ΠΡΑΚΤΙΚΑ ΤΗΣ ΑΚΑΔΗΜΙΑΣ ΑΘΗΝΩΝ

ΣΥΝΕΔΡΙΑ ΤΗΣ 8ΗΣ ΔΕΚΕΜΒΡΙΟΥ 1977

ΠΡΟΕΔΡΙΑ ΠΕΤΡΟΥ ΧΑΡΗ

ΜΑΘΗΜΑΤΙΚΑ.— **On Plateau's problem and a generalization of the Riemann mapping theorem,** *by Themistocles M. Rassias**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φ. Βασιλείου.

1. INTRODUCTION

A critical point theory of a function on a manifold, and of extremals of single integral variational problems has been developed by M. Morse in his Variational Analysis in the large [**9**]. However the basic analytic techniques are not applicable to variational problems formulated on multidimensional varieties. In the case of a single integral problem the Euler - Lagrange equation is an ordinary differential equation, but for a multidimensional problem it is a partial differential equation of a considerable general form. From the geometric viewpoint a given curve can be decomposed by a finite number of points into small segments, leading to a reduction to a finite number of variables. This is not true for multi-dimensional varieties as for example for the Plateau's problem. From the general topological viewpoint the abstract variational analysis becomes a theory of the decomposition of a space V by means of a real valued function, using the order of the real numbers. From the global

* Θ. ΡΑΣΣΙΑ, **Τὸ πρόβλημα τοῦ Plateau καὶ γενίκευσις τῆς προτάσεως ἀπεικονίσεως τοῦ Riemann.**

analysis viewpoint we can study certain variational problems by formulating them in the context of infinite dimensional differentiable manifolds, and then by using techniques of non-linear functional analysis and infinite dimensional Riemannian geometry to obtain solutions of the problem under consideration. It is very frequently an essential step to construct a decomposition of the function space, in which the variational problem has been considered, by lines of steepest descent.

An attempt to resolve this problem was made by M. Morse and C. Tompkins [**10**], M. Shiffman [**17**], and by R. Courant [**3**] who applied various methods from analysis and topology for the development of a special theory for the problem of unstable minimal surfaces bounded by a given closed curve in a Euclidean space. From the Morse viewpoint the critical point theory has mainly been studied for compact manifolds M. However the variational analysis is mostly interesting in the case where M is infinite dimensional. It is clear that the sphere in a Hilbert space H is compact if and only if dim $H < \infty$. Therefore a compactness condition is needed for the study of such problems. For this purpose R. Palais and S. Smale [**12**], [**18**] discovered a very interesting condition, called *«Condition C»* by them which allowed them to prove some very important results of Morse theory for Hilbert (or Banach) manifolds.

A little later J. Schwartz [**16**] by applying the Palais - Smale theory succeeded in carrying over the deformation theorem and the Minimax Principle, therefore getting a generalization of the Lusternik - Schnirelman theory [**7**] for Riemannian manifolds of any dimension. Recently A. Tromba [**20**], [**21**] succeeded in obtaining a generalization of the Morse theory under some weaker assumptions than the Palais - Smale theory and he was able to prove some very interesting results concerning the Plateau's problem. In a recent paper [**14**] I indicated a way to apply the Morse theory on Hilbert manifolds to the Plateau's problem.

The present paper is divided into two parts. In the first part I give a partial answer to the problem of finding the number of minimal surfaces spanning a given contour in R^3. In the second part I give a generalization of the Riemann mapping theorem. The methods used in the proofs of the above theorems are contained in the Morse theoretic approach to Plateau's problem.

2. ON THE NUMBER OF MINIMAL SURFACES SPANNING A CURVE IN R^3

The following theorems give a partial answer to the problem of finding the number of minimal surfaces spanning a given Jordan curve Γ in R^3.

Theorem 1. *For any natural number* n, *there exists a Jordan curve* Γ *in* R^3 *that has exactly* $(2n+1)$ *minimal surfaces spanning* Γ.

Conjecture 1. *For any* C^1*-Jordan curve* Γ *in* R^3 *there exists an odd number of minimal surfaces spanning* Γ.

Remark 1. An affirmative answer to the above conjecture will imply, as a special case, a general solution of Plateau's problem for C^1-Jordan curves Γ.

The natural question now is whether the set of Jordan curves such that the solution to Plateau's problem is unique is in some sense generic. Such curves are referred to as *curves of uniqueness*. The following theorem gives a partial answer to the above question.

Let S be the set of Jordan curves in R^3. A topology is defined on S with a metric d as follows. For any two Jordan curves Γ_1, Γ_2 in S,

$$d(\Gamma_1, \Gamma_2) = \inf \|\gamma_1 - \gamma_2\|_\infty$$

where $\|\cdot\|_\infty$ is the usual sup-norm and where γ_1, γ_2 run over all possible homeomorphisms of S^1 with Γ_1 and Γ_2 respectively.

Theorem 2. *The set of Jordan curves such that the solution to Plateau's problem is unique is dense in* S. *Furthermore, the set of Jordan curves such that the solution to Plateau's problem is not unique is dense in* S.

Corollary. *The curves of uniqueness of the solution to Plateau's problem do not form an open set in* S.

Conjecture 2. *The curves of uniqueness of the solution to Plateau's problem form a set of second category in* S.

3. ON A GENERALIZATION OF THE RIEMANN MAPPING THEOREM

Riemann's problem of mapping a simply connected plane region whose boundary consists of more than a single point conformally on a circle as normal region can be reduced to the study of two problems: (1) *the interior problem* that concerns the map of the interior points and (2) *the boundary problem* that concerns the behavior of the map on the boundary. It was Riemann who studied the first problem by using techniques of the Dirichlet principle and Schwarz and Neumann who gave proofs for the case of regions with restricted boundaries. Later, Osgood gave a satisfactory answer to the general case using methods due to Poincaré. The second problem was solved for analytic boundaries by Schwarz and in other special cases by Picard. The general case was treated by Osgood [**11**] and by Carathéodory [**1**].

It is my purpose nowto indicate how I can apply the theory of minimal surfaces for a generalization of the Riemann mapping theorem.

Consider the minimal surface equation (Lagrange [**5**]) given by

$$(1+\phi_y^2)\,\phi_{xx}-2\phi_x\phi_y\phi_{xy}+(1+\phi_x^2)\,\phi_{yy}=0. \tag{I}$$

The surface is assumed in the non-parametric form and Plateau's problem is regarded as a generalized Dirichlet problem, with (I) replacing Laplace's equation. According to Weierstrass [**22**] a parametric form of the solution for the minimal surface equation (I) is given by

$$\left\{\begin{array}{ll} x = \operatorname{Re} F_1(w) & (1)\\ y = \operatorname{Re} F_2(w) & (2)\\ z = \operatorname{Re} F_3(w) & (3)\end{array}\right.$$

where $F_1(w)$, $F_2(w)$, $F_3(w)$ are any analytic functions satisfying

$$F_1'^2(w)+F_2'^2(w)+F_3'^2(w)=0. \tag{4}$$

Set

$$\psi_1(w)=\frac{1}{2}\left(F_1'(w)+i\,F_2'(w)\right) \tag{5}$$

and

$$\psi_2(w)=\frac{1}{2}F_3'(w). \tag{6}$$

It follows that

$$x + iy = \int \psi_1(w)\,dw - \overline{\int \frac{\psi_2^2(w)}{\psi_1(w)}\,dw}. \qquad (7)$$

Denote

$$U(\psi_1) = \int \psi_1(w)\,dw - \overline{\int \frac{\psi_2^2(w)}{\psi_1(w)}\,dw} \qquad (8)$$

the we can state the following theorem.

T h e o r e m 3. *Let* Γ *be a simple closed analytic curve in the z-plane. Then there exists a regular function* $\psi_1(w)$ *defined in* $\Omega = \{w : |w| \geqslant 1\}$, *such that* $U(\psi_1)$ *maps* Ω *simply onto the closed domain exterior to* Γ *and such that infinity is mapped into infinity, for a fixed regular function* $\psi_2(w)$ *defined in* Ω.

R e m a r k 2. If $\psi_2(w) = 0$, $\forall w$, then the above theorem implies the Riemann mapping theorem, as a special case.

A c k n o w l e d g m e n t. I would like to express my gratitude to Professor S. Smale for introducing me to this field of mathematics and for fruitful conversations.

Π Ε Ρ Ι Λ Η Ψ Ι Σ

Ἡ παροῦσα ἀνακοίνωσις ἀποτελεῖ συνέχειαν τῆς προηγουμένης ἀνακοινώσεως τοῦ συγγραφέως, ἥτις ἀνεκοινώθη εἰς τὴν Ἀκαδημίαν Ἀθηνῶν, ὑπὸ τὸν τίτλον: «Morse theory on Hilbert manifolds and Plateau's problem». Εἰς τὴν παροῦσαν ἀνακοίνωσιν ἀποδεικνύεται μία περίπτωσις τοῦ προβλήματος εὑρέσεως τοῦ ἀριθμοῦ τῶν λύσεων εἰς τὸ πρόβλημα τοῦ Plateau.

Τὸ πρόβλημα αὐτὸ ἔχει ἀπασχολήσει ὡρισμένους τῶν διασημοτέρων μαθηματικῶν τοῦ 19ου καὶ 20οῦ αἰῶνος, ὅπως τοὺς C. Carathéodory, D. Hilbert, B. Riemann, H. Schwarz, K. Weierstrass ὡς ἐπίσης J. Douglass, C. B. Morrey, T. Rado, S. Smale, καὶ ἄλλους πολλούς.

Ἡ περίπτωσις τὴν ὁποίαν ὁ συγγραφεὺς ἔχει ἐπιλύσει, σχετικὰ μὲ τὸ ἀνωτέρω πρόβλημα, εἶναι οὐσιώδους σημασίας, καθόσον δημιουργεῖ τὴν θετικὴν εἰκασίαν ὅτι ὁ ἀριθμὸς τῶν λύσεων τοῦ προβλήματος τοῦ Plateau εἶναι πάντοτε ἕνας περιττὸς ἀριθμὸς δι' ὁποιανδήποτε καμπύλην τοῦ Jordan, εἰς τὸν χῶρον τοῦ

Εὐκλείδου. Ὡς πόρισμα τῆς ἀνωτέρω εἰκασίας εἶναι μία γενικὴ λύσις τοῦ προβλήματος τοῦ Plateau καὶ συγχρόνως ἐφαρμογαὶ εἰς τὴν Φυσικὴν ὡς π. χ. εἰς τὴν θεωρίαν τῆς σχετικότητος.

Εἰς τὴν ἀνακοίνωσιν διατυποῦται μέθοδος τοῦ συγγραφέως, προερχομένη ἀπὸ συμπεράσματα εἰς τὸ πρόβλημα τοῦ Plateau, ὅπου δίδει μίαν γενίκευσιν ἑνὸς τῶν θεωρημάτων τοῦ B. Riemann, τώρα γνωστοῦ ὡς Riemann Mapping Theorem.

REFERENCES

1. C. Carathéodory, Uber die gegenseitige Beziehung der Ränder bei der Konformen Abbildung des Innern einer Jordanschen Kurve auf einen kreis, Mathematische Annalen, vol. 73 (1913), pp. 305 - 320.
2. S. S. Chern and Smale (eds.), Proceedings of the Symposium in Pure Mathematics XIV, XV. Global Analysis, University of California, Berkeley (1970).
3. R. Courant Dirichlet's Principle, Conformal Mappings and Minimal Surfaces, Interscience, New York (1950).
4. J. Eells, A setting for Global Analysis, Bull. Amer. Math. Soc. 72 (1966), 751 - 807.
5. J. L. Lagrange, Méchaniqne Analytique, 3me ed., Mallet-Bachelier, Paris (1853).
6. S. Lang, Introduction to Differentiable Manifolds, New York (1962).
7. L. Lusternik and L. Schnirelman, Méthodes topologiques dans les problèmes variationeles, Paris (1934).
8. C. B. Morrey, The problem of Plateau on a Riemannian manifold, Annals of Math. 2 (49) (1948), 807 - 851.
9. M. Morse, Functional topology and abstract variational theory, Ann. of Math. 38 (1937), 386 - 449.
10. M. Morse and C. Tompkins, The existence of minimal surfaces of general critical types, Ann. of Math. 40 (1939), 443 - 472.
11. W. S. Osgood, On the transformation of the boundary in the case of conformal mapping, Bull. Amer. Math. Soc. 9 (1903), 233 - 235.
12. R. Palais and S. Smale, A generalized Morse theory, Bull. Amer. Math. Soc. 70 (1964), 165 - 172.
13. R. Palais, Foundations of Global Non-linear Analysis, W. A. Benjamin, New York (1968).
14. T. M. Rassias, Morse theory on Hilbert manifolds and Plateau's problem, Praktika Acad. Athens, Greece (1978).
15. ——, On the problem of Plateau for two contours, Notices of the Amer. Math. Soc. 24 (1), Issue 175, # 76T - G6 (Jan. 1977), A22 - A23.

16. J. Schwartz, Generalizing the Lusternik-Schnirelman theory of critical points, Comm. Pure Appl. Math. 17 (1964), 307 - 314.

17. M. Shiffman, The Plateau problem for non-relative minima, Ann. of Math. 40 (1939), 834 - 854.

18. S. Smale, Morse theory and a non-linear generalization of the Dirichlet problem, Ann. of Math. 2 (80) (1964), 382 - 396.

19. ——, An infinite dimensional version of Sard's theorem, Amer. J. of Math. 87 (1965), 861 - 866.

20. A. Tromba, A generalized approach to Morse theory, to appear in J. of Diff. Geometry.

21. ——, On the number of simply connected minimal surfaces spanning a curve in R^3, to appear.

22. K. Weierstrass, Mathematische Werke, 3 Bände, Mayer and Müller, Berlin (1903).

23. K. Yosida, Functional Analysis, Springer - Verlag, Berlin - Göttingen - Heidelberg, Band 123 (1965).

★

Ὁ Ἀκαδημαϊκὸς κ. **Φίλων Βασιλείου,** παρουσιάζων τὴν ἀνωτέρω ἀνακοίνωσιν εἶπε τὰ ἑξῆς:

Ἡ ἐργασία τοῦ κ. Θεμιστοκλέους Ρασσιᾶ, μὲ τίτλον «Τὸ πρόβλημα τοῦ Plateau καὶ γενίκευσις τῆς προτάσεως ἀπεικονίσεως τοῦ Riemann» —ἐργασία τὴν ὁποίαν ἔχω τὴν τιμὴν νὰ ἀνακοινώσω εἰς τὴν Ἀκαδημίαν Ἀθηνῶν — ἀποτελεῖ συνέχειαν προηγουμένης ἀνακοινώσεως τοῦ ἰδίου συγγραφέως γενομένης ἐπίσης εἰς τὴν ἡμετέραν Ἀκαδημίαν. Μὲ τὴν εὐκαιρίαν τῆς σημερινῆς ἀνακοινώσεως, ἐπιτρέψατέ μου νὰ ἀναφέρω τὴν λίαν τιμητικὴν προσφορὰν τῆς ὁποίας ἔτυχεν ὁ κ. Ρασσιᾶς, ὅπως ἀποτελέση μέλος, διὰ τὸ ἐπόμενον ἀκαδημαϊκὸν ἔτος, τοῦ περιφήμου Institute for Advanced Study εἰς Princeton τῶν Ἡνωμένων Πολιτειῶν. Εἰς σχετικὴν ἐπιστολήν του ὁ καθηγητὴς τοῦ ἐν λόγῳ Ἰνστιτούτου Marston Morse ἐκφράζει τὰ συγχαρητήριά του διὰ τὸ πράγματι ἐντυπωσιακόν, ὅπως λέγει, ἐρευνητικὸν ἔργον τοῦ κ. Ρασσιᾶ καὶ παρακινεῖ αὐτὸν ὅπως ἀσχοληθῇ καὶ μὲ ἰδικάς του ἐργασίας, τὰς ὁποίας ὁ Morse, λόγῳ ἡλικίας, ἀδυνατεῖ πλέον νὰ συνεχίσῃ.

Τὸ πρόβλημα, τὸ ὁποῖον φέρει τὸ ὄνομα τοῦ Βέλγου φυσικοῦ J. Plateau (1801 - 1883), ἀναφέρεται εἰς τὴν εὕρεσιν ἐλαχιστικῶν ἐπιφανειῶν, δηλαδὴ ἐπιφανειῶν ἐλαχίστου ἐμβαδοῦ, ἐκτεινομένων ἀπὸ δοθείσας κλειστὰς συνοριακὰς γραμμὰς (σύνορα), ὅπως καὶ κατὰ τὴν πρώτην ἀνακοίνωσιν τοῦ κ. Ρασσιᾶ ἀνεπτύξαμεν. Πρῶτος ὁ Plateau προέβη εἰς σχετικὰ μὲ τὸ πρόβλημα αὐτὸ φυσικὰ πειράματα. Παρ' ὅλον ὅτι ἡ διατύπωσις τοῦ προβλήματος εἶναι ἀρκετὰ σαφὴς καὶ ἁπλῆ, ὅμως ἡ μαθηματικὴ ἐπίλυσίς του, καὶ μάλιστα ὑπὸ γενικευμένην μορφήν, ἀπετέλεσεν ἔκτοτε ἀντικείμενον βαθείας ἐρεύνης — ἐρεύνης σχετιζομένης μὲ πλείστους μαθηματικοὺς κλάδους, ὅπως τῆς «Συμμόρφου Ἀπεικονίσεως», τῶν «Διαφορικῶν Ἐξισώσεων μὲ μερικὰς παραγώγους», τῆς «ἀρχῆς Dirichlet», καὶ αὐτῆς ἀκόμη τῆς «Θεωρίας τῆς Σχετικότητος». Ἂς σημειωθῇ ἐδῶ, ὅτι φυσικὴ ὕπαρξις τῆς λύσεως φυσικοῦ τινὸς προβλήματος δὲν σημαίνει κατ' ἀνάγκην καὶ τὴν ὕπαρξιν λύσεως τοῦ ἀντιστοίχου μαθηματικοῦ προβλήματος, ὅπως, ἐπ' εὐκαιρίᾳ τῆς ἀρχῆς Dirichlet, ἔδειξεν ἐνωρίτερον ὁ πολὺς Weierstrass.

Ἡ ὡς ἄνω γενικευμένη μορφὴ τοῦ προβλήματος Plateau, ἀφορᾷ εἰς ν-διαστάτους χώρους ἀντὶ τοῦ Εὐκλειδείου καὶ εἰς συνοριακὰς γραμμὰς ἀποτελουμένας ἀπὸ σύστημα πεπερασμένων τὸ πλῆθος γραμμῶν Jordan. Πρῶτος ὁ Ἀμερικανὸς μαθηματικὸς J. Douglas εὗρε καὶ μάλιστα τὴν γενικευμένην αὐτὴν λύσιν τοῦ προβλήματος, τιμηθεὶς δι' αὐτὸ μὲ τὸ Fields Medaille, τὸ ὁποῖον ἀποτελεῖ τὴν ἀνωτάτην τιμητικὴν διάκρισιν διὰ τὰ Μαθηματικά, ἀντιστοιχοῦσαν μὲ τὸ βραβεῖον Nobel, διάκρισιν ἀπονεμηθεῖσαν εἰς τὸν Douglas κατὰ τὸ εἰς Oslo, τὸ 1936, Διεθνὲς Μαθηματικὸν Συνέδριον.

Ὅμως, καὶ μετὰ τὸ ἐπίτευγμα τοῦ Douglas, πολλὰ καὶ ἐνδιαφέροντα προβλήματα ἐπὶ τῶν ἐλαχιστικῶν ἐπιφανειῶν παρέμεναν πρὸς ἀπάντησιν. Μεταξὺ αὐτῶν περιλαμβάνοντο ζητήματα ἀναφερόμενα εἰς τὴν μοναδικότητα ἢ μὴ τῶν λύσεων τοῦ προβλήματος Plateau, εἴτε εἰς τὴν συνέχειαν τῶν ἐν λόγῳ λύσεων. Καὶ πράγματι· ἦτο εὔλογον νὰ διερωτηθῇ κανείς, ἂν δὲν εἶναι δυναταὶ περισσότεραι τῆς μιᾶς λύσεις τοῦ προβλήματος, ἐφ' ὅσον δι' ὡρισμένας κλειστὰς συνοριακὰς γραμμάς, ὡς εὑρέθη πειραματικῶς, αἱ λύσεις ὑπερέβαιναν τὴν μίαν. Ἐξ ἄλλου ἡ ἐπέκτασις τοῦ προβλήματος τοῦ Plateau εἰς πολλαπλότητας Hilbert ἀπετέλεσεν ἑτέραν κατεύθυνσιν γενικεύσεως.

Εἰς τὴν προηγηθεῖσαν ἀνακοίνωσιν τοῦ κ. Ρασσιᾶ εἰς τὴν Ἀκαδημίαν Ἀθηνῶν, ὁ συγγραφεὺς εἶχεν ἀσχοληθῇ μὲ τὴν ἐφαρμογὴν ὡρισμένης θεωρίας τοῦ Marson Morse διὰ τὰς πολλαπλότητας Hilbert καὶ εἰς τὸ πρόβλημα τοῦ Plateau. Ἡ ἐν λόγῳ θεωρία Morse ἐκτίθεται εἰς τὸ γνωστὸν σύγγραμμά του Variational Analysis in the Large.

Ἡ περίπτωσις τώρα τὴν ὁποίαν ὁ κ. Ρασσιᾶς πραγματεύεται εἰς τὴν παροῦσαν ἀνακοίνωσιν, ὡδήγησαν αὐτὸν εἰς τὴν εἰκασίαν, ὅτι «Τὸ πλῆθος τῶν λύσεων τοῦ προβλήματος Plateau εἶναι πάντοτε περιττόν, δι' οἱονδήποτε πεπερασμένον σύστημα γραμμῶν Jordan ὡς συνόρων καὶ εἰς τὸν Εὐκλείδειον χῶρον.

Ἡ ἐν λόγῳ ἀνακοίνωσις διαιρεῖται εἰς δύο μέρη : Εἰς τὸ πρῶτον ὁ συγγραφεὺς δίδει μερικὴν ἀπάντησιν εἰς τὸ πρόβλημα εὑρέσεως τοῦ πλήθους τῶν ἐλαχιστικῶν ἐπιφανειῶν, αἱ ὁποῖαι ἐκτείνονται ἀπὸ δοθεῖσαν συνοριακὴν γραμμὴν εἰς τὸν τριδιάστατον εὐκλείδειον χῶρον. Εἰς τὸ δεύτερον μέρος παρέχει γενίκευσιν τοῦ θεμελιώδους θεωρήματος τῆς συμμόρφου ἀπεικονίσεως, λεγομένου καὶ θεωρήματος ἀπεικονίσεως Riemann. Κατ' αὐτό, κάθε ἁπλῶς συνεκτικὸν ἁπλοῦν χωρίον τοῦ ἐπιπέδου, μὲ τουλάχιστον δύο συνοριακὰ σημεῖα, ἀπεικονίζεται μὲ μίαν ὁλόμορφον συνάρτησιν ἀμφιμονοσημάντως καὶ συμμόρφως εἰς τὸ ἐσωτερικὸν ἑνὸς κύκλου, ἰδιαιτέρως εἰς τὸ ἐσωτερικὸν τοῦ μοναδιαίου κύκλου. Ἀξιοσημείωτον εἶναι, ὅτι αἱ μέθοδοι αἱ ὁποῖαι ἀκολουθοῦνται εἰς τὴν παροῦσαν ἀνακοίνωσιν δὲν διαφέρουν ἀπὸ ἐκείνας τὰς ὁποίας χρησιμοποιεῖ ὁ Morse κατὰ τὴν θεωρητικὴν προσπέλασιν τοῦ προβλήματος Plateau.