ΜΑΘΗΜΑΤΙΚΑ.— **A numerical method for solving singular integro-differential equations with variable coefficients,** *by P. S. Theocaris and G. Tsamasfyros* *.

A B S T R A C T

A method for the numerical solution of singular integrodifferential equations is presented where the integrals are discretized by using a convenient quadrature rule. Then, the problem is reduced to a system of linear algebraic equations by applying the discretized functional equation to appropriately selected collocation points. This technique constitutes an extension of an analogous method convenient for solving singular integral equations, which was proposed in ref. [**1**].

1. INTRODUCTION

The solution of a large class of boundary-value problems in physics and engineering can be reduced to a system of singular either integral, or integrodifferential equations along a finite part of the real axis. In particular, singular integrodifferential equations are encountered in the theory of wings of finite span [**2**-**4**], or in contact problems of two elastic bodies [**4**-**6**].

Problems expressed in the form of singular integral equations have up-to-now extensively developed. The solution of such problems was mainly based on the method presented in ref. [**7**], where a series-expansion of the unknown function in Jacobi polynomials was used. Then, with the aid of the orthogonality properties of the Jacobi polynomials an infinite system of equations was obtained, which yielded numerically the solution of the problem. A further development of the solution of integral equations with variable coefficients was presented by Dow and Elliott [**8**], where again a series expansion of the unknown function was used. The same problem was generalized in a recent paper by the authors [**1**]. In this paper and a series of similar papers [**9**] to [**14**] a

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n-node Gauss-quadrature rule [**15** - **16**] was used in connection with a special collocation procedure, which yielded a solution accurate to the (2n—1) - degree.

On the contrary, little attention has been focussed to the numerical solutions of integrodifferential equations. Multhopp [**3**] was the first in 1938 to present a numerical solution for a special class of integrodifferential equations. The method makes use of a variable transformation, similar to the transformation used by Kalandiya [**4**] for singular integral equations. The singular integral in the equation of ref. [**4**] was evaluated by applying techniques of interpolation, instead of the efficient Gauss rules of integration. This method was extensively described in the book by Kalandiya [**4**], where its convergence was also proved. Another application of the method was made by Sharfuddin [**5**].

An extension of Multhopp's method has been recently developed, where the procedures developed in refs [**7** - **12**], as well as in [**1**] for solving singular integral equations were introduced to the solution of integrodifferential equations [**17**].

According to this method two different quadrature rules were introduced with n— and (n+k) - nodes respectively. The collocations points of the n-node quadrature rule were used as abscissas for the (n+k)-node rule and vice versa. It was shown in refs. [**1**] and [**9**] that the method was of a 2n-degree of accuracy. This method was first used for the special case where the unknown function $\varphi(x)$ presents singularities of order of $^1/_2$ at the vicinities of the end points of the [1,1] - interval. Then the number of nodes used in the method was (2n—1) and its accuracy was of the same degree although the number of linear equations of the system was equal to (n—1) [**17**].

In the present paper this method of solution was extended to encompass the most general cases for singular integrodifferential equations. Thus, it was succeeded, in this method, to consider unknown functions, which present in the vicinity of points $x = \pm 1$ real and complex singularities with positive real parts. Moreover, in the present method (2n—1) - nodes have been used, while the accuracy attained was of (2n—1) - degree.

Although the results obtained by using Multhopp's method, which is based on a Lagrange-trigonometric interpolation of the unknown function and performs collocation at the interpolation points, and the methods based on Gauss quadrature rules are of the same order of accuracy, the latter methods present the advantage over Multhopp's method to reduce the final system of (2n—1)-linear algebraic equations to a number of (n—1)-equations by performing convenient substitutions. Finally, it should be mentioned here that the superiority of the method of the present paper over the previous method given in ref. [**17**] is the potentiality of generalization of the procedure to a much larger area of problems.

2. THE INTEGRODIFFERENTIAL EQUATION

Since any finite interval can converted to the interval [—1,1] by means of a linear transformation, we assume without loss of generality that the singular integrodifferential equation is of the form:

$$A(x)\,\omega(x) + \frac{B(x)}{\pi}\int_{-1}^{1}\frac{\omega(t)}{t-x}\,dt + C(x)\,\omega'(x) + \frac{D(x)}{\pi}\int_{-1}^{1}\frac{\omega'(t)}{t-x}\,dt +$$

$$+\int_{-1}^{1}k_1(t,x)\,\omega(t)\,dt + \int_{-1}^{1}k_2(t,x)\,\omega'(t)\,dt = f(x), \quad -1<x<1 \tag{1}$$

where the first two integrals are to be considered in the principal-value sense. In relation (1) the quantities A, B, C, D, k_1, k_2 are known functions, satisfying a Hölder condition in each of the variables x and t. It is also considered that the unknown function $\omega(x)$ and its derivatives with respect to t are required to satisfy a Hölder condition, whereas $\omega'(x)$, or its first derivative, are assumed to present an integrable singularity at the end points $t = \pm 1$. Finally, the following two conditions are assumed valid in order that the solution $\omega(x)$ of Eq. (1) be uniquely determined:

$$\omega(-1) = \omega(1) = 0 \tag{2}$$

Supposing that $k_2(t,x)$ is once differentiable with respect to x, be performing an integration by parts of the last term of Eq. (1), the singular integrodifferential equation may be written in operator form as:

$$\mathbf{L}_0\omega + \mathbf{K}_0\omega' + \mathbf{K}\omega = f \tag{3}$$

where the operators $\mathbf{L}_0$, $\mathbf{K}_0$ and $\mathbf{K}$ are expressed by:

$$\mathbf{L}_0\omega\ A(x)\,\omega(x)+\frac{B(x)}{\pi}\int_{-1}^{1}\frac{\omega(t)}{t-x}\,dt \tag{4}$$

$$\mathbf{K}_0\omega'=C(x)\,\omega'(x)+\frac{D(x)}{\pi}\,\frac{d}{dx}\int_{-1}^{1}\frac{\omega(t)}{t-x}\,dt \tag{5}$$

$$\mathbf{K}\omega=\int_{-1}^{1}k(t,x)\,\omega(t)\,dt \tag{6}$$

and the quantity $k(t,x)$ is given by:

$$k(t,x)=\left[k_1(t,x)-\frac{d}{dt}\,k_2(t,x)\right] \tag{7}$$

The singular behavior of the function $\omega'(x)$ around $x=\pm 1$ may be obtained from the dominant par $K_0\omega'$ of the integrodifferential equation (1) by applying a method given in ref. [**18**]. It can be readily established that the fundamental function $Z(x)$, which characterizes the singular behavior of $\omega'(x)$, is given by:

$$Z(x)=w(x)\,\Omega(x) \tag{8}$$

where:

$$w(x)=(1-x)^{\alpha}\,(1+x)^{\beta} \tag{9}$$

and:

$$\alpha=\frac{1}{2\pi i}\log\left[\frac{C(1)-iD(1)}{C(1)+iD(1)}\right]+\mu' \tag{10}$$

$$\beta=-\frac{1}{2\pi i}\log\left[\frac{C(-1)-iD(-1)}{C(-1)-iD(-1)}\right]+\mu'' \tag{11}$$

Moreover, Ω is a non-vanishing in (1,1), Hölder-continuous function, μ' and μ'' are integers, chosen in such a way that the behavior of the fundemental function Z at points $x=\pm 1$ is compatible with the expected singular behavior of the unknown function $\omega'(x)$ (i.e. either bounded, if $0<\mathrm{Re}\alpha$, $\mathrm{Re}\beta<1$, or infinite, but integrable, if $-1<\mathrm{Re}\alpha$, $\mathrm{Re}\beta<0$).

In the case when both C and D become equal to zero the integro-differential equation (1) is reduced to a typical integral equation, whose solution may be obtained by anyone of the well established methods of numerical solution (see for example refs. [**7** to **16**]). The case where C and D are constants was studied in ref. [**10**]. When $C = 0$ and $D \neq 0$ it can be shown that $\alpha = \beta = -{}^1/_2$ and Eq. (1) is reduced to the case which was studied in ref. [**17**].

Thus, by taking into consideration the conditions (2) we can write:

$$\omega(x) = (1-x)^{\mu+1}(1+x)^{\nu+1} g(x) = (1-x)^{\mu}(1+x)^{\nu}\varphi(x) \tag{12}$$

and

$$\omega'(x) = (1-x)^{\mu}(1+x)^{\nu}\left\{[(\nu-\mu)-(\mu+\nu)x]\varphi(x)/(1-x^2)+\varphi'(x)\right\} \tag{13}$$

$$\mu = \begin{cases} \alpha, & \text{if} \quad \alpha < 0 \\ \alpha-1, & \text{if} \quad \alpha > 0 \end{cases}, \qquad \nu = \begin{cases} \beta, & \text{if} \quad \beta < 0 \\ \beta-1, & \text{if} \quad \beta > 0 \end{cases} \tag{14}$$

3. METHOD OF SOLUTION

3.1. The Formation of the First Set of (n–1)-Equations.

Applying methods similar to the methods used for singular integral equations, the integrodifferential equation (1) may be transformed to a Fredholm equation, whose solution can be obtained only by numerical means. However, in order to avoid the above unnecessary operations, a direct approximation method may be developed, which preserves the correct nature of singularities of the unknown function $\omega(x)$. In this method we consider the *Gauss-Jacobi* integration formula with n-nodes [**7**]:

$$\int_{-1}^{1} W(t)\varphi(t)dt \simeq \sum_{j=1}^{n} \lambda_j \varphi(t_j), \quad P_n^{(\mu,\nu)}(t_j) = 0, \quad j = 1, 2, \ldots, n \tag{15}$$

where:

$$W(x) = (1-x)^{\mu}(1+x)^{\nu} \tag{16}$$

and λ_j are the weight coefficients.

Applying an extension of this formula for singular integrals **[10, 15]** we can write the part $\mathbf{L}_0\omega$ of the operator equation (3) as follows:

$$\mathbf{L}_0\omega \simeq \frac{B(x)}{\pi} \sum_{j=1}^{n} \frac{\lambda_j}{t_j - x} (\varphi t_j) + G_n(x)\,\varphi(x) \tag{17}$$

with

$$G_n(x) = A(x)\,W(x) + \frac{B(x)}{\pi}\, q_n^{(\mu,\nu)}(x) \tag{18}$$

and :

$$q_n^{(\mu,\nu)}(x) = -2\Psi_n^{(\mu,\nu)}(x) \,/\, P_n^{(\mu,\nu)}(x) \tag{19}$$

where the Jacobi associated function $\Psi_n^{(\mu,\nu)}$ is defined by :

$$\Psi_n^{(\mu,\nu)}(x) = \frac{1}{2} \int_{-1}^{1} \frac{w(t)\,P_n^{(\mu,\nu)}(t)}{x-t}\, dt\,, \qquad -1 < x < 1 \tag{20}$$

Furthermore, by differentiating with respect to x the above *Gaussian* quadrature rule for singular integrals, we can write :

$$\mathbf{K}_0\omega' \simeq \frac{D(x)}{\pi} \sum_{j=1}^{n} \frac{\lambda_j}{(t_j - x)^2}\, \varphi(t_j) + \left[C(x)\,W(x)\, \frac{\nu-\mu-(\mu+\nu)}{1-x^2} + \right.$$

$$\left. + \frac{D(x)}{\pi}\, q'^{(\mu,\nu)}(x) \right] \varphi(x) + F_n(x)\,\varphi'(x) \tag{21}$$

where :

$$F_n(x) = C(x)W(x) + \frac{D(x)}{\pi}\, q_n^{(\mu,\nu)}(x) \tag{22}$$

The necessary conditions for the function $F_n(x)$ to become zero are explored in ref. **[9]**, whereas in ref. **[10]** the particular case where C and D are constants was examined. In this case it is valid that :

$$F_n(x) = -2^{-\varkappa}\, \frac{D}{\pi}\, \Gamma(\mu)\,\Gamma(1-\mu)\, P_{n-\varkappa}^{(-\mu,-\nu)}(x) \,/\, P_n^{(\mu,\nu)}(x) \tag{23}$$

where the $F_n(x)$-function presents (n—1) - roots.

If the above-mentioned necessary conditions are satisfied, so that there are at least (n—1)-roots at the collocation points $\{x_i\}_{i=1}^{n-1}$, the operator equation (3) can be written as follows:

$$\left[G_n(x_i) + C(x_i)W(x_i)\,\frac{\nu-\mu-(\mu+\nu)x_i}{1-x_i} + \frac{D(x_i)}{\pi}\,q_n'^{(\mu,\nu)}(x_i)\right]\varphi(x_i) +$$

$$+\sum_{j=1}^{n} S(t_j, x_i)\,\varphi(t_j) \simeq f(x_i), \quad i=1(1)n-1 \tag{24}$$

where:

$$S(t_j, x_i) = \frac{\lambda_j}{\pi}\left[\frac{B(x_i)}{t_j-x_i} + \frac{D(x_i)}{(t_j-x_i)} + \pi K(t_j, x_i)\right] \tag{25}$$

Taking into consideration that:

$$\frac{q^{(\mu,\nu)}(x_i)}{\pi} = -\,\frac{C(x_i)W(x_i)}{D(x_i)}, \quad \text{for } D(x_i) \neq 0 \tag{26}$$

relation (24) takes the form:

$$H_1(x_i)\,\varphi(x_i) + \sum_{j=1}^{n} S(t_j, x_i)\,\varphi(t_j) \simeq f(x_i), \quad i = 1(1)n-1 \tag{27}$$

where:

$$H_1(x_i) = W(x_i)\left[A(x_i) - \frac{C(x_i)B(x_i)}{D(x_i)} + C(x_i)\,\frac{\nu-\mu-(\mu+\nu)x_i}{1-x_i^2} + \right.$$

$$\left. + \frac{D(x_i)}{\pi}\,\frac{q_n'^{(\mu,\nu)}(x_i)}{W(x_i)}\right], \quad i = 1(1)n-1 \tag{28}$$

2.2. The Formation of the Second Set of n-Equations.

Relation (27) gives (n—1)-linear equations for the (2n—1)-unknowns $\{\varphi(t_j)\}_{j=1}^{n}$, $\{\varphi(x_i)\}_{i=1}^{n-1}$. The complementary (n+1)-linear equations necessary for completing the solution can be obtained if we use the *Lobatto - Jacobi integration rule* **[15]** at the (n+1)-nodes $\left[\pm 1 \text{ and } \{t_j\}_{j=n+1}^{2n-1}\right]$. Thus, by taking into consideration the zeroing of the integrand at the points ± 1, this rule yields:

$$\int_{-1}^{1} W(t)\,\varphi(t)\,dt \simeq \sum_{j=n+1}^{2n-1} \lambda_j\,\varphi(t_j), \quad P_{n-1}^{(\mu+1,\nu+1)}(t_j) = 0,$$

$$j = n+1,\ n+2, \ldots, 2n \tag{29}$$

On the other hand, by using again the extension of this rule for singular integrals, as presented in refs [**10**] and [**15**], we obtain for the first term of the operator equation (3) that :

$$\mathbf{L}_0\omega \simeq \frac{B(x)}{\pi} \sum_{j=n+1}^{2n-1} \frac{\lambda_j}{t_j - x} \varphi(t_j) + g_n(x)\varphi(x) \tag{30}$$

with :

$$g_n(x) = A(x)W(x) + B(x)\, q_{n-1}^{(\mu+1,\,\nu+1)}(x) \tag{31}$$

$$g_{n-1}^{(\mu+1,\,\nu+1)}(x) = -\frac{2\psi_{n-1}^{(\mu+1,\,\nu+1)}(x)}{(1-x^2)P_{n-1}^{(\mu+1,\,\nu+1)}(x)} \tag{32}$$

In the same way as previoulsy we can write for the second term of the operator equation (3) that :

$$\mathbf{K}_0\omega' \simeq \frac{D(x)}{\pi} \sum_{j=n+1}^{2n-1} \frac{\lambda_j}{(t_j-x)^2} \varphi(t_j) + \left[C(x)W(x) \frac{\nu-\mu-(\mu+\nu)x}{1-x^2} + \right.$$

$$\left. + \frac{D(x)}{\pi} q_{n-1}^{\prime(\mu+1,\,\nu+1)}(x) \right] \varphi(x) + f_n(x)\varphi'(x) \tag{33}$$

where :

$$f_n(x) = C(x)W(x) + \frac{D(x)}{\pi} q_{n-1}^{(\mu+1,\,\nu+1)}(x) \tag{34}$$

If we suppose again that $f_n(x)$ has n-roots and the necessary conditions are satisfied at the collocation points $\{x_i\}_{i=n}^{2n-1}$, we can reduce again the integral equation (3) to the following system of (n−1) equations :

$$\left[g_n(x_i) + C(x_i)W(x_i) \frac{\nu-\mu-(\mu+\nu)x_i}{1-x_i^2} + \frac{D(x_i)}{\pi} q_{n-1}^{\prime(\mu+1,\,\nu+1)}(x_i) \right] \varphi(x_i) +$$

$$+ \sum_{j=n+1}^{2n-1} S(t_j, x_i)\varphi(t_j) \simeq f(x_i), \quad i = n(1)2n-1 \tag{35}$$

or to :

$$H_2(x_i)\varphi(x_i) + \sum_{j=n+1}^{2n-1} S(t_j, x_i)\varphi(t_j) = f(x_i), \quad i = n(1)2n-1 \tag{36}$$

where :

$$H_2(x_i) = W(x_i) \left[A(x_i) - \frac{\pi C(x_i)B(x_i)}{D(x_i)} + C(x_i) \frac{\nu-\mu-(\mu+\nu)x_i}{1-x_i^2} + \right.$$

$$\left. + \frac{D(x_i)}{\pi} \frac{q_{n-1}^{\prime(\mu+1,\,\nu+1)}(x_i)}{W(x_i)} \right], \quad i = n(1)2n-1 \tag{37}$$

Relations (27) and (36) [or (24) and (35)] constitute a linear system of (2n—1)-equations with 2(2n—1)-unknowns, that is the values of the function $\varphi(x)$ at the points x_i and t_j $\{\varphi(x_i)\}_{i=1}^{2n-1}$ and $\{\varphi(t_j)\}_{j=1}^{2n-1}$.

Taking into consideration the fact that relations (27) and (36) are exact for $\varphi(t) \in P_{2n-1}$ (where P_{2n-1} is the class of polynomials of degree $\leqq 2n-1$) it is easy to express the set $\{\varphi(x_i)\}_{i=1}^{2n-1}$ by the set $\{\varphi(t_j)\}_{j=1}^{2n-1}$ by using a Lagrangian interpolation. In particular, by selecting as interpolation points the points ± 1, $\{t_j\}_{j=1}^{2n-1}$ and taking into consideration relation (12), we have:

$$\varphi(x) \simeq \sum_{j=1}^{2n-1} L_j(x)\,\varphi(t_j) \qquad (38)$$

where:

$$L_j(x) = \frac{\pi_{2n+1}(x)}{\left((x-t_j)\,\pi'_{2n+1}\,t_j\right)}\,, \quad \pi_{2n+1}(x) = (x^2-1)\,P_n^{(\mu,\nu)}(x)\,P_{n-1}^{(\mu+1,\nu+1)}(x) \qquad (39)$$

The last interpolation is exact for any $\varphi(t) \in P_{2n}$, i.e. for polynomials of degree by a unity greater than the degree of accuracy of relations (27) and (36).

3.3. Numerical Solution of the Set of (2n—1)-Equations.

Relations (27) and (36) may be written as follows, if we introduce also to this set of equations relations (38):

$$\sum_{j=1}^{n}\{H_1(x_i)\,L_j(x_i) + S(t_j, x_i)\}\,\varphi(t_j) + H_1(x_i)\sum_{j=n+1}^{2n-1} L_j(x_i)\,\varphi(t_j) = f(x_i)\,,$$

$$i = 1(1)\,n-1 \qquad (40)$$

$$H_2(x_i)\sum_{j=i}^{n} L_j(x_i)\,\varphi(t_j)\sum_{j=n+1}^{2n-1}\{H_2(x_i)\,L_j(x_i) + S(t_j, x_i)\}\,\varphi(t_j) = (f\,x_i)\,,$$

$$i = n(1)\,2n-1 \qquad (41)$$

Eqs. (40) and (41) constitute an algebraic system of (2n—1)-linear equations with an equal number of unknowns, the values of the unknown function $\varphi(t)$ at the points $\{t_j\}_{j=1}^{2n-1}$. This system may be solved and then, using the intepolation formula (38), we can determine the values of $\varphi(t)$ at any point in the interval $[-1,1]$.

In the particular case where C and D are constants and $\mu = \alpha$ and $\nu = \beta$, that is the index of the function $Z(x)$ is unity, the expressions $H_1(x_i)$ and $H_2(x_1)$ may be transformed to :

$$H_1(x_i) = (1-x_i)^\alpha (1+x_i)^\beta \left[A(x_i) - B(x_i)\frac{C}{D}\right] - \frac{D(n+1)}{4\sin\pi\alpha} \cdot \frac{P_{n-2}^{(1-\alpha,\,1-\beta)}(x_i)}{P_n^{(\alpha,\,\beta)}x_i}$$

$$i = 1\,(1)\,n-1 \tag{42}$$

$$H_2(x_i) = (1-x_i)^\alpha (1+x_i)^\beta \left[A(x_i) - B(x_i)\frac{C}{D}\right] - \frac{Dn}{\sin\pi\alpha} \cdot \frac{P_{n-1}^{(-\alpha,\,-\beta)}(x_i)}{(1-x_i^2)\,P_{n-1}^{(\alpha+1),\,(\beta+1)}(x_i)}$$

$$i = n\,(1)\,2n-1 \tag{43}$$

On the other hand, the integration points of the Gauss quadrature rule are given as the roots of the Jacobi polynomial $P_n^{(\alpha,\,\beta)}$, whereas the corresponding collocation points as the roots of the Jacopi polynomial $P_{n-1}^{(-\alpha,\,-\beta)}$. The integration points and the collocation points of the Lobatto-quadrature rule are given respectively as the roots of the Jacobi polynomials $P_{n-1}^{(\alpha+1,\,\beta+1)}$, $P_n^{(-\alpha-1,\,-\beta-1)}$. Taking into consideration that $\varkappa = 1$, the last polynomials coincide with the polynomials $P_{n-1}^{(-\beta,\,-\alpha)}$, $P_{n+1}^{(\beta,\,\alpha)}$. But, the roots of the last polynomials may be obtained by taking respectively the opposite valyes of the roots of the polynomials $P_{n-1}^{(-\alpha,\,-\beta)}$, $P_{n+1}^{(\alpha,\beta)}$. In this way we can write that :

$$\left.\begin{aligned} -t_{2n-i} &= x_i\,, \qquad i = 1\,(1)\,n-1 \\ -t_{2n-j} &= t_j\,, \qquad j = 1\,(1)\,n \end{aligned}\right\} \tag{44}$$

The importance of this remark appears when the $\varphi(x)$— and $\varphi(-x)$— functions are known a priori to be related with a given formula (i.e. if $\varphi(x)$ is an odd or an even function). In this case we can replace the values of $\{\varphi(t_j)\}_{j=1}^{n}$ from relation (36) to relation (27), or, inversely, the values of $\{\varphi(x_i)\}_{i=1}^{n-1}$ from relation (27) to relation (36). Thus, we obtain a system of only n— or (n—1)-linear algebraic equations with an equal number of unknowns.

TABLE

Numerical results from the solution of Eq. (46) by applying the method developed in the paper for several abscissas at the integration interval [—1, 1].

x	φ (x)		
	n = 5	n = 10	n = 15
—1.00	0.000000 (0)	0.000000 (0)	0.000000 (0)
—0.90	—0.815112 (0)	—0.815139 (0)	—0.815122 (0)
—0.80	—0.167040 (1)	—0.166998 (1)	—0.166996 (1)
—0.70	—0.252861 (1)	—0.252777 (1)	—0.252774 (1)
—0.60	—0.336177 (1)	—0.336081 (1)	—0.336076 (1)
—0.50	—0.414541 (1)	—0.414432 (1)	—0.414426 (1)
—0.40	—0.485665 (1)	—0.485536 (1)	—0.485529 (1)
—0.30	—0.547407 (1)	—0.547258 (1)	—0.547250 (1)
—0.20	—0.597797 (1)	—0.597632 (1)	—0.597624 (1)
—0.10	—0.635069 (1)	—0.634890 (1)	—0.634880 (1)
0.00	—0.657688 (1)	—0.657492 (1)	—0.657482 (1)
0.10	—0.664380 (1)	—0.664168 (1)	—0.664158 (1)
0.20	—0.654176 (1)	—0.653963 (1)	—0.653952 (1)
0.30	—0.626474 (1)	—0.626277 (1)	—0.626267 (1)
0.40	—0.581098 (1)	—0.580926 (1)	—0.580917 (1)
0.50	—0.518354 (1)	—0.518194 (1)	—0.518184 (1)
0.60	—0.439073 (1)	—0.438899 (1)	—0.438891 (1)
0.70	—0.344677 (1)	—0.344501 (1)	—0.344494 (1)
0.80	—0.237353 (1)	—0.237245 (1)	—0.237239 (1)
0.90	—0.120529 (1)	—0.120497 (1)	—0.120492 (1)
1.00	—0.397877 (—14)	—0.394468 (—14)	—0.393855 (—14)

4. A. NUMERICAL EXAMPLE

As an illustration of the method we consider the simple singular integrodifferential equation :

$$\omega(x) + \cos(c\pi)\,\omega'(x) - \frac{\sin(c\pi)}{\pi}\int_{-1}^{1}\frac{\omega'(t)}{t-x}\,dt = 1, \quad -1<x<1, \quad c=-0.6 \tag{45}$$

In this equation the derivative $\omega'(x)$ has integrable singularities at $x=\pm 1$, $A(x)=1$, $B(x)=0$, $C(x)=\cos(c\pi)$, $D(x)=\sin(c\pi)$ and $f(x)=1$. Finally the exponents α and β are determined to be $\alpha=-0.6$ and $\beta=-0.4\,(\varkappa=1)$.

This equation was solved numerically by the method proposed in this paper for $n=5, 10, 15$. In order to obtain a comparison of the numerical results derived from this solution we have further used the interpolation formula (38), so that the values of $\varphi(x)$ at arbitrarily selected abscissas x are obtained. The values for the $\varphi(x)$-function are listed in the Table. Since there is no other means of comparing the accuracy of the method, the only indication is the stability of the values for $\varphi(x)$ at the various abscissas as the number of integration points is increased. Indeed, the differences between corresponding values at various x's differ only at the fourth decimal number between steps with $n=5$ and $n=10$, whereas for the steps between $n=10$ and $n=15$ the differences in respective values for $\varphi(x)$ appear only in the fifth decimal number. However, the speed of convergence appears to be similar with the special case of ref. [**17**], where the results were checked directly with Multhopp's solution.

ΠΕΡΙΛΗΨΙΣ

Ἡ λύσις μεγάλης κατηγορίας προβλημάτων συνοριακῶν τιμῶν εἰς τὰς φυσικὰς ἐπιστήμας καὶ τὰς ἐπιστήμας τοῦ μηχανικοῦ δύναται ν' ἀναχθῇ εἰς σύστημα ἰδιομόρφων εἴτε ὁλοκληρωτικῶν, εἴτε ὁλοκληροδιαφορικῶν ἐξισώσεων κατὰ μῆκος πεπερασμένου τμήματος τοῦ πραγματικοῦ ἄξονος. Εἰδικῶς αἱ ἰδιόμορφοι ὁλοκληροδιαφορικαὶ ἐξισώσεις συναντῶνται εἰς τὴν θεωρίαν τῶν πτερύγων πεπερασμένου ἀνοίγματος εἰς τὴν ἀεροδυναμικὴν καθὼς καὶ εἰς τὰ προβλήματα ἐπαφῆς δύο ἐλαστικῶν σωμάτων εἰς τὴν Μηχανικήν.

Προβλήματα ἐκφραζόμενα ὑπὸ μορφὴν ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων ἔχουν μέχρι σήμερον πλήρως ἀναπτυχθῆ. Ἡ λύσις τοιούτων προβλημάτων βασίζεται κυρίως εἰς μέθοδον παρουσιασθεῖσαν ὑπὸ τοῦ Καθηγητοῦ Erdogan κατὰ τὸ ἔτος 1973 καὶ τῶν συνεργατῶν του εἰς τὸ Πανεπιστήμιον Lehigh τῶν Ἡνωμένων Πολιτειῶν.

Κατὰ τὴν μέθοδον αὐτὴν ἡ ἄγνωστος μιγαδικὴ συνάρτησις ἀναπτύσσεται εἰς σειρὰν πολυωνύμων Jacobi. Ἐν συνεχείᾳ, καὶ δι' ἐφαρμογῆς τῶν ἰδιοτήτων ὀρθογωνικότητος τῶν πολυωνύμων αὐτῶν, σύστημα ἀπείρων ἐξισώσεων διαμορφοῦται, τὸ ὁποῖον παρέχει τὴν ἀριθμητικὴν λύσιν τοῦ προβλήματος. Περαιτέρω ἀνάπτυξις τῆς μεθόδου λύσεως ὁλοκληρωτικῶν ἐξισώσεων μὲ μεταβλητοὺς συντελεστὰς εἰσήχθη ὑπὸ τοῦ Elliott εἰς τὴν Αὐστραλίαν τὸ ἔτος 1979. Κατὰ τὴν μέθοδον τοῦ Elliott χρησιμοποιεῖται καὶ πάλιν ἀνάπτυξις εἰς σειρὰν τῆς ἀγνώστου συναρτήσεως. Τὸ αὐτὸ πρόβλημα ἐγενικεύθη τελευταίως ὑπὸ τῶν συγγραφέων. Εἰς τὴν ἀνακοίνωσιν αὐτὴν καθὼς καὶ εἰς σειρὰν σχετικῶν δημοσιεύσεων χρησιμοποιεῖται κανὼν τετραγωνισμοῦ κατὰ Gauss n-ἀριθμοῦ κόμβων ἐν συνδυασμῷ μὲ διαδικασίαν εἰδικῆς ταξιθεσίας, ἡ ὁποία παρέχει λύσιν τοῦ προβλήματος ἀκριβείας $(2n-1)$ - βαθμοῦ.

Ἐν ἀντιθέσει, ἐλαχίστη προσπάθεια κατεβλήθη μέχρι σήμερον διὰ τὴν ἀριθμητικὴν ἐπίλυσιν ὁλοκληροδιαφορικῶν ἐξισώσεων. Ὁ Multhopp ὑπῆρξεν ὁ πρῶτος ὁ ὁποῖος κατὰ τὸ ἔτος 1938 παρουσίασε ἀριθμητικὴν λύσιν εἰδικῆς κατηγορίας ὁλοκληροδιαφορικῶν ἐξισώσεων. Ἡ μέθοδος κάνει χρῆσιν μετασχηματισμοῦ μεταβλητῶν, ὁμοίου μὲ τὸν μετασχηματισμὸν ποὺ χρησιμοποιεῖ ὁ Ρῶσσος ἐπιστήμων Kalandiya διὰ τὴν ἐπίλυσιν ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων. Τὸ ἰδιόμορφον ὁλοκλήρωμα τῆς ἐξισώσεως τοῦ Kalandiya ἐκτιμᾶται δι' ἐφαρμογῆς τεχνικῶν παρεμβολῆς, ἀντὶ τῆς χρησιμοποιήσεως τῶν δυναμικῶν κανόνων ὁλοκληρώσεως κατὰ Gauss. Ἡ μέθοδος αὐτὴ περιγράφεται ἐν ἐκτάσει εἰς τὸ βιβλίον τοῦ Kalandiya, εἰς τὸ ὁποῖον ἀποδεικνύεται καὶ ἡ σύγκλισις τῶν χρησιμοποιουμένων σειρῶν.

Ἐπέκτασις τῆς μεθόδου τοῦ Multhopp παρουσιάσθη τελευταίως εἰς τὴν διεθνῆ βιβλιογραφίαν, ὅπου αἱ διαδικασίαι αἱ ἀναπτυχθεῖσαι ὑπὸ τοῦ ὁμιλοῦντος καὶ τῆς ὁμάδος του διὰ τὴν λύσιν ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων ἐπεξετάθησαν καὶ διὰ τὴν λύσιν ὁλοκληροδιαφορικῶν ἐξισώσεων.

Συμφώνως πρὸς τὴν μέθοδον αὐτὴν δύο διάφοροι κανόνες τετραγωνισμοῦ εἰσάγονται μὲ n — καὶ $(n+k)$ - κόμβους ἀντιστοίχως. Τὰ σημεῖα ταξιθεσίας τοῦ κανόνος τετραγωνισμοῦ n - κόμβων ἐχρησιμοποιήθησαν ὡς τετμημέναι τοῦ κανόνος (n+k)-κόμβων καὶ ἀντιστρόφως. Ἀπεδείχθη ἐπίσης, ὅτι ἡ μέθοδος ἦτο ἀκριβὴς μὲ ἀκρίβειαν 2n-βαθμοῦ. Ἡ μέθοδος αὐτὴ ἐχρησιμοποιήθη κατὰ πρῶτον εἰς τὴν εἰδικὴν περίπτωσιν, ὅπου ἡ ἄγνωστος συνάρτησις παρουσιάζει ἰδιομορφίας τῆς τάξεως $^1/_2$ εἰς τὴν γειτονίαν τῶν ἀκραίων σημείων τοῦ διαστήματος $[-1,1]$. Ὁ ἀριθμὸς τῶν κόμβων ποὺ ἐχρησιμοποιήθησαν εἰς τὴν μέθοδον ἦτο $(2n-1)$ καὶ ἡ ἀκρίβεια τῆς μεθόδου ἦτο τοῦ αὐτοῦ βαθμοῦ, ἂν καὶ ὁ ἀριθμὸς τοῦ προκύπτοντος συστήματος γραμμικῶν ἐξισώσεων ἦτο ἴσος πρὸς $(n-1)$.

Εἰς τὴν παροῦσαν ἀνακοίνωσιν ἡ μέθοδος ἐπιλύσεως αὐτὴ ἐπεκτείνεται διὰ νὰ συμπεριλάβῃ τὴν γενικὴν περίπτωσιν ἰδιομόρφων ὁλοκληροδιαφορικῶν ἐξισώσεων. Κατ' αὐτὸν τὸν τρόπον διὰ τῆς ἐπεκτάσεως αὐτῆς, ἐπετεύχθη νὰ μελετηθοῦν ἄγνωστοι συναρτήσεις, αἱ ὁποῖαι παρουσιάζουν εἰς τὴν γειτονίαν τῶν σημείων $x=\pm 1$ πραγματικὰς ἢ καὶ μιγαδικὰς ἰδιομορφίας μὲ θετικὰ πραγματικὰ μέρη. Περαιτέρω, κατὰ τὴν παροῦσαν μέθοδον ὁ ἀριθμὸς τῶν χρησιμοποιηθέντων κόμβων ἦτο $(2n-1)$ καὶ ἡ ἀκρίβεια τῆς μεθόδου ἦτο τῆς τάξεως $(2n-1)$.

Παρ' ὅλον ὅτι τὰ ἀποτελέσματα ποὺ ἐπιτυγχάνονται διὰ χρήσεως τῆς μεθόδου Multhopp, ἡ ὁποία στηρίζεται εἰς τριγωνομετρικὴν παρεμβολὴν τύπου Lagrange τῆς ἀγνώστου συναρτήσεως καὶ ἐκτελεῖ ταξιθεσίαν εἰς τὰ σημεῖα παρεμβολῆς, καὶ τὰ ἀποτελέσματα τὰ προκύπτοντα δι' ἐφαρμογῆς τῶν μεθόδων ποὺ στηρίζονται στοὺς κανόνας τετραγωνισμοῦ κατὰ Gauss παρουσιάζουν ἀκρίβειαν τῆς αὐτῆς τάξεως, αἱ τελευταῖαι μέθοδοι, αἱ βασιζόμεναι εἰς τοὺς κανόνας τετραγωνισμοῦ Gauss, παρουσιάζουν τὸ πλεονέκτημα, ἐν συγκρίσει μὲ τὴν μέθοδον Multhopp, ὅτι ὑποβιβάζουν τὸν ἀριθμὸν τῶν ἐξισώσεων τοῦ τελικοῦ συστήματος ἀπὸ $(2n-1)$-γραμμικὰς ἀλγεβρικὰς ἐξισώσεις εἰς τὸν ἀριθμὸν τῶν $(n-1)$-ἐξισώσεων, δι' εἰσαγωγῆς καταλλήλων ἀντικαταστάσεων.

Ἐν κατακλεῖδι ἀναφέρεται ἡ ὑπεροχὴ τῆς παρούσης μεθόδου, ἐν συγκρίσει πρὸς τὴν ἤδη παρουσιασθεῖσαν εἰδικὴν περίπτωσίν της ὡς πρὸς τὴν γενίκευσίν της, διὰ τὴν ἐπίλυσιν μεγάλου ἀριθμοῦ προβλημάτων.

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