ἐπιπέδου, ὁριζομένων ὑφ' ἑκάστου τῶν συστημάτων τούτων καὶ περιγραφόντων δύο καμπύλας τοῦ ἐπιπέδου, συνιστᾷ μίαν σχέσιν ἰσοδυναμίας. Ἡ τελευταία προκαλεῖ κατανομὴν τῶν καμπυλῶν τοῦ ἐπιπέδου εἰς διακεκριμένα ἀλλήλων σύνολα. Ἀποδεικνύεται ἐν συνεχείᾳ κατὰ τρόπον γενικόν, ὅτι τὰ σύνολα ταῦτα συνιστοῦν ὁμάδας ὑπὸ ὡρισμένον νόμον συνθέσεως.

Ἡ διαφορὰ καὶ ἡ δυσκολία τῆς ἐργασίας ταύτης ἐν σχέσει πρὸς τὴν ἀνακοινωθεῖσαν, ὡς ἀνωτέρω, κατὰ τὴν συνεδρίαν τῆς 28ης Φεβρ. 1957 ἔγκειται εἰς τὴν λῆψιν τυχούσης καμπύλης ὡς ἀρχικῆς καὶ εἰς τὴν χρησιμοποίησιν συστημάτων συντεταγμένων, τῶν ὁποίων οἱ τύποι μετασχηματισμοῦ εἶναι ἐπίσης γενικευμένοι. Κατὰ τὴν πορείαν τῆς ἀποδείξεως τοῦ θεωρήματος κατανομῆς ἐν τῇ γενικῇ του μορφῇ προέκυψαν θέματα συμβολικῆς παραστάσεως ὡρισμένων τύπων, τῶν ὁποίων τὸ μέγεθος καθίστα ἄκρως δύσκολον τὴν ἀνάπτυξιν τῆς ἀποδείξεως. Διὰ τῆς χρησιμοποιήσεως καταλλήλου συμβολισμοῦ παρεκάμφθησαν αἱ δυσκολίαι καὶ ἐπετεύχθη ἡ ἀπόδειξις τοῦ θεωρήματος ὑπὸ τὴν γενικήν του μορφήν.

Τέλος γίνεται γεωμετρικὴ ἑρμηνεία τοῦ γενικευμένου τούτου θεωρήματος ἐπὶ τοῦ Εὐκλειδείου ἐπιπέδου. Αἱ παριστώμεναι καμπύλαι κατατάσσονται εἰς δύο ὑποσύνολα. Ἑκάστη καμπύλη τοῦ ἑνὸς ὑποσυνόλου ἔχει τὴν γεωμετρικῶς «ἀντίστροφόν» της εἰς τὸ ἄλλο. Ἀξιοσημείωτον εἶναι, ὅτι ἡ γεωμετρικὴ πρᾶξις αὕτη ἐφαρμοζομένη ἐπὶ δύο οἱωνδήποτε ἀντιστρόφων καμπυλῶν ὁδηγεῖ εἰς τὴν κατασκευὴν τῆς ἀρχικῆς καμπύλης, ἡ ὁποία ἀποτελεῖ ἓν εἶδος «στοιχείου ταυτότητος» τῆς ὅλης ὁμάδος καμπυλῶν.

ΜΑΘΗΜΑΤΙΚΗ ΑΝΑΛΥΣΙΣ.— **On the evaluation of double integrals containing a large parameter,** *by* ***N. Chako*** *. Ἀνεκοινώθη ὑπὸ τοῦ κ. Ἰωάνν. Ξανθάκη.

I. INTRODUCTION

In this paper we shall be concerned with the evaluation of double integrals of the type

$$\text{(A)} \qquad U(k) = \iint_D g(x,y)\, e^{ik\Phi(x,y)}\, dxdy,$$

for large values of the real parameter k. The amplitude and phase functions, g(x,y) and $\Phi(x,y)$ are real in the real variables (x,y) subject to certain restrictions which will be specified later, and D is a finite domain of integration. Integrals of this type occur often in mathematical physics, especially in diffraction and scattering problems (1 - 3).

The method which will be developed here for evaluating integrals of this kind will follow closely the method developed by Poincaré (4) and

* N. CHAKO, Ἐπὶ τοῦ προσδιορισμοῦ διπλῶν ὁλοκληρωμάτων μὲ μεγάλας παραμέτρους.

by Picard (5) for evaluating rational functions in two variables. The integrals considered by them are of the form

$$(B) \qquad I = \iint F(x,y)\, dxdy = \iint \frac{g(x,y)}{h(x,y)}\, dxdy$$

where g and h are polynomials in x and y.

In section II the method of Poincaré and Picard for finding the residues of such integrals is briefly outlined. In sect. III a procedure is given for extending their results to double integrals of exponential type (A), when k is large. Explicit expressions for the leading term of (A) are given for cases when the phase function has stationary points within or on the boundary of the domain of integration, which in most practical applications is the required approximation. These expressions are the same as the leading terms found by the method of stationary phase. Furthermore, we have indicated a procedure of obtaining the higher order terms of the asymptotic expansion of U(k). The coefficients of the higher terms of the asymptotic series are expressed by Abelian integrals of exponential type in a single variable. In the last section two typical examples taken from the diffraction theory of aberrations are evaluated by this method.

II. THE METHOD OF POINCARÉ AND PICARD.

The method developed by Poincaré and by Picard for finding the residues of double integrals of the form (B) is briefly as follows. They consider (B) as an integral in a two-dimensional complex space by letting $x = u + iv$, $y = w + it$. The non-vanishing values of (B) arise from certain points which make the integrand infinite or discontinuous. These points are given by the solutions of the equation

$$(2.1) \qquad h(x,y) = 0$$

or from its discontinuities.

Introducing the complex representation of x and y in (2.1), we have

$$(2.2) \qquad h(x,y) = h_1(u,v,w,t) + ih_2(u,v,w,t) = 0$$

hence,

$$(2.3) \qquad h_1(u,v,w,t) = 0, \quad h_2(u,v,w,t) = 0.$$

Relation (2.3) represents a surface in the four-dimensional space of u,v,w, and t. The set of points which makes the integrand infinite or discontinuous are called *singular points*, and from (2.3) these form in general a surface, called the *singular surface* of the integral.

Let α, β, γ be three real variables which may be regarded as coordinates of an ordinary space. Let S be an algebraic surface and α, β, γ be a point on S. We take.

(2.4) $u = \Phi_1(\alpha, \beta, \gamma), \quad v = \Phi_2(\alpha, \beta, \gamma), \quad w = \Phi_3(\alpha, \beta, \gamma), \quad t = \Phi_4(\alpha, \beta, \gamma),$

where $\Phi_i, (i=1,2,3,4)$ are rational functions and are finite for all α, β, γ. As α, β, γ trace in ordinary space the surface S or portions of it, the point u,v,w,t describes a hypersurface S* or portions of it in the hyperspace of the u,v,w,t system, in such a way that it is defined by the surface S and the four functions Φ_1. The singular surface given by (2.3) is expressed by the relations

(2.5) $h_1(\alpha, \beta, \gamma) = 0, \quad h_2(\alpha, \beta, \gamma) = 0,$

which define certain curves in ordinary space. These curves are the *singular curves* Γ along which the integrand of (B) becomes infinite or discontinuous.

If the surface S is closed and does not contain within it a singular curve, the value of (B) vanishes. If S*, belonging to the ordinary space, is closed and lies entirely within S then the integral (B) over S* is the same as that over S, provided there is no point of Γ between S* and S. On the other hand if in the interior region between S* and S there are singular curves or portions of these, then the integral over S* is equal to that over S. The surface S represents the surface of integration in the space of α, β and γ.

Now consider the following cases. In the first case let the function h(x, y) be of the form

(2.6) $h(x,y) = P(x,y)\ Q(x,y),$

where P and Q are irreducible polynomials. One wishes to find the value of the integral,

(2.7) $$I = \int\int \frac{g(x,y)}{P(x,y)Q(x,y)}\, dxdy.$$

By considering x and y complex with u, v, w, and t given by (2.4), one takes for the closed surface S in ordinary space the surface defined by equation (2.4). The value of the integral will depend on the singular curves Γ which are in the interior of S. These curves are of two kinds: (a) Those arising from P=0, (b) those arising from Q=0. A third case arises when both (a) and (b) are satisfied simultaneously. Since the singular curves are closed, the quantities u,v,w,t are periodic of some parameter ω with period. 2π. We write

(2 8) $u=\psi_1(\omega)$, $v=\psi_2(\omega)$, $w=\psi_3(\omega)$, $t=\psi_4(\omega)$.

Introducing two new parameters ρ and θ and letting

(2.9) $\alpha=\cos\omega\,(1+\rho\cos\theta)$, $\beta=\sin\omega\,(1+\rho\sin\theta)$, $\gamma=\rho\sin\theta$,

one then takes the following expressions for u,v,w ant t

(2.10) $u=\psi_1(\omega)$, $v=\psi_2(\omega)$, $w=\psi_3(\omega)+\rho\cos\theta$ $t=\psi_4(\omega)+\rho\sin\theta$

If $\rho<1$, a point in the α,β,γ space lies inside a torus. To every such point corresponds one and only one point in the hyperspace. By taking S the surface of the torus given by $\rho=\rho_o$ $0<\rho_o<1$, the curves Γ are given by $\rho=0$, or $\alpha^2+\beta^2=1$, $\gamma=0$.

One integrates (B) first with respect to y by regarding x as a parameter. Keeping x fixed one has $y=\psi_3+i\psi_4+\rho_o\,e^{i\theta}$; therefore y describes a circle of radius ρ_o with center at $\psi_3+i\psi_4$. The value of the integral with respect to y is equal to $2\pi i$ times the residue of the integrand evaluated at $\psi_3+i\psi_4$. Hence we get

(2.11) $$I=2\pi i\int_\Gamma \frac{g(x,y_1)}{P_{y_1}\,Q(x,y_1)}\,dx,$$

where P_{y_1} denotes the derivative of P with respect to y_1, and Γ is the singular curve defined by $P(x,y_1)=0$, with $x=\psi_1+i\psi_2$, $y=\psi_3+i\psi_4$. This is an Abelian integral relative to the curve Γ. In case (b) one obtains a similar expression with Q_{y_1} instead of P_{y_1}.

On the other hand if (a) and (b) are satisfied simultaneously at $x=x_o$, $y=y_o$ located within S, then from the above arguments one finds after integrating with respect to y

(2.12) $$I=2\pi i\int_\Gamma \frac{g(x,y_1)}{P_{y_1}\,Q(x,y_1)}\,dx$$

and since $Q(x,y_1)=0$ for $y=y_o$, the value of the integral is

(2.13) $$I=(2\pi i)^2\left[\frac{g(x_o,y_o)}{P_{y_o}Q_{x_o}-P_{x_o}Q_{y_o}}\right].$$

A more interesting case is when h(x,y) has a double point at $x=x_o$, $y=y_o$. The singular curves Γ are given by the relations, $h(x_o,y_o)=0$, $h_x(x_o,y_o)=0$, $h_y(x_o\,y_o)=0$. From the result of case (a), one has.

(2.14) $$I=2\pi i\int_\Gamma \frac{g(x,y_1)}{h_{y_1}\,x,y_1)}\,dx,$$

and since h_{y_1} vanishes at $y=y_0$, the value of the integral (2.14) is $2\pi i$ times the residue of the integrand, namely

$$(2.15)\qquad I=(2\pi i)^2\left[\frac{g(x_0,y_0)}{h_{y_1y_1}y_{1x}+h_{x_1x}}\right]$$

for $x=x_0$. The quantities h_{xx}, h_{yx}, h_{yy} are the partial derivatives of h with respect to the subscripts, and y_{1x} is the slope function. Since (x_0, y_0) is a double point of $h(x,y)$, y_{1x} is found by solving the equation

$$(2.16)\qquad h_{y_1y_1}\, y^2_{1x} + h_{y_1x}\, y_{1x} + h_{xx}=0.$$

Introducing the values of y_{1x} id (2.15), one obtains the value of the double integral

$$(2.17)\qquad I=(2\pi i)^2 \frac{g(x_0,y_0)}{\sqrt{h^2_{x_0y_0}-h_{x_0x_0}h_{y_0y_0}}}.$$

This is the formula derived by Poincaré which we shall find useful in the next section.

III. EVALUATION OF DOUBLE INTEGRALS OF EXPONENTIAL TYPE

In various branches of physics and also in problems of celestial mechanics one often has to evaluate double integrals involving one or more large parameters. More specifically, in diffraction and scattering problems the double integrals can be generally expressed in the form announced in the introduction, namely

$$(A)\qquad U(k)=\iint_D g(x,y)\, e^{ik\phi(x,y)}\, dxdx$$

where $k=\frac{2\pi}{\lambda}$ and D is a finite domain of integration. The so called amplitude and phase functions, $g(x y)$ and $\phi(x,y)$ are in general non-analytic. However, in many cases of interest they can be considered analytic, or infinitely differentiable in D. Here it is assumed that g and ϕ are at least n-times differentiable in D.

The purpose of this paper is to evaluate (A) for large k by following closely the method of Poincaré and Picard which has been outlined in section II.

However, in order to apply their method to integrals of exponential type (A), one must first reduce this integral in a suitable form so that the procedure outlined in the previous section can be applied to the reduced form. This can by easily carried out if one transforms (A) into the form

$$(3.1)\qquad U(k)=\iint_D \frac{g(x,y)}{ik\phi_y(x,y)}\, d_x - d_y\,\left(e^{ik\phi(x,y)}\right)\, dy.$$

Now consider a point (x_o, y_o) in the interior of D. This point is called a stationary point if the partial derivatives ϕ_x and ϕ_y vanish simultaneously at $x=x_o$, $y=y_o$. Therefore, at every stationary point in the interior of D, the denominator of (3.1) $\phi_y(x,y)$ vanishes. By partial integration with respect to y (3.1) reduces to

$$(3.2)\qquad U(k)=\frac{1}{ik}\int_\Gamma \frac{g(x,y_1)}{\phi_{y_1}(x,y_1)}\, e^{ik\phi(x,y_1)}\, dx-\frac{1}{ik}\iint_D e^{ik\phi(x,y)}\, d_y\left(\frac{g(x,y)}{\phi_y(x,y)}\right) dxdy,$$

where Γ is a closed contour in the x-plane surrounding the point x, and y is a root of ϕ_y for values of x in the neighborhood of $x=x_o$ i.e., $\phi_y(x,y_1)=0$.

Now we consider the first integral of (3.2)

$$(3.3)\qquad U_1(k)=\frac{1}{ik}\int_\Gamma \frac{g(x,y_1)}{\phi_{y_1}(x,y_1)}\, e^{ik\phi(x,y_1)}\, dx.$$

As y_1 depends on x, $\phi_y(x,y_1)$ vanishes for $x=x_o$. Since $g(x,y_1)$ and $\phi(x,y_1)$ are different from zero when x approaches x_o, the integral (3.3) is different from zero. Apart from the exponential part it is similar to the integral (2.1) of section II. The contribution to the value of the integral comes from the stationary point (x_o, y_o) and its value is equal to $2\pi i$ times the residue of the integrand at this point. Hence, we have (9)

$$(3.4)\qquad U_1(k)=\frac{2\pi}{k}\,\varepsilon\, e^{ik\phi(x_o,y_o)}\left[\frac{g(x_o,y_o)}{\sqrt{|\phi^2_{x_o y_o}-\phi_{x_o x_o}\phi_{y_o y_o}|}}\right],$$

where ε stand for the expression

$$\varepsilon=\begin{cases} 1 \text{ if } \Delta>0,\ \phi_{x_o x_o}>0, \\ -1 \text{ if } \Delta>0,\ \phi_{y_o y_o}<0, \quad \Delta=\phi_{x_o x_o}\phi_{y_o y_o}-\phi^2_{x_o y_o}\,. \\ -1 \text{ if } \Delta<0, \end{cases}$$

Therefore, one has

$$(3.6)\qquad U(k)=U_1(k)-\frac{1}{ik}\iint_D e^{ik\phi(x,y)}\, d_y\left(\frac{g(x,y)}{\phi_y(x,y)}\right) dxdx.$$

For large k the leading term of U(k) is $U_1(k)$. This is the required approximation to U(k) which finds many applications. Until recently this formula has been derived by heuristic methods (see van Kampen, ref. 1). A rigorous derivation of this formula as well as of the higher order terms of the asymptotic expansion of U(k) have been recently obtained by Focke (6) and the author (7) using the method of stationary phase. Another derivation based on a different procedure was given by Jones and Kline (8).

If more than one stationary point exists in the interior of the domain of integration, then the leading term of U(k) is given by the sum of the contributions arising from each of these points. These expressions are of similar form as the right hand side of (3.4).

In order to obtain the higher order terms of the asymptotic series of U(k), one integrates the double integral in (3.6) by parts by merely repeating the same transformation which was performed for (A). The results lead to the evaluation of certain Abelian integrals of exponential type in one variable taken along the singular curve Γ. For instance, the coefficient of the term in $(ik)^{-n}$ of the asymptotic series of U(k) is given by the following expression

$$(3.7)\qquad (-1)^n \int_\Gamma e^{ik\phi(x,y)}\; D^{n-1}\,[g(x,y)]\;\frac{dx}{\phi_y(x,y)},$$

with y replaced by y_1, the latter satisfying the relation, $\phi_y(x\,y_1)=0$. The operator D^n stands for the expression

$$(3.8)\qquad D^n = D^{n-1}\,D + D^{-n2}\,D^2 = \ldots\ldots = D^{n-k}\,D^k$$

with the operator D given by

$$(3.9)\qquad D = d_y\left(\frac{1}{\phi_y}\right).$$

The coefficient of the second term of the asymptotic expansion, namely of the term $(ik)^{-2}$ is

$$(3.10)\qquad \int_\Gamma e^{ik\phi(x,y)}\left(\frac{g_y\,\phi_y - g\,\phi_{yy}}{\phi^3_y}\right)dx,\quad \text{with } y=y_1.$$

This integral is of a similar kind as the one found by Poincaré, when h(x,y) is replaced by $h^2(x,y)$ in the integral (B). The evaluation of (3.10) leads to the same expression which is derived by the method of stationary phase.

The procedure outlined above can be extended to cover the case when the tangential derivative of ϕ vanishes at some point on the boundary of the domain of integration D.

This is analytically expressed by $\phi_s = 0$, s being the arc parameter measured along the boundary curve of D. The integration of (A) can be carried out first by finding a transformation which replaces ϕ_y by ϕ_s in (A) and then integrating by parts. Without going into the details of the calculations, we find for the leading term of U(k) the expression, (9)

$$(3.11)\quad U_1(k) = \left(\frac{1}{2}\right) e^{\,ik\varphi(x,y)} \left[\frac{\varphi_{xx}\,\varphi^2_y - 2\varphi_{xy}\,\varphi_x\,\varphi_y + \varphi_{yy}\,\varphi^2_x}{(\varphi^2_x + \varphi^2_y)^{\frac{1}{2}}} - \frac{1}{\rho}\,(\varphi^2_x + \varphi^2_y)^{\frac{3}{2}} \right]^{-\frac{1}{2}} (ik)^{-\frac{3}{2}}$$

evaluated at $x=x_o, y=y_o$, where the tangential derivative $\varphi_s(x_o,y_o)$ vanishes. The quantity ρ is the radius of curvature of the boundary curve at this point. This formula was found by the author and presented at the McGill University Symposium on Micro-Wave Optics in June 1953, (currently in press, Cambridge Air Force Research Center).

This formula is in agreement with the leading term which is obtained by the method of stationary phase. The higher order terms of the asymptotic series can also be explicitly evaluated provided the amplitude and the phase functions are infinitely differentiable. These results will appear elsewhere.

The two cases which have been treated here form only a part of the general contribution to the asymptotic value of the double integral (A). Besides the contributions coming from the infinities of the integrand—the stationary points of the phase function within the domain of integration and the boundary points - one must also take account of the contributions arising from the discontinuous points of the integrand, that is, the contributions due to the non-analyticity of the boundary, of the amplitude and of the phase functions, as well as, of the higher order infinities of the phase function, such as the vanishing of the discriminant Δ within the domain D and, of the higher tangential derivatives of φ on the boundary of D. These cases are at present under investigation.

IV. APPLICATIONS.

Here we shall illustrate the method of the previous section by considering some scalar diffraction problems in connection with optical instruments.

From Kirchhoff's theory the light distribution at some fixed point in image space is, except for some constant factors, expressed by a double integral of type (A). In this theory the amplitude g (x,y) is assumed to be constant over the aperture of the optical instrument D and the phase function is a polynomial in x and y. In the first two examples g is assumed constant. In the first example we consider diffraction by an optical instrument possessing defocusing properties and spherical aberration only. The phase function then takes the form

$$(4.1) \qquad \phi(x,y) = ax^2 + by^2 + 2ex^2y^2 + c(x^2+y^2)^2.$$

The constants **a**, **b** and **e** are the defocusing parameters and **c** is the spherical aberration constant. The stationary points are located at $x=0$, $y=0$ and $x_o = (ac-bd)(d^2-c^2), ^{-1/2}$, and $y_o = (cb \cdot ad)(d^2-c^2)^{-1/2}$. A simple calculation leads to the following expressions for $U_1(k)$:

$$(4.2) \qquad U_1(k) = \frac{2\pi i}{k} 4 (ab)^{-1/2} + \frac{\pi i}{k} \frac{e^{ik(ax^2_0 + by^2_0 + ex^2_0 y + {}^2_0 c(x^2_0 + y^2_0)}.}{x_0 y_0 (c^2-d^2)^{\frac{1}{2}}}$$

where $d=e+c$.

As a second example we take the phase function to be of the form

$$(4.3) \qquad \phi(x,y) = a(x^2+y^2) + c(x^2y + xy^2),$$

where the second term represents pure coma and c is called the coma aberration coefficient.

The stationary points of the phase function are located at (0,0) and $x_o = y_o = -\frac{2a}{3c}$. In this case the leading term $U_1(k)$ is the sum of two contributions, one from the point (0,0) and the other from x_o, y_o point. The result is

$$(4.4) \qquad U_1(k) = i(ak)^{-1} - 3(15)^{-\frac{1}{2}} (ak)^{-1} e^{ik} \left(\frac{8a^3}{27c^2}\right).$$

In this example the second stationary point has a negative discriminant. In both examples $g(x,y)$ is assumed to be 1.

Finally, we consider a case where the amplitude and phase functions are as follows:

$$(4.5) \qquad g(x,y) = (|x|\,|y|)^{-\frac{1}{2}}, \quad \phi(x,y) = ax + by + 2exy + c(x^2+y^2).$$

The method of section III is valid only when $g(x,y)$ is a regular function of x and y. However, as the absolute values of x and y appear in (4.5), the method is still applicable to this case. If we take as new variables u and v given by the relations, $u^2=x$, $v^2=y$, the integral (A) is transformed to four times the integral of the first example. Therefore, the value of the double integral for this case is four times the value found in the first example.

REFERENCES

1. VAN KAMPEN N.G., An asymptotic treatment of diffraction problems, Part I; Physica, **14** (1948), 575 - 590. Part II, ibid **16** (1950), 817 - 821.
2. ZERNIKE F., Diffraction and optical image formation; Proc. Phys. Soc. London, **61** (1948), 158 - 164.
3. WOLF E., The diffraction theory of aberrations; Reports on Progress of Physics, London, **14** (1951), 95 - 120.
4. POINCARÉ H., Sur les résidues des intégrales doubles; Acta Mathematica, **9** (1886 - 87,) 321-380.
5. PICARD E. et SIMART G., Théorie des Fonctions Algébriques de deux Variables Indépendentes. Tome I, Gauthier-Villars, Paris 1897.
6. FOCKE JOACHIM., Asymptotische entwicklungen mittels der Methode der stationären Phase. Berichte Säch. Akad. der Wiss. zu Leipzig, No 3. 1954.
7. CHAKO N., Asymptotic expansion of double integrals occurring in diffraction theory (in press). See also, C. R., Acad. d. Sci., Paris, t. 247 (1950), p. 436.
8. JONES D. S. and MORRIS KLINE, Asymptotic expansion of multiple integrals and the method of stationary phase. J. Math. & Physics, 37 (1958) 1.
9. CHAKO N., C. R., Acad. d. Sci., Paris, t. 247 (1951) p. 580.

ΠΕΡΙΛΗΨΙΣ

Ὁ συγγραφεὺς ἀσχολεῖται ἐνταῦθα περὶ τὸν προσδιορισμὸν διπλῶν ὁλοκληρωμάτων τῆς μορφῆς $U(k)=\iint_D g(x,y)\,e^{ik\varphi(x,y)}dxdy$, ὅπου $g(x,y)$ καὶ $\varphi(x,y)$ παριστοῦν ἀντιστοίχως τὰς συναρτήσεις πλάτους καὶ φάσεως, αἱ ὁποῖαι εἶναι πραγματικαὶ διὰ πραγματικὰς μεταβλητὰς τῶν x,y μέ τινας περιορισμούς. Τὸ D παριστᾷ ἓν πεπερασμένον πεδίον ὁλοκληρώσεως καὶ τὸ k λαμβάνει μεγάλας τιμάς.

Ὁλοκληρώματα τοῦ τύπου τούτου συναντῶνται εἰς τὴν μαθηματικὴν φυσικὴν καὶ ἰδίως εἰς τὰ προβλήματα διαθλάσεως καὶ διαχύσεως. Ἀκολουθεῖται ἡ μέθοδος Poincaré καὶ Picard διὰ ρητὰς συναρτήσεις μὲ δύο μεταβλητάς. Αἱ ἐν λόγῳ μέθοδοι ἀναπτύσσονται συντόμως εἰς τὰς παραγρ. I καὶ II. Εἰς τὴν παράγρ. III γίνεται ἐπέκτασις τῶν ἐργασιῶν Poincaré καὶ Picard εἰς διπλᾶ ὁλοκληρώματα τῆς μνημονευθείσης μορφῆς ὅπου τὸ k λαμβάνει μεγάλας τιμάς. Ὑποδεικνύεται ἕτι τρόπος εὑρέσεως ἀνωτέρων ὅρων τῆς ἀσυμπτωτικῆς ἐπεκτάσεως τοῦ ἀνωτέρω διπλοῦ ὁλοκληρώματος $U(k)$. Εἰς τὴν τελευταίαν τέλος παράγρ. δίδονται δύο τυπικὰ παραδείγματα ἔνθα ἡ ἐν λόγῳ μέθοδος εὑρίσκει τὴν ἐφαρμογήν της.