbf{X})$ is the solution of the problem:

$$A(\partial_x)\, \upsilon(\mathbf{X}) = 0, \qquad \mathbf{X} \in D^+, \tag{17}$$

$$\overset{+}{p}(\mathbf{Z}) \equiv \Omega^{(k)}(\mathbf{Z}), \tag{18}$$

and finally we deriwe:

$$\{[T(\partial_z, n(\mathbf{Z})) + \sigma(\mathbf{Z})]\, q(\mathbf{Z})\}^+ = \Omega^{(k)}(\mathbf{Z}). \tag{19}$$

Then, following the developments of Podil'chuk [**92**] it can be concluded that the solution vector of the mixed boundary-value problem, which is stated by Eqs. (10) and (11), is described by the relations:

$$u(\mathbf{X}) = \lim_{N\to\infty} u^{(N)}(\mathbf{X}), \qquad \mathbf{X} \in D^+, \tag{20}$$

where:

$$u^{(N)}(\mathbf{X}) = \sum_{k=1}^{N} \sum_{j=1}^{k} \Omega_k f_{k,j} K^{(l_j)} (\mathbf{X}^{[(j+2)/3]} - \mathbf{X}) + \frac{1}{2}\int_{D^+} K(\mathbf{X}-\mathbf{Y})\, H(\mathbf{Y})\, d\mathbf{Y} \tag{21}$$

holds uniformly.

Furthermore, the methods for obtaining approximate solutions for the mixed boundary-value problem of Eq. (11) may be extended to boundary-contact problems in inhomogeneous media. Then, the vector equation, which solves the corresponding boundary-value problem, may be written in the following form:

$$\oint_S H(\mathbf{X}^k_{(1)}, \mathbf{X}^k_{(2)}; \mathbf{Y})\, u(\mathbf{Y})\, d_y S = \Phi(\mathbf{X}^k_{(1)}), \qquad k = 1, 2, \ldots \tag{22}$$

where:

$$H(\mathbf{X}^k_{(1)}, \mathbf{X}^k_{(2)}; \mathbf{Y}) \equiv \begin{Vmatrix} [T^{(1)}(\partial_{y,n}) K^{(1)}(\mathbf{X}^k_{(2)} - \mathbf{Y})], & -K^{(1)}(\mathbf{X}^k_{(2)} - \mathbf{Y}) \\ [T^{(2)}(\partial_{y,n}) K^{(2)}(\mathbf{X}^k_{(1)} - \mathbf{Y})], & -K^{(2)}(\mathbf{X}^k_{(1)} - \mathbf{Y}) \end{Vmatrix} \tag{23}$$

and the vectors:

$$u(\mathbf{Y}) \equiv (u^+, (T^{(1)} u)^+);$$

$$\Phi \equiv (O; F); \text{ F denotes a given vector,} \tag{24}$$

while the finite domains bounded by S_1, S_2 are denoted by D_1 and D_2 respectivelly and it was further assumed that the countable sets of points $\{\mathbf{X}^k_{(1)}\}^\infty_{k=1}$, $\{\mathbf{X}^k_{(2)}\}^\infty_{k=1}$ are dense everywhere on the surfaces S_1, S_2. In order to obtain a closed-form solution of the integral equation (22) the vector $u(\mathbf{Y})$ may be expanded in Fourier series:

$$u(\mathbf{Y}) \approx \sum_{m=1}^{\infty} A_m u^{(m)}(\mathbf{Y}), \tag{25}$$

where:

$$A_m = \int_S u^{(m)}(\mathbf{Y})\, u(\mathbf{Y})\, dS, \tag{26}$$

are the Fourier coefficients, which can be calculated as:

$$A_m = \sum_{i=1}^{\infty} A^{mi}\, \Phi(\mathbf{X}^i_{(1)}), \qquad m = 1, 2, \ldots \tag{27}$$

The quantities A^{mt} are determined by:

$$A^{mt} = \begin{cases} l_{m,m}\, i^1, & \text{for } t = \dfrac{m+5}{6}, \\[2mm] l_{m,m-1}\, i^1 + l_{m,m}\, i^2, & \text{for } t = \dfrac{m+4}{6}, \\[2mm] l_{m,m-2}\, i^1 + l_{m,m-1}\, i^2 + l_{m,m}\, i^3, & \text{for } t = \dfrac{m+3}{6}, \\[2mm] l_{m,m-3}\, i^1 + l_{m,m-2}\, i^2 + l_{m,m-1}\, i^3 + l_{m,m}\, i^4, & \text{for } t = \dfrac{m+2}{6}, \\[2mm] l_{m,m-4}\, i^1 + l_{m,m-3}\, i^2 + l_{m,m-2}\, i^3 + l_{m,m-1}\, i^4 + l_{m,m}\, i^5, & t = \dfrac{m+1}{6}, \\[2mm] l_{m,m-5}\, i^1 + l_{m,m-4}\, i^2 + l_{m,m-3}\, i^3 + l_{m,m-2}\, i^4 + l_{m,m-1}\, i^5 + l_{m,m}\, i^6, & t = \dfrac{m}{6}, \end{cases} \tag{28}$$

where $l_{t,k}$ are the orthonormalization coefficients and i^k, $k = 1, \ldots, 6$, are the coordinate unit-vectors in a six-dimensional space. Obviously, for the exact solution of the problem we have:

$$u(\mathbf{X}) = \lim_{N\to\infty} u^{(N)}(\mathbf{X}), \tag{29}$$

where:

$$u^N(\mathbf{X}) = \begin{cases} -\left(U_1^{(N)}(\mathbf{X}),\ U_2^{(N)}(\mathbf{X}),\ U_3^{(N)}(\mathbf{X})\right), & \mathbf{X} \in D^+, \\ \left(U_4(\mathbf{X}),\ U_5(\mathbf{X}),\ U_6(\mathbf{X})\right), & \mathbf{X} \in D^-, \end{cases} \tag{30}$$

and

$$u^N(\mathbf{X}) = \frac{1}{2}\int_S H(\mathbf{X}, \mathbf{X}; \mathbf{Y}) \sum_{k=1}^{N} A_k u^{(k)}(\mathbf{Y})\, d_y S - \frac{1}{2}\Phi(\mathbf{X}). \tag{31}$$

As a special case of the abovementioned solutions (31) we consider the boundary-value problem of an infinite domain with the constants λ_0, μ_0, carrying m elastic inclusions of different materials having no common points. Here, we denote with E_3 the 3-D space, D_k is the domain of the k-th inclusion, $k = 1, 2, \ldots, m$, S_k the boundary of the k-th inclusion, $D^+ = U_{k=1}^{m} D_k$, $D^- \equiv E_3 / D^+$.

The boundary conditions are described by the following system of equations (or in another wording it is required to find a regular vector satisfying these conditions):

$$\mu_k \Delta u + (\lambda_k + \mu_k)\ \mathrm{graddiv} u = F^k(\mathbf{X}), \quad \forall \mathbf{X} \in D_k, \tag{32}$$

$$\mu_0 \Delta u + (\lambda_0 + \mu_0)\ \mathrm{graddiv} u = F^0(\mathbf{X}), \quad \forall \mathbf{X} \in D^-, \tag{33}$$

and finally:

$$\begin{aligned} &\forall \mathbf{Y} \in S_k : u^-(\mathbf{Y}) - u^+(\mathbf{Y}) = f^k(\mathbf{Y}), \\ &(T^0 u)^- - (T^k u)^+ = F^k(\mathbf{Y}), \quad (k = 1, \ldots, m), \end{aligned} \tag{34}$$

where the quantities f^k, F^k on the right-hand side of Eqs. (32) to (34) are given.

According to the developments of Ustinov and Shleven **[39]** or Chernopiskii **[94, 95]**, or Shapiro **[96]**, Sheinin [97], the general representation formulas are of the form:

$$\mathbf{AX} \in D_k : 2u(\mathbf{X}) = \int_{S_k} K^k(\mathbf{X}\text{-}\mathbf{Y})\,\overline{(Tu)}\, d_y S - \int_{S_k} \left(T^k K^k(\mathbf{Y} - \mathbf{X})\right) u^-(\mathbf{Y})\, d^y S + R^k(\mathbf{X}) \tag{35}$$

while for $\forall \mathbf{X} \in E_3 - D_k$ we have:

$$\int_{S_k} K^k (\mathbf{X}-\mathbf{Y})(Tu)^- d_y S - \int_{S_k} \left(T^k K^k (\mathbf{Y}-\mathbf{X})\right) u^- (\mathbf{Y})\, d_y S + R_k (\mathbf{X}), \tag{36}$$

The quantity $R_k(\mathbf{X})$ can be determined as:

$$\left.\begin{aligned} R_k (\mathbf{X}) = &- \int_{D_k} K^k (\mathbf{X}-\mathbf{Y})\, \Phi^k (\mathbf{Y})\, d\mathbf{Y} + \int_{S_k} \left(T^k K^k (\mathbf{Y}-\mathbf{X})\right) f^k (\mathbf{Y})\, d_y S - \\ &- \int_{S_k} K^k (\mathbf{X}-\mathbf{Y})\, F^k d_y S\,; \quad k = 1, 2, \ldots, m. \end{aligned}\right\} \tag{37}$$

Furthermore, for every $\mathbf{X} \in D^-$, it is valid that:

$$2u(\mathbf{X}) = -\int_{\bigcup_{i=1}^{m} S_i} K(\mathbf{X}-\mathbf{Y})(Tu)^- d_y S + \int_{\bigcup_{i=1}^{m} S_i} \left(TK(\mathbf{Y}-\mathbf{X})\right) u(\mathbf{Y})\, d_y S - \Pi_{m+1}(\mathbf{X}), \tag{38}$$

while for $\forall \mathbf{X} \in \bigcup_{i=1}^{m} D_i$:

$$\int_{\bigcup_{i=1}^{m} S_i} K(\mathbf{X}-\mathbf{Y})(Tu)^- d_y S - \int_{\bigcup_{i=1}^{m} S_i} TK(\mathbf{Y}-\mathbf{X})\, u^-(\mathbf{Y})\, d_y S + \Pi_{m+1}(\mathbf{X}), \tag{39}$$

with:

$$\Pi_{m+1}(\mathbf{X}) = \int_{D^-} K(\mathbf{X}-\mathbf{Y})\, \Phi^0 (\mathbf{Y})\, d\mathbf{Y}. \tag{40}$$

We introduce now the matrices [**98** to **110**]:

$$Q_{(k)}(\mathbf{X}, \mathbf{Y}) = \begin{cases} \| (T^k K^k (\mathbf{Y}-\mathbf{X}), \quad -K^k (\mathbf{X}-\mathbf{Y}) \|_{3\times 6}\,; \\ \qquad\qquad 0\,; \end{cases}$$

$$\left.\begin{aligned} &\exists \mathbf{X} \in E_3\,; \quad k = 1, \ldots, m\,; \\ &\mathbf{Y} \in S_j\,; \quad j \neq k\,; \quad j = 1, \ldots, m\,; \end{aligned}\right\}, \tag{41}$$

$$Q_{(0)}(\mathbf{X}, \mathbf{Y}) = \| (T^0 K^0 (\mathbf{Y}-\mathbf{X}), \quad -K^0 (\mathbf{X}-\mathbf{Y}) \|_{3\times 6}\,;$$

$$\mathbf{Y} \in \bigcup_{i=1}^{m} S_i\,, \quad \forall \mathbf{X} \in E_3. \tag{42}$$

We now have the system of equations:

$$\int\limits_{\bigcup_{i=1}^{m} S_i} Q_{(k)}(\mathbf{X}_j^0, \mathbf{Y})\, \bar{u}(\mathbf{Y})\, d_y S = \Pi_k(\mathbf{X}_j^0), \quad k = 1, 2, \ldots, m; \tag{43}$$

$$\int\limits_{\bigcup_{i=1}^{m} S_i} Q_{(0)}(\mathbf{X}_j^1, \mathbf{Y})\, \bar{u}(\mathbf{Y})\, d_y S = \Pi_{k+1}(\mathbf{X}_j^1), \tag{44}$$

and finally:

$$\int\limits_{\bigcup_{i=1}^{m} S_i} Q_{(0)}(\mathbf{X}_j^m, \mathbf{Y})\, \bar{u}(\mathbf{Y})\, d_y S = \Pi_{k+1}(\mathbf{X}_j^m). \tag{45}$$

The left-hand sides of these equations involve vector quantities, while the right-hand sides involve given quantities $\left(\bar{u}(\mathbf{Y}) \equiv (u^-, (T^0 u)^-)\right)$. From the direct solution of the linear system of equations (43) to (45) the vector quantity $\bar{u}(\mathbf{Y})$ follows as:

$$\bar{u}(\mathbf{Y}) = \lim_{N\to\infty} \sum_{n=1}^{N} a_n\, u^{(n)}(\mathbf{Y}), \tag{46}$$

from which the scalar component u^- may be determined. By substituting u^- in Eqs. (38), (39) and (40) and denoting the result by $u^N(\mathbf{X})$ we have that the solution vector of the boundary-value problem has the form:

$$u(\mathbf{X}) = \lim_{N\to\infty} u^N(\mathbf{X}). \tag{47}$$

Furthermore, the general thermoelastic problem is evidently treated analogously with oscillation problems without any appreciable modifications. However, in order to emphasize some specific features concerning the 3—D thermoelastic boundary-value problems (which are systematically described in Table III) we shall consider the second basic problem.

According to the developments of Shvets and Eleiko [**111**, **112**] it is necessary to construct in D^+ a regular solution of the differential equation:

$$B(\partial_x, \omega)\, U(\mathbf{X}) + H(\mathbf{X}) \equiv 0, \tag{48}$$

where:

$$B(\partial_x, \omega) = \| B_{ij}(\partial_x, \omega) \|_{4\times 4}, \tag{49}$$

with:

$$B_{ij}(\partial_x, \omega) = \delta_{ij}(\mu\Delta + \varrho\omega^2) + (\lambda+\mu)\frac{\partial^2}{\partial_{x_i}\partial_{x_j}}, \tag{50}$$

when $i, j = 1, 2, 3$ and:

$$B_{i4}(\partial_x, \omega) = -\gamma\partial / \partial x_i, \tag{51}$$

when $i = 1, 2, 3$;

$$B_{4j} = i\omega n\partial / \partial x_j, \tag{52}$$

when $j = 1, 2, 3$; and finally:

$$B_{44}(\partial_x, \omega) = \Delta + i\omega / k. \tag{53}$$

Here, according to the developments of Carslaw [**113**] and Landau and Lifshitz [**114**] it is valid that:

$$K = \frac{k}{\delta}, \tag{54}$$

where k denotes the heat-conduction coefficient, δ is the specific heat per unit volume and:

$$n = \gamma T_0 / k, \tag{55}$$

where $\gamma = (2\mu + 3\lambda)a$, a is the coefficient of linear heat expansion, λ, μ the Lamé constants, ϱ is the density of the medium and ω is the angular velocity.

Let us now solve the second basic problem, the boundary conditions of which are expressed by the relations:

$$(PU)^+ = \varphi, \tag{56}$$

$$[\partial u_4 / \partial n]^+ = \varphi_4, \tag{57}$$

where the matrix differential operator:

$$P(\partial_x, (\mathbf{X})) = \| P_{ij}(\partial_x, n(\mathbf{X})) \|_{3\times 4}, \tag{58}$$

is determined by:

$$\left.\begin{aligned} P_{ij}(\partial_x, n(\mathbf{X})) &= T_{ij}(\partial_x, n(\mathbf{X})), \quad && i, j = 1, 2, 3 \\ P_{i4}(\partial_x, n(\mathbf{X})) &= -\gamma n_i(\mathbf{X}) && i = 1, 2, 3 \end{aligned}\right\} \tag{59}$$

Furthermore, $U = (U_i)$ denotes a four-component vector:

$$\left.\begin{aligned} U_i &= u_i \qquad && i = 1, 2, 3 \\ \text{and:}\quad U_4 &= \Theta(\mathbf{X}) \end{aligned}\right\}, \tag{60}$$

where $\Theta(\mathbf{X})$ is the temperature distribution function and $\varphi \equiv (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ denotes a four-component known function.

Let us introduce the matrix differential operator of dimension 4×4 [**114**]:

$$R(\partial_x, n, \gamma) = \| R_{kj}(\partial_x, n, \gamma) \|_{4\times4} = \left\| \begin{array}{c|c|c|c} T_{11} & T_{12} & T_{13} & -\gamma n_1 \\ T_{21} & T_{22} & T_{23} & -\gamma n_2 \\ T_{31} & T_{32} & T_{33} & -\gamma n_3 \\ 0 & 0 & 0 & \dfrac{\partial}{\partial n} \end{array} \right\|, \tag{61}$$

where the symbol $\| T_{kj}(\partial_x, n) \|_{3\times3}$ stands for the components of the classical stress operator, Magnaradze [**115** - **117**]. On the other hand, the fundamental solutions of the homogeneous equation $B(\partial_x, \omega) U(\mathbf{X}) \equiv 0$ are denoted by G_i, while the matrix $G(\mathbf{X}, \omega)$ is called «the fundamental thermoelastic matrix» and practically corresponds to Kelvin's matrix K (Eq. 13) [**118** to **125**]:

$$G(\mathbf{X}, \omega) \equiv \| G_1, G_2, G_3, G_4 \| . \tag{62}$$

Let now $U(\mathbf{X})$ be a solution of the abovementioned second problem of thermoelasticity. Then, according to the developments of Napetvaridze [**126** to **131**], Natroshvili [**132** to **135**], Mikhlin [**136** to **142**], the following equations are valid:

$$2U(\mathbf{X}) = -\int_S [R(\partial_x, n)\, G(\mathbf{Y}-\mathbf{X}, \omega)]\, U^+(\mathbf{Y})\, d_y S + F(\mathbf{X}), \quad \forall \mathbf{X} \in D^+, \tag{63}$$

and:

$$\int_S [R(\partial_y, n)\, G(\mathbf{Y}-\mathbf{X}, \omega)]\, U^+(\mathbf{Y})\, d_y S = F(\mathbf{X}), \quad \forall \mathbf{X} \in D^-, \tag{64}$$

where:

$$F(\mathbf{X}) = \int_S G(\mathbf{X}-\mathbf{Y}, \omega)\, \varphi(\mathbf{Y})\, d_y S + \int_{D^+} G(\mathbf{X}-\mathbf{Y}, \omega)\, H(\mathbf{Y})\, d\mathbf{Y}, \tag{65}$$

with $H(\mathbf{Y})$ denoting the total force vector. By properly enumerating the elements of the set $\{RG_{(i)}(\mathbf{Y}-\mathbf{X}^k, \omega)\}_{k=1}^{\infty}$, $i = 1, 2, 3, 4$, as follows:

$$\psi^{(k)}(\mathbf{Y}) = R(\partial_y, n)\, G_{(l_k)}(\mathbf{Y}-\mathbf{X}^{[(k+3/4]}, \omega), \tag{66}$$

$$l_k = k - 4\left[\frac{k-1}{4}\right], \quad k = 1, 2, 3, \ldots \tag{67}$$

we can orthonormalize the system $\{\psi^{(k)}(\mathbf{Y})\}_{k=1}^{\infty}$ on S, in order to obtain the set:

$$\{U^+_{(k)}(\mathbf{Y})\}_{k=1}^{\infty}, \quad \{U^+_{(k)}(\mathbf{Y}) = \sum_{t=1}^{k} h_{k,t}\, \psi^{(t)}(\mathbf{Y})\}, \quad (k = 1, 2, \ldots), \tag{68}$$

where $h_{k,t}$ are the coefficients of orthonormalization (Rukhadze [**143** to **148**]). Furthermore, the unknown vector $U(\mathbf{Y})$ may be represented in Fourier series in the complex conjugate system $\{\overline{U}^+_{(k)}(\mathbf{Y})\}$ of Eq. (68):

$$U(\mathbf{Y}) \approx \sum_{s=1}^{\infty} P_m\, \overline{U}^+_{(k)}(\mathbf{Y}), \tag{69}$$

where:

$$P_k = \int_S U^+_{(k)}(\mathbf{Y})\, U^+(\mathbf{Y})\, d_y S, \quad (k = 1, 2, \ldots) \tag{70}$$

These coefficients can be evaluated from the functional equation (64) (Kvinikadze, [**149** to **151**]):

$$\int_S \sum_{k=1}^{4} \{[R(\partial_y, n)\, G(\mathbf{Y}-\mathbf{X}^i, \omega)]\, U^+(\mathbf{Y})\}\, j^k\, d_y S = F(\mathbf{X}_i), \quad i = 1, 2, 3, \ldots \tag{71}$$

where j^k, $k = 1, 2, 3, 4$, are the axial unit vectors in the four-dimensional space. It is easy to verify that, Nemish [**152** to **160**]:

$$\int_S \sum_{k=1}^{4} \sum_{i=1}^{s} \lambda_{m, 4(i-1)+k}\, \psi^{(4(i-1)+k)}(\mathbf{Y}) = \sum_{n=1}^{m} \lambda_{m,n}\, \psi^{(n)}(\mathbf{Y}), \tag{72}$$

from which it follows that:

$$P_k = \sum_{i=1}^{s} S^{ki}\, F(\mathbf{X}^i), \tag{73}$$

where:

$$S^{ki} = \lambda_{k,4i-3}\, j + \lambda_{k,4i-2}\, j^2 + \lambda_{k,4i-1}\, j^3 + \lambda_{k,4i}\, j^4, \tag{74}$$

with $\lambda_{m,k} = 0$ for $k > m$ and $m = 1, 2, \ldots;\ s = [(m+3)/4]$.

Then, the solution of the boundary-value problem is given by:

$$U^{(N)}(\mathbf{X}) = -\frac{1}{2}\int_S [R(\partial_y, n)\, G(\mathbf{Y}-\mathbf{X}, \omega)] \sum_{k=1}^{N} G_k\, \overline{U}_{(k)}(\mathbf{Y})\, d_y S + \frac{1}{2} F(\mathbf{X}), \quad \mathbf{X} \in D^+, \tag{75}$$

while the exact solution $U(\mathbf{X})$ of the second thermoelastic problem is given by:

$$U(\mathbf{X}) = \lim_{N \to \infty} U^{(N)}(\mathbf{X}). \tag{76}$$

Generally speaking, it is commonly thought that the Fourier method connected with the separation of variables is effective only for some specific domains. Actually, the idea of applying the generalized Fourier method in solving 3—D boundary-value problems is close to the so-called method of Fischer-Rietz equations (see Miranda [**161**]) and therefore can be always expressed effectively, if sufficiently general assumptions are accepted for the domain and other data of the problem.

The first exposition of this method is due to Kupradze and Alexidze [**162**], while the method was subsequently extended to the solution of elliptic mixed problems by Burchuladze [**163** to **165**] and to the solution of boundary contact problems by Rukhadze [**166**].

Furthermore, the extension of the method to parabolic equations is due to Domanski et al. [**167**], Domanjski, Piscovek and Rvek [**168**], while the extension to equations of hydrodynamics and electrodynamics is due to Polischuk [**169**], Pham The Lai [**170**] etc.

3. THREE-DIMENSIONAL BOUNDARY VALUE PROBLEMS FOR NONORTHOGONAL BOUNDARY SURFACES

By the terminology non-orthogonal boundary surfaces we understand surfaces at which, for the unit vectors $\mathbf{e}_j$ of the curvilinear orthogonal system of coordinates **i**, **j**, **k** used, the known conditions of orthogonality are not satisfied [**1** to **5**]:

$$\mathbf{n}_i \cdot \mathbf{n}_j \neq 0 \ (i \neq j), \quad n_i n_j \neq 1 . \tag{77}$$

Let us consider now a 3—D boundary-value problem for a 3—D deformable body bounded by a non-orthogonal, close-to-canonical surface S. It can be postulated that the surface S is close to a round cylindrical surface and therefore its equation can be described by:

$$r = r_0 + \varepsilon g(\vartheta, z) . \tag{78}$$

Here for convenience the round cylindrical r, ϑ, z - coordinates are considered, while the analytical function $g(\vartheta, z)$ and the small parameter ε characterize the form of the non-orthogonal surface S and the value of its deviation from the corresponding canonical surface $r = r_0$ ($r_0 = \text{constant} > 0$, $|\varepsilon| \ll 1$).

We consider now, a stress and deformation state of the body D, by following the boundary conditions which are defined over its surface S:

$$u_j |_S = u_j^0 , \tag{79}$$

$$\sum_{i=1}^{3} \sigma_{ij} n_i = \tau_j^0 , \tag{80}$$

where n_i are the directional cosines of the unit vector **k** of the normal **n** to S.

In view of the complexity of the non-orthogonal surface S, described by Eq. (78), the variables in the boundary conditions (79) and (80) are not separable and therefore the problem cannot be solved directly. Then, by assuming that the preassigned displacements or stresses permit the expansion of the solution sought in a Taylor series in the vinicity of the area $r = r_0$, the components of the stress- and deformation-state can be expressed in the form:

$$\left\{\sigma_{ts}, u_s\right\}_s = \sum_{i=0}^{\infty} \varepsilon^i \sum_{j=0}^{i} \frac{g^j}{j!} \frac{\partial^j}{\partial r^j} \left\{\varepsilon_{ts}^{(i-j)}, u_s^{i-j}\right\}_{r=r_0}. \tag{81}$$

In the final analysis the boundary conditions can be represented as:

$$\sum_{s=0}^{n} L_i^{(s)} u_j^{(n-s)} \Big|_{r=r_0} = u_j^{0(n)}, \tag{82}$$

and:

$$\sum_{i=1}^{3} \sum_{s=0}^{n} D_i^{(s)} \sigma_{ij}^{(n-s)} \Big|_{r=r_0} = \tau_j^{0(n)}, \tag{83}$$

here $L_i^{(s)}$, $D_i^{(s)}$ are differential operators which depend on the $g(\vartheta, z)$-function and in the case of non-orthogonal, close to round cylindrical surfaces, have the form:

$$L_i^{(n)} = \frac{1}{n!} g^n \frac{\partial^n}{\partial r^n}, \quad n \geqq 0 \tag{84}$$

and:

$$D_3^n = \mp \frac{\partial f}{\partial z} \sum \left(\gamma_{s-1} + \frac{g}{r_0} \gamma_{s-z}\right) L_i^{(n-s)}, \quad \gamma_{-j} \equiv 0, \tag{85}$$

where $\gamma_s(\vartheta, z)$ are the coefficients of the expansions of the directional cosines n_i in series in terms of ε.

4. APPLICATIONS

A general method for the numerical solution of the systems of 2—D singular integral equations, which are included in Tables I, II, is also presented in [**171**] by Theocaris and collaborators. The method is based on Fredholm integral equations of the first kind, which are obtained from Kupradze's functional equations. Contrary to the general considerations about the regular integral equations, the method presented in [**171**] seems very promissing. In fact, rather stable and accurate results are obtained in all the examples considered. Especially in the case of the sphere, where an analytical solution is known, the solution presents a difference of only 0.5 percent. On the other hand, the proposed in ref. [**171**] method, when compared to the already known methods, has some advantages [**172**]:

(i) It is simpler and easier in programming.
(ii) It is much faster than the classical ones described in refs. [**173** to **175**] and especially Lachat's method [**176**].
(iii) A high-polynomial approximation of the unknown function is obtained, so that the method performs very well with problems, where a high gradient of stresses or displacements is encountered.

More precisely, in 3—D elasticity problems the singular integral equation method was applied mainly to static problems using Somigliana's identity and subdividing the surface to plane triangular elements [**173**], into which the unknown distribution is assumed as either constant [**174**], or having a linear variation [**176**]. An essential improvement of the boundary integral equation methods, mainly used by Cruse [**174**], is obtained by introducing the isoparametric elements by Lachat [**176**]. In this case the unknown distribution can have a quadratic or cubic variation.

The technique, which is proposed by Cruse [**174**], is based on Stokes' theorem and allows an analytical integration of the fundamental solutions Γ^0 and $T_n\Gamma^0$ into each element:

$$\mathbf{\Gamma}^0 \equiv \Gamma^0_{kj}(\mathbf{X}, \mathbf{Y}) \equiv \frac{1}{2\mu(\lambda+2\mu)}\left[(\lambda+\mu)\, r_{,j}\, r_{,k} + 2(\lambda+\mu)\delta_{jk}\right]\left(\frac{1}{r}\right), \quad (86)$$

where:

$$r = \left[(x_i - y_i)(x_i - y_i)\right]^{1/2} \quad (87)$$

and $r_{,j}$ means differentiation with respect to the field variable y_j.

An analogous expression is valid for the tensor component $\mathbf{T}_n\mathbf{\Gamma}^0$:

$$\left.\begin{aligned} \mathbf{T}^{(n)}_j \mathbf{\Gamma}_k \equiv \mathbf{T}_n \mathbf{\Gamma}^0 \equiv 2\mu \frac{\partial\left(\Gamma_{ki}(\mathbf{X}, \mathbf{Y})\right)}{\partial n} + \frac{\lambda}{\lambda+2\mu}\cos(ny_j)\left(\frac{e^{ik_1 r}}{r}\right)_{,k} + \\ + \cos(ny_k)\left(\frac{e^{ik_2 r}}{r}\right)_{,j} - \delta_{jk}\frac{\partial}{\partial n}\left(\frac{e^{ik_2 r}}{r}\right), \end{aligned}\right\} \quad (88)$$

where k_1, k_2 are properly defined as:

$$k_1^2 = \omega^2 / (\lambda + 2\mu), \tag{89}$$

and

$$k_2^2 = \omega^2/\mu. \tag{90}$$

Here r again expresses the distance between the field point $\mathbf{Y}(y_1, y_2, y_3)$ and the load point $\mathbf{X}(x_1, x_2, x_3)$.

However, such an analytic integration is only applicable in the case where the unknown distribution is supposed to be constant, or to vary linearly into each element. In all other cases a numerical integration is needed. In order to overpass some of these difficulties, complicated schemes are proposed by Lachat [**176**], which allow the elements to be subdivided, in a special manner, to sub-elements, while in each sub-element a product Gaussian formula is used. The number of necessary integration points is then chosen in each individual problem in such a way that some inequalities and conditions are fulfilled. Even with this special scheme, which is very difficult in programming, the convergence of the cubature formula is always slow, because of the singularities of the integrands. In addition, the most of the execution time, for such a solution is spent integrating for the formulation of the linear system, than solving it. Therefore, the need of a higher-polynomial approximation of the unknown distribution is obvious.

Furthermore, for the solution of some steady-state dynamic problems, the above-referred methods have to overpass the difficulties created by the oscillatory character of the fundamental solutions of steady-state problems. Thus, Stokes' theorem is not anymore applicable and therefore Cruse's method, described in refs. [**173** to **175**], falls down. On the other hand, in Lachat's method [**176**], a numerical integration of oscillatory integrands is needed. But, Gaussian quadrature formulas do not handle very well neither oscillatory integrands, non singular ones. As a consequence, an increase of the number of the integration points is resulted, a fact which enlarges strongly the amount of the computer cost.

There are also problems, where a higher-polynomial approximation of the unknown distribution is necessary. Such situations arise when a concentrated force or a couple is applied on the surface S of the body, when the body has re-entrant corners, cracks or other geometrical

T A B L E I

Closed Form Solutions for the General Three - Dimensional Problem.

The Boundary-Value Problem	The System of S.I.E. Solving B.V.P.	Surface	Closed-form Solution:	References
First basic problem of Statics $(I)^{\pm}$ Boundary conditions: $U^{\pm}(Y) = f(Y)\ ,\quad Y \in S$ here f denotes a given Hölder continuous function of **Y**.	$2f(Y) = \int_S K(X-Y)\,TU(X)\,d_xS - F(Y)\ ,\quad X \in D^+$ $F(Y) = \int_S [TK(X-Y)]\,f(Y)\,d_xS - \int_D K(X-Y)\,H(Y)\,dY$	A Closed fully symmetrical Lyapunov surface	$2U_N(X) = \int_S K(X-Y) \times \sum_{n=1}^{m} S_n \varphi^n(Y)\,dS - H(X)$ $U(X) = \lim_{N\to\infty} U_N(X)$	Kupradze [60-65] Basheleishvili [70-74]
Second basic problem of Statics $(II)^{\pm}$ Boundary conditions: $\{T(\partial_y, n)\,U(Y)\}^{\pm} = \Omega(Y)\ ,\quad Y \in S,$ $\Omega(Y)$ is any Hölder continuous function of Y.	$2u(X) = -\int_S [TK(X-Y)]\,U(Y)\,d_yS + F(X)\ ,\quad X \in D^+$ $F(X) = \int_S K(X-Y)\,\Omega(Y)\,d_yS - \int_{D^+} K(X-Y)\,H(Y)\,dY$	- " -	$U_N(X) = -\frac{1}{2}\int_S TK(X-Y) \times$ $\times \sum_{k=1}^{N} S_k \varphi^{(k)}(Y)\,d_yS + \frac{1}{2}F(X) +$ $+ \sum_{k=1}^{N} S_{-k} X(x)$ $U(X) = \lim_{N\to\infty} U_N(X)$	Kupradze [65-67]
Third basic problem of Statics $(III)^{\pm}$ Boundary conditions: $[U(Z)]_n^{+} = f_4(Z)\ ,\quad Z \in S\ ,$ $\{TU(Z)\}^{+} - n(Z)\,[TU(Z)]_n^{+} = f(Z)$ $g(z) = f(z) + if_4(z)\ ,$ $g(Z)$ is a Hölder function of $Z \in S$	$2u(X) = \int_S [HK(X-Y)]\,RU(Y)\,d_yS - F(X)\ ,\quad X \in D^+\ ,$ $F(X) = \int_S [RK(X-Y)]\,g(Y)\,dY - \int_{D^+} K(X-Y)\,H(Y)\,dY$	- " -	$U_N(X) = \sum_{k=1}^{N} \sum_{s=1}^{K} \Omega_k a_{ks} \times$ $\times K^{(\ell s)}(X^{[(s+2)/3]} - X)$	Polozhii [27]

Table I (continued)

The Boundary-Value Problem	The System of S.I.E. Solving B.V.P.	S	Closed-form Solution	References
$[Tu(\mathbf{Z})]^{+} + \sigma(\mathbf{Z})u^{+}(\mathbf{Z}) = f(\mathbf{Z})$, $\sigma'(\mathbf{Z}) = \sigma(\mathbf{Z})$, $\mathbf{Z} \in S$ where $\sigma(\mathbf{Z})$, $f(\mathbf{Z})$ are known functions defined for every point $\mathbf{Z}$ of the Lyapunov boundary S.	$2u(\mathbf{X}) = -\int_S \{[T(\partial_y, n)K(\mathbf{X}-\mathbf{Y})] + \sigma(\mathbf{Y})K(\mathbf{X}-\mathbf{Y})\}u^{+}(\mathbf{Y})d_yS + F(\mathbf{X})$ $\forall \mathbf{X} \in D^{+}$ $\int_S \{[T(\partial_y, n)K(\mathbf{X}-\mathbf{Y})] + \sigma(\mathbf{Y})K(\mathbf{X}-\mathbf{Y})\}u^{+}(\mathbf{Y})d_yS = F(\mathbf{X})$, $\forall \mathbf{X} \in D^{-}$, $F(\mathbf{X}) = \int_S K(\mathbf{X}-\mathbf{Y})f(\mathbf{Y})d_yS + \int_{D^{+}} K(\mathbf{X}-\mathbf{Y})H(\mathbf{Y})d\mathbf{Y}$.	Closed Lyapunov Surface	$u(\mathbf{X}) = \lim_{N\to\infty} \upsilon^{(N)}(\mathbf{X}) + \frac{1}{2}\int_{D^{+}} K(\mathbf{X}-\mathbf{Y})H(\mathbf{Y})d\mathbf{Y}$, $\mathbf{X} \in D^{+}$, where: $\upsilon^{(N)}(\mathbf{X}) = \sum_{k=1}^{N} \sum_{s=1}^{k} \Omega_k a_{ks} \times K^{(\ell_s)}(\mathbf{X}^{[s+2/3]} - \mathbf{X})$	[21] [25] [27] [35] [81 to 90]

TABLE II

Effective Solutions of Boundary Value Problems for a Sphere and a Spherical Cavity in an Infinite Medium.

Boundary-value problem	Singular Integral Equations	Surface	Closed-form Solution	References
<u>Problems $(I)^{\pm}$</u>: $u(\mathbf{Y}) = f(\mathbf{Y}) \quad \forall \mathbf{Y} \in S$ where f is a given vector on S	$-g(\mathbf{Z}) + -\!\!\!\!\!\int_S [T(\partial_y, n) K(\mathbf{Z}-\mathbf{Y})] \times$ $g(\mathbf{Y}) d_y S = f(\mathbf{Z}), \quad \mathbf{Z} \in S$	Spherical surface	$u(\mathbf{X}) = \int_S \overline{K}^{+}(\mathbf{X},\mathbf{Y}) f(\mathbf{Y}) d_y S, \quad \mathbf{X} \in D^{+}$ where: $\overline{K}^{+}(\mathbf{X},\mathbf{Y}) = \lVert \overline{K}^{+}_{ij}(\mathbf{X},\mathbf{Y}) \rVert_{3\times 3}$ $\overline{K}^{+}(\mathbf{X},\mathbf{Y}) = \frac{1}{4\pi R}\left\{ \delta_{ij} \frac{R^2-\rho^2}{\lvert \mathbf{X}-\mathbf{Y} \rvert^3} + \right.$ $\left. \beta(R^2-\rho^2) \cdot \frac{\partial^2 \Omega(\mathbf{X},\mathbf{Y})}{\partial x_i \partial x_j} \right\}$ and $\Phi(\mathbf{X},\mathbf{Y}) = \int_0^1 \left[\frac{R^2-\rho^2 t^2}{(R^2-2R\rho t\cos\gamma+\rho^2 t^2)^{3/2}} - \right.$ $\left. -\frac{1}{R} - \frac{3\rho t \cos\gamma}{R^2} \right] \frac{dt}{t^{1+a}}, \quad \beta = \frac{\lambda+\mu}{2(\lambda+3\mu)}$ $a = (\lambda+2\mu)/(\lambda+3\mu) < 1, \quad i,j = 1,2,3$ R = the radius of the sphere $0 \le \rho \le R, \quad \cos\gamma = \frac{1}{\lvert \mathbf{X} \rvert \lvert \mathbf{Y} \rvert} \sum_{k=1}^{3} X_k Y_k$	[132-135]

Table II (continued)

Boundary-value problem	Singular Integral Equations	Surface	Closed-form Solution	References
Problems $II^{\pm}$ $\{T(\partial_z,v)u(Z)\}^{+} = F(Z)$, $Z \in S$ where F is a sufficiently smooth vector given on S.	$g(Z) + \int_S [T(\partial_z,v)K(Z-Y)]g(Y) d_yS$, $Z \in S$, $u(X) = \int_S K(X-Y)g(Y)d_yS$, $X \in D^{+}$	The spherical surface S $(D^{\pm})$	$u(X) = \int_S K^{(2)}_{\mp}(X,Y)F(Y)d_yS + D + [xD^{(1)}]$ where: $K^{(2)}(X,Y) = \| K^{(2)}_{ij\mp}(X,Y) \|_{3\times 3}$ $K^{(2)}_{ij}(X,Y) = \frac{1}{8\pi\mu}\Big\{(\Phi_1+\Phi_2)\delta_{ij} + \frac{R^2-3\rho^2}{2}\frac{\partial^2\Phi_3}{\partial x_i \partial x_j} + x_j\Big(\frac{\partial\Phi_1}{\partial x_j} - \frac{\partial\Phi_2}{\partial x_i}\Big) - 2x_i\frac{\partial\Phi_1}{\partial x_j} + x_i\frac{\partial}{\partial x_j}\Big(2\rho\frac{\partial\Phi_3}{\partial\rho} - \Phi_3\Big) + \rho^2\Big(\frac{\partial^2\Phi_2}{\partial x_i\partial x_j} - \frac{\partial^2\Phi_1}{\partial x_i \partial x_j}\Big)\Big\}$ $i,j = 1,2,3.$ $\Phi_1 = \int_0^1 \Big[\frac{R^2-\rho^2t^2}{Q(t)} - \frac{1}{R}\Big]\frac{dt}{t}$, $\Phi_2 = \int_0^1 \Big[\frac{R^2-\rho^2t^2}{Q(t)} - \frac{1}{R} - \frac{3\rho t\cos\gamma}{R^2}\Big]\frac{dt}{t^2}$, $\Phi_3 = b_1^{-1}\lim\Phi_0$, $\Phi_0 = \int_0^1 \Big[\frac{R^2-\rho^2t^2}{Q(t)} - \frac{1}{R}\Big]\frac{dt}{t^{1+a_1}}$ $\Phi_4 = Re[b_2\Phi_0]$, $a_1 = b_0+ib_1 = \frac{\mu+i\sqrt{2\lambda^2+6\lambda\mu+3\mu^2}}{2(\lambda+\mu)}$ $b_2 = \frac{1}{2} + i\frac{3\lambda+4\mu}{2\sqrt{2\lambda^2+6\lambda\mu+3\mu^2}}$, $Q(t) = (R^2-2R\rho t\cos\gamma+\rho^2t^2)^{3/2}$	[132-135] [60 to 67]

Table II (continued)

Boundary-value problem	Singular Integral Equations	Surface	Closed-form Solution	References
Problem $III^{\pm}$: Boundary conditions: $(\nu u)^{+}=f_4$, $\{T(\partial_z,\nu)u-\nu(T(\partial_z,\nu)u\nu)\}^{+}=f$, where f_4 and $f\equiv(f_1,f_2,f_3)$ are given quantities on S.	$\mp U(Z)+\int_S H(\partial_z,\nu)$ $[R(\partial_y,n)K(Y-Z)]U(Y)d_yS=$ $=f(Z)$, $Y\in S$ where the Kernels H, R are properly defined in Ref.[40] and f(Z) is the given from the boundary conditions vector (f_1,f_2,f_3,f_4).	The spherical surface S **X** belongs to the interior of the sphere $\Sigma^{+}_{\Phi}(0,R)$	$u(X)=\int_S \overset{(3)}{K}_{+}(X,Y)f(Y)d_yS+[XC^{(1)}]$ $X\in\Sigma^{+}_{\Phi}(0,R)$ where $C^{(1)}$ represents an arbitrary vector and $\overset{(3)}{K}_{+}(X,Y)=\Vert\overset{(3)}{K}_{ij}^{+}(X,Y)\Vert_{3\times4}$ $\overset{(3)}{K}_{ij}^{+}(X,Y)=(1-\delta_{j4})\Big\{\overset{(2)}{K}_{ij}^{+}(X,Y)+$ $+2\delta_{ij}\dfrac{[(2\lambda+3\mu)\rho^2+(\mu-\lambda)R^2]-2(\lambda+4\mu)x_ix_j}{8\pi\mu(3\lambda+2\mu)R^3}\Big\}+$ $+\dfrac{\delta_{j4}}{4\pi}\Big\{\dfrac{R^2-\rho^2}{R^3}\dfrac{\partial}{\partial x_i}[(\lambda+\mu)\rho\dfrac{\partial}{\partial\rho}+3\lambda+5\mu]\times$ $\times\left(\rho\dfrac{\partial}{\partial\rho}-1\right)\Phi^*+\dfrac{1}{R}\dfrac{\partial}{\partial x_i}\left[3\lambda+4\mu-2\mu\rho\dfrac{\partial}{\partial\rho}\right]\Phi^*+$ $+2\dfrac{x_i}{R^3}\left((\lambda+3\mu)\rho\dfrac{\partial}{\partial\rho}+\mu\right)\left(\rho\dfrac{\partial}{\partial\rho}-1\right)\Phi^*+\dfrac{3}{R^3}y_i+$ $\dfrac{1}{R^3}x_i\Big\}$ $i=1,2,3$ $j=1,2,3,4$ and $\Phi^*=\dfrac{R}{\sqrt{\lambda^2+16\lambda\mu+32\mu^2}}\int_0^1\left[\dfrac{R^2-\rho^2t^2}{Q(t)}-\dfrac{1}{R}-\dfrac{3\rho t\cos\gamma}{R^2}\right]\times$ $\times\dfrac{t^{\beta_2}-t^{\beta_1}}{t^{1+\beta_1+\beta_2}}dt$, $\beta_{1,2}=\dfrac{-\lambda\pm\sqrt{\lambda^2+16\lambda\mu+32\mu^2}}{4(\lambda+2\mu)}$, $\vert\beta_i\vert<1$, $i=1,2$, $Q(t)=(R^2-2R\rho t\cos\gamma+$ $+\rho^2t^2)^{3/2}$.	[66 to 69]

Table II (continued)

Boundary-value problem	Singular Integral Equations	Surface	Closed-form Solution	References
Problems $IV^{\pm}$: Boundary conditions: $\{u - v(uv)\}^{+} = f$ $\{(\lambda+2\mu)\operatorname{div}u - \frac{4\mu}{R}(uv)\}^{+} = f_4$ where $f \equiv (f_1, f_2, f_3)$ and f_4 are given quantities on S.	$u(Z) + \int_S R(\partial_z, v) \times$ $\times [H(\partial_y, n)\Gamma(Y-Z)]u(Y) \times$ $d_y S = f(Z), \quad Y \in S$ where the Kernels H,R are defined in Ref.[40] and f(Z) is the given boundary vector $f = (f_1, f_2, f_3, f_4)$	The Spherical surface S X belongs to the interior of the Sphere $\Sigma^{+}_{\varphi}(0,R)$	$u(\mathbf{X}) = \int_S \overset{(4)}{K^{+}}(\mathbf{X},\mathbf{Y}) f(\mathbf{Y}) d_y S \quad \mathbf{X} \in \Sigma^{+}_{\varphi}(0,R)$ $\overset{(4)}{K^{+}}(\mathbf{X},\mathbf{Y}) = \|K^{+}_{ij}(\mathbf{X},\mathbf{Y})\|_{3\times 4}$, $\overset{(4)}{K^{+}_{ij}}(\mathbf{X},\mathbf{Y}) = (1-\delta_{j4})\overset{(1)}{K^{+}_{ij}}(\mathbf{X},\mathbf{Y}) + \frac{\delta_{ij}}{4\pi\mu R} \times$ $\times \Big\{ x_i \frac{\lambda}{(3\lambda+2\mu)} + x_i \Big[(\lambda+3\mu)\rho\frac{\partial}{\partial\rho} + \mu\Big]\Phi_A + \frac{R^2-\rho^2}{2} \times$ $\frac{\partial}{\partial x_i} \times \Big[(\lambda+\mu)\rho\frac{\partial}{\partial\rho} + (3\lambda+5\mu)\Big]\Phi_A \Big\}$ i=1,2,3, j=1,2,3,4 $\overset{(1)}{K^{+}_{ij}}(\mathbf{X},\mathbf{Y}) = \frac{1}{4\pi R}\Big\{ \delta_{ij}\frac{R^2-\rho^2}{\lvert \mathbf{X}-\mathbf{Y} \rvert^3} + \beta(R^2-\rho^2) \times$ $\times \frac{\partial^2 \Phi(\mathbf{X},\mathbf{Y})}{\partial x_i \partial y_j} \Big\} \qquad \beta = \frac{\lambda+\mu}{2(\lambda+3\mu)}$, Φ is the same as for the $(I^{\pm})$ for the sphere $\Sigma^{+}_{\varphi}(0,R)$ (see Table II) and: $\Phi_A = \frac{2}{\sqrt{23\lambda^2+68\lambda\mu+28\mu^2}} \operatorname{Im}\Big\{ \int_0^1 \Big[\frac{R^2-\rho^2 t}{Q(t)} - \frac{1}{R}\Big] \frac{dt}{t^{1+\beta_0}}$ $\beta_0 = \frac{-\lambda+2\mu+i\sqrt{23\lambda^2+68\lambda\mu+28\mu^2}}{4(\lambda+2\mu)}$	[66 to 70]

TABLE III

Boundary - Value Problems of Thermoelasticity.

Boundary-value problem	Equations Solving the Problem	Surface, space	Closed-form Solution	References
$\{PU\}^+ = f, \quad \left\{\frac{\partial u_4}{\partial n}\right\}^+ = f_4$ where $PU = Tu - \gamma n u_4$ and $g = (f_1, f_2, f_3, f_4)$ is a given vector on the surface S.	$2U(X) = -\int_S [R\Phi(Y-X)]U^+(Y)$ $d_y S = F(X), \quad X \in D^+, \; Y \in S,$ $F(X) = \int_S \Phi(X-Y)U^+(Y)d_y S +$ $+\int_{D^+} \Phi(X-Y)H(Y)dY$	Lyapunov Surface (S)	$U^N(X) = -\frac{1}{2}\int [R\Phi(Y-X)] \sum_{k=1}^{N} \Phi_k U^{(k)}(Y) d_y S$ $+\frac{1}{2}F(X), \quad X \in D^+$ $U(X) = \lim_{N\to\infty} U^{(N)}(X)$	[101] [120] [121] [122] [123]
<u>The Dirichlet Boundary Value Problem:</u> $x_3=0 \quad 0 \le x_2 < \infty \quad \varphi = f_1^{(1)}(x_1, x_2)$ $f_2^{(1)}(x_1, x_2) \equiv 0,$ $x_2=0 \quad 0 \le x_3 < \infty \quad \varphi = f_1^{(2)}(x_1, x_3)$ $f_2^{(2)}(x_1, x_3) \equiv 0$	$(\Delta + k^2)\varphi \equiv 0$ $k^2 = \frac{\omega^2}{\lambda + 2\mu}$	Quarter of Space (D^+)	$\varphi(X) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dz_1 \int_0^{+\infty}\left[\frac{\partial}{\partial n(Z)} \frac{e^{i\lambda_1 \lvert X-Z\rvert}}{\lvert X-Z\rvert} - \frac{\partial}{\partial n(Z)} \frac{e^{i\lambda_1 \lvert X-Z\rvert}}{\lvert X-Z\rvert}\right] \times$ $\times f_1^{(1)}(z_1 - z_2)dz_2 + \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\zeta_1 \int_0^{+\infty} \frac{\partial}{\partial n(\zeta)} \frac{e^{i\lambda_1 \lvert X-Z\rvert}}{\lvert X-Z\rvert}$ $-\frac{\partial}{\partial n(\zeta)} \frac{e^{i\lambda_1 \lvert X-Z\rvert}}{\lvert X-Z\rvert} f_1^{(2)}(\zeta_1, \zeta_3)d\zeta_3$	[125]

Table III (continued)

Boundary-value problem	Equations Solving the Problem	Surface, space	Closed-form Solution	References
The Neumann Boundary Value Problem: $x_3=0,\ 0\leq x_2<\infty\ \frac{\partial\varphi}{\partial n}=$ $=\frac{\partial\upsilon_1}{\partial n}=f_1^{(1)}(x_1,x_2)$ $f_2^{(1)}(x_1,x_2)\equiv 0$ $x_2=0,\ 0\leq x_3<\infty\ \frac{\partial\varphi}{\partial n}=$ $=\frac{\partial\upsilon_1}{\partial n}=f_1^{(2)}(x_1,x_3)$ $f_2^{(2)}(x_1,x_3)=0$	$(\Delta+k^2)\varphi\equiv 0$	- " -	$\varphi(X)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dz_1\int_0^{+\infty}\Big[\frac{e^{i\lambda_1\lvert X-Z\rvert}}{\lvert X-Z\rvert}+\frac{e^{i\lambda_1\lvert X-\overline{Z}\rvert}}{\lvert X-\overline{Z}\rvert}\Big]\times$ $\times f_1^{(1)}(z_1,z_2)dz_2+\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\zeta_1\int_0^{+\infty}\Big[\frac{e^{i\lambda_1\lvert X-Z\rvert}}{\lvert X-Z\rvert}+$ $+\frac{e^{i\lambda_1\lvert X-\overline{Z}\rvert}}{\lvert X-\overline{Z}\rvert}\Big]f_1^{(2)}(\zeta_1,\zeta_2)d\zeta_3$	[126]

discontinuities and, more generally, when a high gradient of stresses or displacements is encountered. These reasons justify why in such cases the higher-polynomial approximation of the unknown function can increase the accuracy very sensibly.

Generally speaking, by properly using the Somigliana identity we obtain the following functional equation:

$$\int_S \Gamma(\mathbf{X}, \mathbf{Y})\, f(\mathbf{X})\, dS = \int_S T^{(n)} \Gamma(\mathbf{X}, \mathbf{Y})\, \varphi(\mathbf{X})\, dS\,, \tag{91}$$

which describes either the first, or the second fundamental problem of elasticity, depending on whether the function $f(\mathbf{X})$, or $\varphi(\mathbf{X})$, is considered as known along the boundary.

If the reference points $\mathbf{Y}$ are taken along a definite surface S_0, surrounding S, the functional equation (91) is transformed into a Fredholm regular integral equation. If this surface S_0 is taken sufficiently far from the boundary, the integrals appearing in this equation may be evaluated by using the classical cubature formulas.

A standard method to solve an integral equation is to discretize the integrals by using convenient integration rules and to apply the resulting equation to a number of appropriate collocation points $\{y_p\}_{p=1}^{l} \in S_0$, y_p being the point with coordinates $y_{p_1}, y_{p_2}, y_{p_3}$. Thus, the integral equation (91) may be reduced in a simple way to the system of linear equations:

$$\left.\begin{aligned} &\sum_{k=1}^{l} A_k \Gamma(\mathbf{X}_k, \mathbf{Y}_p)\, f(\mathbf{X}_k) = B_p\,, \\ &B_p = \int_S T^{(n)} \Gamma(\mathbf{X}, \mathbf{Y}_p)\, \varphi(\mathbf{X})\, dS\,, \\ &p = 1, 2, \ldots, n\,; \end{aligned}\right\} \tag{92}$$

with unknowns the values of the distribution $f(\mathbf{X})$ at the integration points $\mathbf{X}_k(X_{k1}, X_{k2}, X_{k3})$, A_k being the weights of the integration rule. The right-hand side, B_p, of equation (92) may be evaluated either analytically, if it is possible, or by using any appropriate integration rule.

In order to examine the validity of the proposed method, the above-described procedure has been applied for the solution of some static and dynamic steady-state problems concerning a prolate spheroid [**171**]. As material of the spheroid was considered a material which has the following Lamé's constants:

$$\lambda = 5600\,\mathrm{MPa}, \quad \mu = 26000\,\mathrm{MPa} \tag{93}$$

and a density:

$$\varrho = 2700\,\mathrm{kgm^{-3}}. \tag{94}$$

The collocation points are selected at positions which correspond to the integration-points positions, on the surface of self-similar spheroids S_0. It is worthwhile noting that the above described choice of collocation points improves the stability of the linear system. This can be explained by the fact that the diagonal terms of the linear system become in this way greater, due to the presence of the singularity $1/r$ in the kernel of the integral equation and to the correspondence between integration and collocation points. Several boundary-value problems concerning a spheroidal surface, which is deformed by a constant or a cosinusoidal strain, applied on its surface along a given direction, are systematically investigated in ref. [**171**].

CONCLUSIONS

A review of certain static and dynamic problems of three-dimensional elasticity is presented. The general method is based on the analytical solutions of certain singular integral equations, which are directly obtained from Kupradze's functional equations.

Analytic solutions have been obtained for the solution of a large number of axisymmetric and non-axisymmetric boundary-value problems of the 3—D theory of elasticity and thermoelasticity.

On the other hand, the methods of solution already presented here can be extended to include 3—D crack problems [**177** to **181**].

In the future, the following directions of research seems to be of potential interest:

a. Extension of the analytical methods, which are valid for the solution of the 3—D problems concerning smooth surfaces, in order to include crack problems.

b. Proof of the convergence of the numerical methods, which are presented in this paper.

c. Experimental investigations of the deformation of cracked 3—D bodies.

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ΠΕΡΙΛΗΨΙΣ

Ἡ ἔρευνα εἰς τὸν τομέα τῆς Τριδιαστάτου Ἐλαστικότητος ἀρχίζει εἰς τὰς ἀρχὰς τοῦ 19ου αἰῶνος μὲ τὰς ἐργασίας τῶν Cauchy, Lamé καὶ Poisson. Διὰ τῶν ἐργασιῶν αὐτῶν, ἐπιτυγχάνεται ἡ ἀναγωγὴ τοῦ γενικοῦ τριδιαστάτου προβλήματος εἰς τὴν ἐπίλυσιν καθωρισμένων συστημάτων διαφορικῶν ἐξισώσεων, ὁριζομένων καταλλήλως ἐπὶ τῆς λείας ἐπιφανείας τριδιαστάτου ἐγκλείσματος. Ἡ μετέπειτα ἔρευνα παρέσχε τὴ δυνατότητα διαμορφώσεως συστημάτων διαφορικῶν ἐξισώσεων καταλλήλων διὰ τὰς περιπτώσεις στατικῶν, δυναμικῶν, στατικῶν - θερμοελαστικῶν καὶ δυναμικῶν - θερμοελαστικῶν τρισδιαστάτων συνοριακῶν προβλημάτων τῆς συμμέτρου ἢ ἀσυμμέτρου ἐλαστικότητος (ἐλαστικότητος τύπου Cosserat).

Παρατηρεῖται ὅτι ἡ μορφὴ τῶν ἐξισώσεων αὐτῶν παραμένει ἀναλλοίωτος ἀνεξαρτήτως τοῦ ἐπιλυομένου προβλήματος. Τοιουτοτρόπως καθίσταται ἐφικτὴ ἡ ἑνοποιημένη ἀντιμετώπισις τῶν διαφόρων τύπων τρισδιαστάτων συνοριακῶν προβλημάτων (τύπου $I^{\pm}$, $II^{\pm}$, $III^{\pm}$, $IV^{\pm}$), οἱ ὁποῖοι ἀφοροῦν τὴν λείαν ἐπιφάνειαν S τοῦ τυχόντος ἐγκλείσματος D.

Διὰ τὴν ἐπίλυσιν τῶν ἀναφερθέντων συστημάτων διαφορικῶν ἐξισώσεων προετάθησαν διάφοροι ἀριθμητικαὶ μέθοδοι, αἱ σημαντικώτεραι τῶν ὁποίων εἶναι ἡ μέθοδος τῶν πεπερασμένων στοιχείων καὶ ἡ μέθοδος τῆς ἀναγωγῆς τοῦ προβλήματος εἰς τὴν ἐπίλυσιν συστημάτων ἰδιομόρφων διδιαστάτων ὁλοκληρωτικῶν ἐξισώσεων.

Εἰς προηγουμένην ἀνακοίνωσιν εἰς τὴν Ἀκαδημίαν, κατὰ τὴν συνεδρίαν τῆς 24 Ἰανουαρίου 1980, ἐδόθησαν συνοπτικοὶ πίνακες περιέχοντες τὰ συστήματα διδιαστάτων ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων, τὰ ὁποῖα ἐπιλύουν τὰ ἀνωτέρω ἀναφερθέντα συνοριακὰ προβλήματα διὰ τὰς περιπτώσεις στατικῶν, δυναμικῶν, στατικῶν - θερμοελαστικῶν καὶ δυναμικῶν - θερμοελαστικῶν προβλημάτων τῆς συμμέτρου ἢ ἀσυμμέτρου ἐλαστικότητος.

῎Ηδη, γεννᾶται τὸ ἐρώτημα τῆς ἐπιλύσεως ὑπὸ κλειστὴν μορφὴν τῶν ἀνωτέρω προταθέντων συστημάτων διδιαστάτων ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων.

῾Η γνῶσις τοῦ ἀριθμοῦ καὶ τῆς ποικιλίας τῶν προβλημάτων, τὰ ὁποῖα δύνανται νὰ δεχθοῦν κλειστὴν λύσιν εἶναι ἄκρως ἐνδιαφέρουσα διότι:

(i) ᾿Επιτρέπει τὴν μελέτην τῆς εὐσταθείας τῆς ἐπιλύσεως τῶν τριδιαστάτων συνοριακῶν προβλημάτων διὰ τῆς χρήσεως προσεγγιστικῶν μεθόδων.

(ii) ᾿Εκ τῶν κλειστῶν λύσεων συζυγῶν συνοριακῶν προβλημάτων, αἱ ὁποῖαι ἀφοροῦν τὴν ἐπιφάνειαν τριδιαστάτου ἐγκλείσματος προκύπτει εὐκόλως κλειστὴ λύσις διὰ τὸ ἀντίστοιχον πρόβλημα τῆς τριδιαστάτου ρωγμῆς. ῾Η μεθοδολογία αὐτὴ ἀναπτύσσεται διὰ πρώτην φορὰν εἰς τὴν παροῦσαν ἀνακοίνωσιν καὶ ἐφαρμόζεται εἰς συγκεκριμένα προβλήματα ρωγμῶν.

(iii) Παρέχει συστηματικὸν ἐργαλεῖον διὰ τὸν ἔλεγχον τῆς ὀρθότητος καὶ τῆς ἀκριβείας τῶν νέων προσεγγιστικῶν ἀριθμητικῶν μεθόδων ποὺ εἶναι πιθανὸν νὰ προταθοῦν μελλοντικῶς.

(iv) ᾿Επιτρέπει κατὰ τρόπον ἀόριστον, τὸν διαμερισμὸν τῶν τριδιαστάτων προβλημάτων εἰς «εὔκολα προβλήματα» καὶ εἰς «δύσκολα προβλήματα» καὶ τοιουτοτρόπως προστατεύει τὸν νέον ἐρευνητὴν ἀπὸ τὴν ἀπότομον ἐνασχόλησίν του μὲ δυσχερέστατα προβλήματα τῆς τριδιαστάτου ἐλαστικότητος.

Διὰ τὴν ὑλοποίησιν τῶν ἀνωτέρω σκοπῶν καταστρώνεται σειρὰ τριῶν πινάκων, ἐκ τῶν ὁποίων ἡ πρώτη κατηγορία περιλαμβάνει τὸ γενικὸν στατικὸν ἢ δυναμικὸν συνοριακὸν πρόβλημα ποὺ λύεται μὲ κλειστὴν μορφὴν δι᾿ ἐφαρμογῆς τῶν γενικευμένων σειρῶν Fourier. ῾Η δευτέρα κατηγορία πινάκων περιλαμβάνει τὰς κλειστὰς λύσεις τῶν ἰδίων προβλημάτων ποὺ ὁρίζονται ἐπὶ τῆς ἐπιφανείας τῆς σφαίρας. ῾Η δὲ τρίτη κατηγορία πινάκων περιλαμβάνει ὡρισμένα κλασσικὰ συνοριακὰ προβλήματα τῆς θερμοελαστικότητος ποὺ δέχονται κλειστὰς λύσεις.

῾Η ταξινόμησις τῶν συνοριακῶν προβλημάτων τῆς Τριδιαστάτου ᾿Ελαστικότητος ποὺ δέχονται κλειστὴν λύσιν ἐπιτρέπει ἀφ᾿ ἑνὸς μὲν νὰ καταστήσωμε συγκεκριμένας τὰς διαφόρους ἀναλυτικὰς μεθόδους ποὺ δύνανται νὰ ἐφαρμοσθοῦν, ἀφ᾿ ἑτέρου δὲ παρέχει τὴν δυνατότητα τῆς ἐξ ἀντικειμένου ἐκτιμήσεως τοῦ βαθμοῦ δυσχερείας διὰ τὴν ἐπίλυσιν πολυπλόκου προβλήματος, ἔστω καὶ ἂν δὲν δέχεται λύσιν κλειστῆς μορφῆς.

Τέλος δι᾿ ἐπεκτάσεως τῶν κλασσικῶν μεθόδων ποὺ ἰσχύουν διὰ τὴν ἐπίλυσιν τοῦ προβλήματος τοῦ τριδιαστάτου ἐγκλείσματος κατέστη δυνατὸν νὰ ἐπιτευχθῇ κλειστὴ λύσις εἰς τὸ πρόβλημα τῆς σφαιρικῆς ρωγμῆς ἐντὸς ἀπείρου καὶ ἰσοτρόπου μέσου.

Ἐκ παραλλήλου, παρέχονται πλήρη βιβλιογραφικὰ στοιχεῖα εἰς τὰ ὁποῖα δύναται νὰ ἀνατρέξῃ ὁ μελετητὴς ὅταν τὰ χρειασθῇ. Ἕκαστον τῶν ἀνωτέρω προβλημάτων ἐπαναλαμβάνεται διὰ διάφορα εἴδη φορτίσεων (στρέψις, κάμψις, ἐφελκυσμὸς κ.λ.π.). Ἐπιπλέον, δίδεται ἡ κλειστὴ λύσις τοῦ προβλήματος τῆς σφαιρικῆς ρωγμῆς δι᾽ ἀναπτύξεως τῆς ἀγνώστου συναρτήσεως εἰς σειρὰς ὀρθογωνίων πολυωνύμων.

Ἄλλη δυνατότης, προκύπτουσα ἐκ τῆς χρήσεως τῶν ἀνωτέρω πινάκων, εἶναι ἡ διατύπωσις τῆς λύσεως τῶν προβλημάτων ποὺ δὲν ἔχουν ἀκόμη ἐρευνηθῆ, διὰ τῆς ἐφαρμογῆς παρεμφερῶν μεθόδων πρὸς ἐκείνας ποὺ ἐμπεριέχονται εἰς τοὺς πίνακας.

Τέλος, ἡ προτεινομένη βιβλιογραφία καλύπτει τὰς πλείστας τῶν δημοσιεύσεων ποὺ ἀφοροῦν κλειστὰς λύσεις τριδιαστάτων συνοριακῶν προβλημάτων, ἐπιλυομένων διὰ τῆς μεθόδου τῶν ἰδιομόρφων διδιαστάτων ὁλοκληρωτικῶν ἐξισώσεων μέχρι καὶ τὸ 1981. Κατὰ συνέπειαν εἶναι χρήσιμος διὰ τὸν ἐρευνητὴν ποὺ ἀσχολεῖται μὲ τὴν περιοχὴν αὐτὴν τῆς Τριδιαστάτου Ἐλαστικότητος.

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