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ΜΑΘΗΜΑΤΙΚΑ.—**The spectra of some singular integral operators for solving three-dimensional boundary value problems,** *by P. S. Theocaris and J. G. Kazantzakis* *. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Π. Θεοχάρη.

ABSTRACT

In the present paper the systems of two-dimensional singular integral operators which solve the dynamic and static boundary value problems of three-dimensional elasticity and thermoelasticity are formulated by using the singular integral operators technique. In this way, the singular integral operator method may be established as a means for solving practical engineering problems in many fields (three-dimensional inclusion and crack problems etc.).

1. INTRODUCTION

Several numerical techniques have been developed into efficient tools for the solution of problems of continuum mechanics in the last fifteen years. In particular, the finite element method, although is now well established as a means for solving practical engineering problems in many fields, presents certain application problems. Even where the most sophisticated data generation and checking facilities are available, the expenditure of the data preparation is considerable. The finite element method requires the solution of a large number of simultaneous

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equations which results a high computer time for the solution of these systems. Nevertheless, the use of most of the information obtained by a finite-element program is useless, thus the engineer finds it difficult to select the results of interest to him. Finally, the aforementioned technique gives good results for the displacements, but less accurate results for the stresses, which are in most cases of greater interest to the engineer.

On the other hand, the difficulties associated with the finite - element method may be overcome by the use of the singular integral operator method. It should be mentioned that the integral methods are an extension of the method of Fredholm integral equations, which can be considered as a direct application of the results of potential theory [**1** to **9**]. This method seems to be without merit today because only special problems are analyzed and the mathematical treatment lacks rigor.

Furthermore, only plane and axisymmetric elastostatic boundary value problems had been thoroughly studied until 1950. This was due to the fact that the aforementioned method seems to be inadequate for the investigation of the second and the third fundamental boundary value problems.

More recently, by using the method of potential, Kupradze [**6**], as well as Kinoshita and Mura [**9**] have obtained systems of singular integral equations for the first and second fundamental problems. In 1963, Kupradze applied his method to the more general case of periodic oscillations of an elastic body. In this work general and rigorous proofs based on the theory of multidimensional singular integral equations are given in order to establish the existence and uniqueness of solutions for homogeneous and piecewise homogeneous bodies.

In 1967, plane inclusion problems were investigated by Rizzo, Cruse and Shippy [**10** to **13**] by using the singular integral operator method. Swedlow and Cruse [**14**] presented an elastoplastic analysis, the results of which were generalized in the case of anisotropic compressible materials subjected to strain-hardering by Mendelson [**15**]. Green and Sneddon [**16**] applied the method of potential to the three-dimensional fracture analysis while Irwin extended this solution [**17**] to the case of a flat elliptical-shaped crack in an infinite body and estimated the stress intensity factor (S. I. F.) and crack extension force for a part through crack in an infinite plate. Thus, the method of Irwin provides a powerful technique for solving both three-dimensional inclusion and fracture mechanics problems.

The work amount in the area of three-dimensional analysis of stresses near cracks due to thermal loading is rather limited. In 1960, Olesiak and Sneddon [**18**], presented the solution for a penny-shaped crack in an infinite elastic solid with either the temperature or heat flux prescribed on the crack surface. The solution was restricted to being axisymmetric about an axis passing through the center of the crack and perpendicular to the crack surface.

In 1963, Florence and Goodier [**19**] obtained the solution for an insulated penny-shaped crack embedded in an infinite medium perpendicular to a uniform flow of heat. Their solution was obtained by superimposing upon the undisturbed heat flow, the results for a problem having the opposite temperature gradient prescribed upon the crack surface. In 1966 Smith [**20**] used Schwarz's alternating method and calculated opening mode stress intensity factors of a semi-circular edge crack located at a free surface for both thermal and isothermal loading conditions.

Various authors have considered thermoelastic solutions pertaining to the effects of change the specific volume for elastic inclusions of various shapes. Goodier [**21**] has reduced the general problem of an inclusion of arbitrary shape embedded in an infinite matrix to the solution of a problem of potential theory provided that both the inclusion and matrix have the same material properties.

Gatewood [**22**] solved the problem of a matrix in the shape of a circular cylinder with an eccentric circular inclusion embedded within. By using Goodier's method Gatewood obtained and presented a particular solution for the case where the composite body is at a uniform temperature above a stress free reference state.

Mindlin and Cooper [**23**], as well as Edwards [**24**], Goodier and Florence [**25**] and Segedin [**26**] have obtained thermoelastic solutions for the case when the elastic constants of the inclusion and matrix are different. Using Goodier's theory Mykelstad [**27**] solved the case of a spheroidal inclusion and a semi-infinite cylinder embedded in infinite matrices. Goodier's solutions to the semi-infinite matrix were extended by Mindlin and Cheng [**28**] and the problem of a spheroidal inclusion located at any depth below the surface was also solved.

Edwards [**24**] obtained the thermoelastic solution for a spheroidal inclusion in an infinite matrix (of which a special case in the solution of Mykelstad) by superimposing particular solutions which were generated using the Boussinesq-Papkovitch formulation.

Liu [**29**] has recently solved the problem of an infinitely long cylindrical inclusion embedded in a half space where the cylindrical axis is parallel to the surface and the depth of the cylinder is not restricted except to insure that the inclusion is within the matrix. The method used was an extension of Goodier's and the composite body was assumed to be at a constant temperature above a stress free reference state.

Goodier and Florence also solved the three-dimensional problem of a spherical inclusion in an infinite matrix by using Goodier's thermoelastic potential to find a particular solution for both the sphere and the matrix and by using the Boussinesq-Papkovitch-Neuber formulation to solve the isothermal homogeneous equations. As in two-dimensional case the solutions are superimposed using appropriate constants to insure compatibility at the inclusion-matrix interface.

In 1978 Theocaris *et al.* has considered [**30**, **31**] a general method for the numerical solution of the aforementioned boundary value problems by constructing a cubature formula for the evaluation of two dimensional singular integrals, which are frequently encountered in them. It should be noticed that for the construction of the above referenced cubature formula one did not need to discretize the surface on which the integration was performed into pieces, while the surface data of the density function did not need to be considered as constant. This is the main advantage of the method reported in [**30**, **31**] in comparison with the B.I.E-method developed by Cruse, [**12** to **14**]. The same method was extended by Theocaris *et al.* in solving three-dimensional crack problems [**32** to **34**].

2. SETS OF DIFFERENTIAL EQUATIONS

In an orthogonal rectilinear system of reference $O(x_1 x_2 x_3)$ we introduce the notation $\mathbf{U}(u_1, u_2, u_3)$ for the elastic displacement vector, ε_{ij} for the components of the deformation tensor and σ_{ij} for the components of the stress tensor corresponding to the displacements **u** [**1**]. Tensor σ_{ij} is related to ε_{ij} by Hooke's law:

$$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl}\,, \tag{1}$$

the Hooke constants c_{ijkl} satisfy the relation [**2**]:

$$c_{ijkl} = c_{klij} = c_{jikl}\,, \tag{2}$$

because of which it is valid that:

$$\sigma_{ij} = \sigma_{ji}\,. \tag{3}$$

If the influence of couple-stresses on stress concentrations is taken into account [**3**, **4**] relation (1) must be modified as:

$$\mu_{ij} = c'_{ijkl}\,\omega_{kl}\,, \tag{4}$$

where μ_{ij} denotes the couple-stress tensor and ω_{kl} the local rolation tensor.

Let us now investigate the case where heat sources act inside the region under consideration, constituting a part of a body V. Then, a change in temperature T of the body results in a corresponding change of the state of stress, which is calculated by:

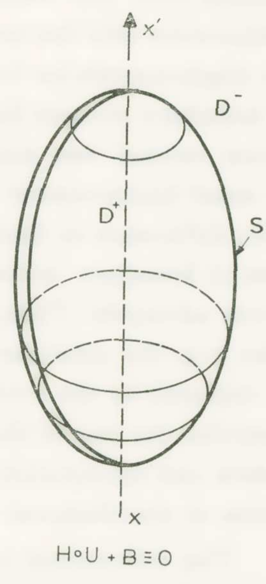
$$\sigma_{ij} = c_{ijkl}\,\varepsilon_{kl} + c_{ij}\,T, \tag{5}$$

where the Duhamel Neumann constants c_{ij} [**5**] satisfy the condition:

$$c_{ij} = c_{ji}\,. \tag{6}$$

In textbooks [**2**] on the theory of elasticity it is shown that the equations of equilibrium for an elastic body, on which the body forces F act, have the form:

$$\sigma_{ij,j} + F_i = 0 \qquad i, j = 1, 2, 3. \tag{7}$$



$H \circ U + B \equiv 0$

Inserting here the values of σ_{ij} from (1), (4), (5) we get the basic set of differential equations in the displacements, whose general from is represented by:

$$\mathbf{H} \circ \mathbf{U} + \mathbf{B} \equiv 0\,. \tag{8}$$

The operators **H, B** are tabulated in Table I, while **U** denotes the unknown displacement vector.

3. BASIC FORMULATION OF THE BOUNDARY INTEGRAL EQUATIONS

The integral method which is analyzed here consists in the transformation of the sets of differential equations which are presented in Table I to sets of two-dimensional singular integral equations relating the unknown function and some of its derivatives to the given values on the boundary. In the case where analytic solutions are contemplated, the transformation of the field equations to singular integral equations is of little interest. However, if numerical techniques are to be used, this transformation is necessary for the following reason.

For each $x \in S$ (where S denotes the boundary of the region under consideration) the solution of the systems of singular integral equations is the displacement function $\mathbf{U}(x)$. From this information the values of $\mathbf{U}(y)$ are calculated directly for each interior point $y \in V$. The main advantage is therefore that, since the problem was transformed from a problem over the volume V to a problem over its boundary S, we need to discretize only the boundary and solve the resulting equations. Stresses and displacements for bodies with non-smooth boundaries with non-singular boundary stresses have been shown to be accurate within the internal region located one nodal spacing from any boundary. In contrast to the usual improvement of accuracy, associated with the refinement of finite difference or finite element nodal spacings, an analogous refinement of boundary nodes for the singular integral operator method is always advisable. Thus, the dimension of the problem is reduced by one order (e.g. the solutions of the singular integral equations over the body are replaced by the study of its behaviour on the surface), which greatly simplifies the use of the computer program, especially the specification of data and interpretation of results, and also reduces the order of the system of simultaneous equations to be solved.

The differential equation (8) (Table I) can be readily transformed into an integral relationship by using Betti's theorem and a limiting process [**6**]. In the present study, the relationship is established in terms of generalized functions, by use of the distribution theory. This technique is an elegant and powerful tool for the transformation of an elliptic operator, for which a fundamental solution is known, into an integral relationship [**7**]. For convenience, operator symbols are used in the following derivation: ∇ is the gradient operator, $\nabla .$ is the divergence operator. Bold-type symbols without indices represent tensors, while with indices represent their components. We consider in the following, the first fundamental boundary value problem ($I^{\pm}$, [**6**]) and we investigate in detail the derivation of the related integral relationships. The collective results for the second ($II^{\pm}$), third ($III^{\pm}$) and fourth ($IV^{\pm}$) fundamental boundary value problems are derived by applying a quite similar technique and are summarized in Table II.

To derive a formulation of the aforementioned boundary value problem ($I^{\pm}$) in terms of distributions, one applies in the sense of distribution theory the operator $\mathbf{H}$ (Table I) to $\mathbf{R}(S)\,\mathbf{U}$ where $\mathbf{R}(S)$ is the characteristic function of the domain V with boundary S at every point $\mathbf{X}(x_1, x_2, x_3)$ of the domain V being considered. That is, $\mathbf{R}(S) = 1$ if $\mathbf{X} \in V$, $\mathbf{R}(S) = 0$ if $\mathbf{X} \notin V$. Now let $\mathbf{E}$ be an elementary solution corresponding to the operator $\mathbf{H}$, then:

$$\mathbf{H} \circ \mathbf{E} \equiv \delta, \tag{9}$$

where $\delta(S)$ is the Dirac distribution on the surface S, then:

$$\mathbf{H}\langle \mathbf{R}(S)\,\mathbf{U}\rangle = \mathbf{R}(S)\{\mathbf{H} \circ \mathbf{U}\} - T_n \langle \mathbf{U}\delta(S)\rangle - \langle T_n\rangle\,\delta(S), \tag{10}$$

where the sign $\langle\ \rangle$ denotes a distribution and $\{\}$ represents a regular function and T_n is the operator defined by:

$$T_n = \lambda \nabla n \cdot + \mu \nabla \cdot n + \mu \nabla \cdot n^T, \tag{11}$$

here λ, μ are the Lamé constants, n is the outward normal to S. Recalling that $\mathbf{H} \circ \mathbf{E} \equiv \delta$, relation (10) may be rewritten as:

$$\begin{aligned} U(\mathbf{X}) = &- \int_V E(\mathbf{X}, \mathbf{Y})\, B(\mathbf{Y})\, d\mathbf{Y} + \int_S T_n\big(E(\mathbf{X}, \mathbf{Y})\big)\, U(\mathbf{Y})\, dS - \\ &- \int_S E(\mathbf{X}, \mathbf{Y})\, T_n(\mathbf{U})\, dS, \quad \mathbf{Y} \in S. \end{aligned} \tag{12}$$

For the particular case considered here, the elementary solution of equation (9) was derived as [**19**]:

$$E_{ij}(\mathbf{X}, \mathbf{Y}) = \left[\frac{k}{r}\right] \left\{(3 - 4\nu)\,\delta_{ij} + \frac{(x_i - y_i)\,(x_j - y_j)}{r^2}\right\}, \tag{13}$$

where $k = -(1+\nu)/8\pi E(1-\nu)$, E = Young's modulus, ν = Poisson's ratio, while r denotes the distance between the points $\mathbf{X}, \mathbf{Y} \in S$. Let us consider the case of zero body force $(B(\mathbf{Y}) = 0)$. The point $\mathbf{X}$ may be on a smooth surface, or at the intersection of several smooth surfaces. Let $\{\mathbf{U}_i\}_{i=1,2,3}$ denote the displacement field and $\sigma_{ij}(\mathbf{U})$, its corresponding stress field. If $\mathbf{U}_i$ is continuously differentiable in the domain $V - V_\varepsilon$ and on its surface, the divergence theorem may be used to give (the subregion V_ε surrounds $\mathbf{X}$ and tends to zero as $\varepsilon \to 0$, with S_ε we denote the surface of it):

$$c_{ij}(\mathbf{X}, \mathbf{Y})\, U_j(\mathbf{X}) + \int_S T_{ij}(\mathbf{X}, \mathbf{Y})\, U_j(\mathbf{Y})\, dS = \int_S E_{ij}(\mathbf{X}, \mathbf{Y})\, T_n(\mathbf{U})\, dS, \quad (14)$$

where,

$$c_{ij}(\mathbf{X}) = \lim_{\varepsilon \to 0} \int_{S_\varepsilon} T_{ij}(\mathbf{X}, \mathbf{Y})\, dS, \quad (15)$$

and

$$T_{ij}(\mathbf{X}, \mathbf{Y}) = -\frac{k}{r^2}\left[\frac{\partial r}{\partial n}(\delta_{ij} + 3/(1-2\nu)\, r_{,i} r_{,j}) - n_j r_{,i} + n_i r_{,j}\right] \quad (16)$$

here the principal value of the integral on the left side of Eq. (14) is taken n_i, n_j denote the components of the outward normal to S. It is interesting to note that the value of $c_{ij}(\mathbf{X})$ would change if the shape of region of exclusion used to define this principal value is changed. The two-dimensional singular integral equations which solve the second $(II^{\pm})$, third $(III^{\pm})$ and fourth $(IV^{\pm})$ boundary value problem are summarized in Table II. They are derived by using a quite similar limiting process and their form is analagous to that of Eqn (14).

4. DISCUSSION

The results of the present study demonstrate the application of the singular integral operators method to three-dimensional static, dynamic and thermodynamic boundary value problems. In order to solve the systems of the related singular integral equations, Cruse et al. discretized the surface of the three-dimensional region under consideration into subregions. This technique allows inhomogeneous bodies to be analyzed without appreciable reduction of the accuracy. Then it would be possible, by a proper elimination of the traction unknowns, to create a stiffness matrix for each subregion.

In most of the practical problems it appears that the aforementioned integral operators method requires less execution time than the equivalent finite-element analysis provided that the results are required at not too many interior points. Practically, it is the results on the surface which are of primary interest, while results at interior points can be readily obtained by a judicious positioning of interfaces. However, as the complexity of the problem to be solved increases, more tests must be carried out before the conditions under which the proposed method is advantageous (compared with the finite element method) can be precisely defined. The optimal algorithm is probably a mixed-mode one, in which both formulations are used according to their suitability. The generalization to the analysis of time-dependent phenomena presents no additional analytical problems although the numerical procedure to be used is greatly complicated. Material anisotropy, inhomogeneity and non-linearity provide great numerical difficulties and we conclude that the entire range of problems can be solved economically by using the singular operator method. It is anticipated that at least the problem of the permeable anisotropic elastic material, a fair approximation to soil, can be considered.

Finally, a worth-to-mention conclusion of the above analysis is the fact that, independently of the physical problem which is solved, the form of the resulting differential equation (Table I) remains the same. Furthermore, in Table II the aforementioned differential equations are transposed, by following a suitable limiting process, to their corresponding equivalent systems of singular integral equations. Thus, it seems that the complexity of the physical problem solved, af far as the variation of the boundary conditions are concerned, does not mainly affect the form of the system of singular, integral equations to be solved, but rather, it results in a more complicated structure in their kernels.

TABLE I

The integral operators H, B for various physical problems.

The physical problem which is solved with the aid of Eqn(8).	The integral operator H	The integral operator B	List of symbols used for the definition of H,B.
The three-dimensional elastostatic boundary value problem.	$H(\partial_x) = \Vert H_{ij}(\partial_x) \Vert$ $H_{ij}(\partial_x) = \delta_{ij}\mu\Delta\cdot + (\lambda+\mu)\frac{\partial^2\cdot}{\partial x_i \partial x_j}$	$B = \rho F$	λ,μ are the Lamé constants, ρ=the density at the point X, F=the body force per unit volume.
The three-dimensional dynamic boundary value problem.	$H(\partial_x,\omega) = \Vert H_{ij}(\partial_x,\omega) \Vert$ $H_{ij}(\partial_x,\omega) = H_{ij}(\partial_x) + \delta_{ij}\rho\omega^2$	$B = \rho F$	ω = vibration frequency.
The three-dimensional thermoelastic dynamic boundary value problem (for ω=0 the static problem is obtained).	$H(\partial_x,\omega) = \Vert H_{ij}(\partial_x,\omega) \Vert_{4\times 4}$ $H_{kj}(\partial_x,\omega) = H_{kj}(\partial_x) + \delta_{kj}\rho\omega^2$ $H_{k4}(\partial x,\omega) = -\gamma\partial/\partial x_k$ $H_{4j}(\partial_x,\omega) = i\omega\eta\frac{\partial\cdot}{\partial x_j}$ $(k,j=1,2,3)$.	$B_i = \rho F_i$, $(i = 1,2,3)$ $B_4 = Q/\beta$	α = coefficient of expansion, $\gamma = (2\mu+3\lambda)\alpha$, T_0=the initial absolute temperature K=coefficient of thermoconductivity $\beta = K/\delta$, Q = heat quantity, δ = specific heat.
The three-dimensional elastostatic boundary value problem of unsymmetrical elasticity (couple stresses are taken into account).	$H(\partial_x) = \Vert H_{ij}(\partial_x) \Vert_{6\times 6}$ $H(\partial_x) = \begin{Vmatrix} H^{(1)}(\partial_x), & H^{(2)}(\partial_x) \\ H^{(3)}(\partial_x), & H^{(4)}(\partial_x) \end{Vmatrix}$ where $H^{(k)}(\partial_x) = \Vert H^{(k)}_{ij}(\partial_x) \Vert_{3\times 3}$ with $H^{(1)}_{ij}(\partial_x) = (\mu+\alpha)\delta_{ij}\Delta + (\lambda+\mu-\alpha)\frac{\partial^2\cdot}{\partial x_i \partial x_j}$ $H^{(2)}_{ij}(\partial_x) = H^{(3)}_{ij}(\partial_x) = -2\alpha\varepsilon_{ijk}\partial/\partial x_k$ $H^{(4)}_{ij}(\partial_x) = \delta_{ij}[(\upsilon+\beta)\Delta - 4\alpha] + (\varepsilon+\upsilon-\beta)\frac{\partial^2\cdot}{\partial x_i \partial x_j}$	$H_i = F_i$, $(i = 1,2,3)$ $H_i = \Gamma_{i-3}$, $(i = 4,5,6)$	$\alpha,\varepsilon,\upsilon,\beta,\lambda,\mu$ are the elastic constants of the Cosserat material while Γ_i is a properly defined integral operator [4]

Table I (continued)

The physical problem which is solved with the aid of Eqn (8).	The integral operator H. ,	The integral operator B.	List of symbols used for the definition of H,B
The three-dimensional dynamic boundary value problem of unsymmetrical elasticity.	$H(\partial_x,\sigma) = \lVert H_{ij}(\partial_x,\sigma)\rVert_{6\times 6}$ $H_{ij}(\partial_x,\sigma) = H_{ij}(\partial_x)+\delta_{ij}\sigma^2 r_{ij}$, $r = \lVert r_{ij}\rVert_{6\times 6}$ where , $r_{ij} = 0 \quad \forall\ i \neq j$, $r_{ii} = \rho \quad (i = 1,2,3)$, $r_{ii} = J \quad (i = 4,5,6)$.	$B_i = F_i \ (i = 1,2,3)$, $B_i = \Gamma_{i-3}(i = 4,5,6)$.	J,Γ are properly defined integral operators [4] σ is a properly defined real constant [3,4].

TABLE II

The systems of two-dimensional singular integral equations for various physical problems.

The three-dimensional static boundary value problem of unsymmetrical elasticity	The systems of two-dimensional singular integral equations (S.I.E).	List of quantities contained in the corresponding systems of S.I.E.
$(\mathrm{I}^{\pm})$ $\forall y \in S$: $u^{\pm}(y) = f(y)$.	$\mp\varphi(z)+\fint_S [T(\partial_y,\eta)\Psi(y-z)]\varphi(y)d_yS=f(z)$ $\forall\ z,y\in S$.	$\Psi(y-z) = \left\Vert \begin{matrix} \Psi^{(1)}(y-z) & \Psi^{(2)}(y-z) \\ \Psi^{(3)}(y-z) & \Psi^{(4)}(y-z) \end{matrix} \right\Vert ,$
$(\mathrm{II}^{\pm})$ $\forall y \in S$: $[T(\partial_y,\eta)u(y)]^{\pm} = f(y)$.	$\mp\psi(z)+\fint_S [T(\partial_y,\eta)\Psi(y-z)]\psi(y)d_yS=f(z)$ $\forall\ z,y\in S$.	$\Psi^{(\ell)}_{(y-z)} = \left\Vert \Psi^{(\ell)}_{kj}(y-z) \right\Vert_{3\times 3} ,\ \ell = 1,4 ,$ $\Psi^{(1)}_{kj}(y-z) = \frac{\delta_{kj}}{2\pi}\left[\frac{1}{\mu}\frac{1}{\lvert y-z\rvert} - \frac{\alpha}{\mu(\mu+\alpha)}\frac{\exp(-\sigma_4\lvert y-z\rvert)}{\lvert y-z\rvert}\right]$
$(\mathrm{III}^{\pm})$ $\forall y \in S$: $[H(\partial_y,\eta)u(y)]^{\pm} = f(y)$.	$\mp\varphi(z)+\fint_S H(\partial_z,\eta)[R(\partial_y,\eta)\Psi(y-z)]\varphi(y)d_yS=f(z)$ $\forall\ z,y\in S$.	$+\frac{1}{2\pi\mu}\frac{\partial^2}{\partial x_k\partial x_j}\left[\frac{\beta+\upsilon}{4\mu}\frac{(\exp(-\sigma_4\lvert y-z\rvert)-1)}{\lvert y-z\rvert} - \frac{(\lambda+\mu)\lvert y-z\rvert}{2(\lambda+2\mu)}\right] ,$ $\Psi^{(2)}_{kj}(y-z) = \Psi^{(3)}_{kj}(y-z) =$
$(\mathrm{IV}^{\pm})$ $\forall y \in S$: $[R(\partial_y,\eta)u(y)]^{\pm} = f(y)$.	$\pm\psi(z)+\fint_S R(\partial_z,\eta)[H(\partial_y,\eta)\Psi(y-z)]\psi(y)d_yS=f(z)$ $\forall\ z,y\in S$.	$= \frac{1}{4\pi\mu}\sum_{p=1}^{3}\varepsilon_{jkp}\frac{\partial}{\partial x_p}\left[\frac{1-\exp(-\sigma_4\lvert y-z\rvert)}{\lvert y-z\rvert}\right] ,$ $\Psi^{(4)}_{kj}(y-z) = \frac{\delta_{kj}}{2\pi(\beta+\upsilon)}\frac{\exp(-\sigma_4\lvert y-z\rvert)}{\lvert y-z\rvert} +$ $+\frac{1}{8\pi}\frac{\partial^2}{\partial x_k\partial x_j}\left\{\left[\frac{\exp(-\sigma_3\lvert y-z\rvert)-\exp(-\sigma_4\lvert y-z\rvert)}{\alpha\lvert y-z\rvert}\right] -\right.$ $\left.-[\exp(-\sigma_4\lvert y-z\rvert)-1]/\mu\lvert y-z)\right\}.$

Table II (continued)

The dynamic boundary value problem of the three-dimensional elasticity.	The systems of two-dimensional singular integral equations (S.I.E).	List of quantities contained in the corresponding systems of S.I.E.
$\overset{\omega}{(\mathrm{I})}{}^{\pm}$: $u^{\pm}(y,t) = f(y,t)$, $\forall\ y\in S$.	$\mp\varphi(z)+\mathop{{\int\!\!\!\!\!\!-}}_S [T(\partial_y,n)U(y-z,\omega)]\varphi(y)d_yS = f(z)$ $\forall\ z,y\in S$.	$T(\partial_y,n)U(y-z,\omega)=\frac{\delta_{kj}}{2\pi}\sum\limits_{\ell=1}^{3} n_\ell(x)\frac{\partial}{\partial x}\frac{\exp(ik_2\|y-z\|)}{\|y-z\|}$ $+\frac{\lambda}{2\pi(\lambda+2\mu)}n_k(x)\frac{\partial}{\partial x_j}\frac{\exp(ik_1\|y-z\|)}{\|y-z\|}$ $+2\mu\sum\limits_{p=1}^{2}\sum\limits_{\ell=1}^{3}\beta_p\frac{\partial^3}{\partial x_\ell\partial x_k\partial x_j}n_\ell$ $\exp[ik_p\|y-z\|]/\|y-z\|$, $k_1^2=\rho\omega^2(\lambda+2\mu)^{-1}$ $k_2^2=\rho\omega^2\mu^{-1}$, $\alpha_\ell=\delta_{z\ell}(2\pi\mu)^{-1}$, $\beta_\ell=(-1)^\ell(2\pi\rho\omega^2)^{-1}$ $U_{kj}(y-z,\omega)=\sum\limits_{\ell=1}^{2}(\delta_{kj}\alpha_\ell+\beta_\ell\frac{\partial^2}{\partial x_k\partial x_j})\frac{\exp(ik_\ell\|y-z\|)}{\|y-z\|}$
$\overset{\omega}{(\mathrm{II})}{}^{\pm}$: $[T(\partial_y,n)u(y)]^{\pm} = f(y,t)$, $\forall\ y\in S$.	$\mp\psi(z)+\mathop{{\int\!\!\!\!\!\!-}}_S [T(\partial_z,n)U(z-y,\omega)]\psi(y)d_yS = f(z)$ $\forall\ z,y\in S$.	
$\overset{\omega}{(\mathrm{III})}{}^{\pm}$: $\{u(y,t)n\}^{\pm} = g(y,t)$, $\{T(\partial_y,n)u(y,t)-n[nT(\partial_y,n)u(y,t)]\}^{\pm} = h(y,t)$, $\forall\ y\in S$.	$\pm\varphi(z)+\mathop{{\int\!\!\!\!\!\!-}}_S H(\partial_z,v)[R(\partial_y,n)U(y-z,\omega)]\varphi(y)d_yS = f(z)$ $\forall\ z,y\in S$	$[R(\partial_y,n)U(y-z,\omega)]_{kj}=(1-\delta_{j4})\Big\{U_{kj}(y-z)-n_j\sum\limits_{i=1}^{3}n\ U_{ik}(y-z)\Big\}+\delta_{j4}\Big[-\frac{1}{2\pi}\frac{\partial}{\partial x_k}\frac{1}{\|y-z\|}-2\mu K(y)\sum\limits_{i=1}^{3}n_iU_{ik}(y-z)\Big]$
$\overset{\omega}{(\mathrm{IV})}{}^{\pm}$: $\{T(\partial_y,n)u(y,t)\pm\sigma(y,t)u(y,t)\}^{\pm} = f(y,t)$, $\forall\ y\in S$.	$\mp\psi(z)+\mathop{{\int\!\!\!\!\!\!-}}_S R(\partial_z,v)[H(\partial_y,n)U(y-z,\omega)]\psi(y)d_yS = f(z)$ $\forall\ z,y\in S$.	$[H(\partial_y,n)U(y-z)]_{kj}=(1-\delta_{j4})\Big[-\frac{1}{2\pi}[\delta_{kj}\frac{\partial}{\partial n}\frac{1}{\|y-z\|}+n_k\frac{\partial}{\partial x_j}\frac{1}{\|y-z\|}]+2\mu\sum\limits_{i=1}^{3}U_{ik}(y-z)D_j(\partial_y,n)n_i\Big]+\delta_{j4}\sum\limits_{i=1}^{3}n_iU_{ik}(y-z)$

Table II (continued)

he dynamic boundary value problem of he three-dimensional thermoelasticity.	The systems two-dimensional singular integral equations (S.I.E).	List of quantities contained in the corresponding systems of S.I.E.
$(\overset{\omega}{I}{}^{\pm}):\ u^{\pm} = f\ ,\ \vartheta^{\pm} = g\ .$	$\mp\psi(z)+\int_S [R[\partial_y,\eta\ \Phi(z-y,\omega)]\psi(y)d_yS=$ $= F(z) - \frac{1}{2}\int_{D^{\pm}} \Phi(z-y,\omega)H(y)dV$ $\forall\ z,y\in S\ .$	$R(\partial_x,\eta,\gamma)=\lVert R_{kj}(\partial_x,\eta,\gamma)\rVert_{4\times 4} = \left\lVert \begin{matrix} \boxed{T} & & & \begin{matrix}-\gamma\eta_1\\ -\gamma\eta_2\\ -\gamma\eta_3\end{matrix} \\ 0 & 0 & 0 & \partial\cdot/\partial_\eta \end{matrix} \right\rVert\ ,$ $T \equiv T(\partial_x,\eta) = \lVert \overset{k=\mu}{T_{ij}}(\partial_x,\eta)\rVert_{3\times 3} =$
$(\overset{\omega}{II}{}^{\pm}):\ \{P(\partial_y,\eta)U\}^{\pm} = f, \left\{\frac{\partial\vartheta}{\partial\eta}\right\} = g\ .$	$\pm\varphi(z)+\int_S [R(\partial_z,\nu)\Phi(z-y,\omega)]\varphi(y)d_yS=$ $= F(z) - \frac{1}{2}\int_{D^{\pm}} R(\partial_z,\nu)\Phi(z-y,\omega)H(y)dV$ $\forall\ z,y\in S\ .$	$= (\lambda+\mu)\eta_i(x)\frac{\partial\cdot}{\partial x_j} + k\{\eta_j(x)\frac{\partial\cdot}{\partial x_i} - \eta_i(x)\frac{\partial\cdot}{\partial x_j}\} + \delta_{ij}\mu\frac{\partial\cdot}{\partial\eta(x)}$ $\Phi(z-y,\omega)=\sum_{\ell=1}^{3}\{(1-\delta_{k4})(1-\delta_{j4})[\frac{\delta_{kj}}{2\pi\mu}\delta_{3\ell}-\alpha_\ell\frac{\partial^2\cdot}{\partial x_k\partial x_\ell}]$ $+\beta_\ell[i\omega\eta\delta_{k4}(1-\delta_{j4})\frac{\partial\cdot}{\partial x_j}-\gamma\delta_{j4}(1-\delta_{k4})\frac{\partial\cdot}{\partial x_k}]+\delta_{k4}\delta_{j4}\gamma_\ell\}\cdot$ $\cdot\{\exp[i\lambda_\ell\lvert z-y\rvert]\lvert z-y\rvert\}$
$(\overset{\omega}{III}\ \pm):u^{\pm} = f\quad \left\{\frac{\partial\vartheta}{\partial\eta}\right\}^{\pm} = g\ .$	$\mp\lambda(z)+\int_S \{Q(\partial_z,\nu)[P(\partial_y,\eta)\Phi(z-y,\omega)]\}\times$ $\varphi(y)d_yS=F(z)-\frac{1}{2}\int_{D^{\pm}} Q(\partial_z,\nu)\Phi(z-y,\omega)H(y)dV$ $\forall\ z,y\in S\ .$	$\alpha_\ell = \frac{(-1)^\ell(1-i\omega k^{-1}\lambda_\ell^{-2})(\delta_{1\ell}+\delta_{2\ell})}{2\pi(\lambda+2\mu)(\lambda_2^2-\lambda_1^2)} - \frac{\delta_{3\ell}}{2\pi\rho\omega^2}\ ,\ \sum_{\ell=1}^{3}\alpha_\ell = 0$ $\beta_\ell = \frac{(-1)^\ell(\delta_{1\ell}+\delta_{2\ell})}{2\pi(\lambda+2\mu)(\lambda_2^2-\lambda_1^2)}\ ,\ \sum_{\ell=1}^{3}\beta_\ell = 0\ ,$
$(\overset{\omega}{IV}{}^{\pm}):\ \{P(\partial_y,\eta)U\}^{\pm} = f\ ,\ \vartheta^{\pm} = g.$	$\pm\mu(z)+\int_S \{P(\partial_z,\nu)[Q(\partial_y,\eta)\Phi(z-y,\omega)]\}\times$ $\mu(y)d_yS=F(z)-\frac{1}{2}\int_{D^{\pm}} P(\partial_z,\nu)\Phi(z-y,\omega)H(y)d_yS$ $\forall\ z,y\in S\ .$	$\gamma_\ell = \frac{(-1)^\ell(\lambda_\ell^2-k_1^2)(\delta_{1\ell}+\delta_{2\ell})}{2\pi(\lambda_2^2-\lambda_1^2)}\quad \sum_{\ell=1}^{3}\gamma_\ell = 1\ .$ $Q_{kj}(\partial_x,\eta(x)) = (1-\delta_{k4})(1-\delta_{j4})\delta_{kj}+\delta_{k4}\delta_{j4}\frac{\partial\cdot}{\partial\eta(x)}\ ,$ $\partial\cdot/\partial\eta(x) = \sum_{\ell=1}^{3}\eta_\ell(x)\frac{\partial\cdot}{\partial x_\ell}\ ,$ $P_{kj}(\partial_x,\eta(x),\omega)=(1-\delta_{k4})(1-\delta_{j4})\left[\delta_{kj}\mu\frac{\partial\cdot}{\partial\eta(x)} + \lambda\eta_k(x)\frac{\partial\cdot}{\partial x_j} + \right.$ $\left. + \mu\eta_j(x)\frac{\partial\cdot}{\partial x_k}\right] - i\omega\eta\delta_{j4}\ (1-\delta_{k4})\eta_k(x) - \delta_{k4}\delta_{j4}\ .$

Table II (continued)

The three-dimensional dynamic boundary value problem of unsymmetrical elasticity	The systems of two-dimensional singular integral equations (S.I.E)	List of quantities contained in the corresponding systems of S.I.E.
$(\overset{\omega}{III^{\pm}})$: $\forall y \in S$: $u^{\pm}(y,t) = f(y,t)$ $\left\{T^{(4)}(\partial_y,\eta)\omega(y,t)\right\}^{\pm} = g(y,t)$.	$\mp\varphi(z) + \int_S H(\partial_z,\eta)[R(\partial_y,\eta)\Psi(y-z,\sigma)]\varphi(y)d_yS =$ $= f(z)$, $\forall\ z,y \in S$.	$T \equiv T(\partial_x,\eta(x)) = \| T_{ij}(\partial_x,\eta(x)) \|_{6\times 6} =$ $= \begin{Vmatrix} T^{(1)}(\partial_x,\eta(x)) & T^{(2)}(\partial_x,\eta(x)) \\ T^{(3)}(\partial_x,\eta(x)) & T^{(4)}(\partial_x,\eta(x)) \end{Vmatrix}$, $T^{(k)}(\partial_x,\eta(x)) = \| T^{(k)}_{ij}(\partial_x,\eta(x)) \|_{3\times 3}$, $T^{(1)}(\partial_x,\eta(x))\cdot = \lambda\eta_i(x)\frac{\partial\cdot}{\partial x_j} + (\mu-\alpha)\eta_j(x)\frac{\partial\cdot}{\partial x_i} + (\mu+\alpha)\delta_{ij}\frac{\partial\cdot}{\partial\eta(x)}$, $T^{(2)}_{ij}(\partial_x,\eta(x))\cdot = -2\alpha\sum_i \varepsilon_{ij\ell}\eta_\ell(x)$, $T^{(3)}_{ij}(\partial_x,\eta(x)) = 0$, $T^{(4)}_{ij}(\partial_x,\eta(x))\cdot = \varepsilon\eta_i(x)\frac{\partial\cdot}{\partial x_j} + (\upsilon-\beta)\eta_j(x)\frac{\partial\cdot}{\partial x_i} + (\upsilon+\beta)\delta_{ij}\frac{\partial\cdot}{\partial\eta(x)}$ $H(\partial_x,\eta(x)) = \begin{Vmatrix} E & 0 \\ 0 & -T^{(4)}(\partial_x,\eta(x)) \end{Vmatrix}$, $R(\partial_x,\eta(x)) = \begin{Vmatrix} T^{(1)}(\partial_x,\eta(x)) & T^{(2)}(\partial_x,\eta(x)) \\ 0 & E \end{Vmatrix}$, $E = \|E_{ij}\|_{3\times 3}$ $E_{ij} = \delta_{ij}$, $E_{i3} = E_{3j} = 0$.
$(\overset{\omega}{IV^{\pm}})$: $\forall\ y \in S$: $\omega^{\pm}(y,t) = f(y,t)$ $\left\{T^{(1)}(\partial_y,\eta)u(y,t) + T^{(2)}(\partial_y,\eta)\omega(y,t)\right\}^{\pm} = g(y,t)$.	$\pm\psi(z) + \int_S R(\partial_z,\eta)[H(\partial_y,\eta)\Psi(y-z,\sigma)]\psi(y)d_yS =$ $= f(z)$, $\forall\ z,y \in S$.	

Table II (continued)

The three-dimensional elastostatic boundary value problem.	The systems of two-dimensional singular integral equations (S.I.E).	List of quantities contained in the corresponding systems of S.I.E.
$(I^{\pm})$: $\forall z \in S$: $\varphi(z) = u^{\pm}(z)$.	$f(z) = \mp\varphi(z) + \int_S [T(\partial_y,\eta)U(y-z)]\varphi(y)\,d_yS$ $\forall z,y \in S$.	$[T(\partial_y,\eta)U(y-z)]_{kj} = \frac{\delta_{kj}}{2\pi}\frac{\partial}{\partial\eta}\frac{1}{\lvert y-z\rvert} + \mu\sum_{i=1}^{3} M_{ji}(\partial_y,\eta)\left[(\lambda'-\mu')\frac{\delta_{ik}}{\lvert y-z\rvert} + 2\mu'\frac{(y_i-z_i)(y_k-z_k)}{\lvert y-z\rvert^3}\right]$ $\lambda' = (\lambda+3\mu)/4\pi\mu(\lambda+2\mu)$, $\mu' = (\lambda+\mu)/4\pi\mu(\lambda+2\mu)$ $M_{kj}(\partial_y,\eta) = \eta_j\frac{\partial\cdot}{\partial y_k} - \eta_k\frac{\partial\cdot}{\partial y_j}$
$(II^{\pm})$: $\forall y \in S$, $[T(\partial_y,\eta)u(y)]^{\pm} = f(y)$.	$f(z) = \pm\psi(z) + \int_S [T(\partial_z,\eta)U(y-z)]\psi(y)\,d_yS$ $\forall z,y \in S$.	$U_{kj}(y-z) = \frac{\delta_{kj}}{2\pi\mu\lvert y-z\rvert} - \mu'\frac{\partial^2\lvert y-z\rvert}{\partial z_k\partial y_j}$
$(III^{\pm})$: $\forall z \in S$: $\{T(\partial_z,\eta)u-\eta(\eta T(\partial_z,\eta)u)\}^{\pm} = f(z)$, $\{u_\eta\}^{\pm} = f_4(z)$.	$f(z) = \mp\varphi(z) + \int_S H(\partial_z,\eta)[R(\partial_y,\eta)\Gamma(y-z)]\varphi(y)\,d_yS$ $\forall z,y \in S$.	$[R(\partial_y,\eta)U(y-z)]_{kj} = (1-\delta_{j4})\left\{U_{kj}(y-z) - \eta_j\sum_{i=1}^{3}\eta_i U_{ik}\right\} + \delta_{j4}\left\{-\frac{1}{2\pi}\frac{\partial}{\partial x_k}\frac{1}{\lvert y-z\rvert} - 2\mu K(z)\sum_{i=1}^{3}\eta_i U_{ik}(y-z)\right\}$
$(IV^{\pm})$: $\forall z \in S$: $\{u-\eta u_\eta\}^{\pm} = f(z)$, $\{\eta T(\partial_z,\eta)u\}^{\pm} = f_4(z)$.	$f(z) = \pm\psi(z) + \int_S R(\partial_z,\eta)[H(\partial_y,\eta)\Gamma(y-z)]\psi(y)\,d_yS$ $\forall z,y \in S$.	$\{H(\partial_y,\eta)U(y-z)\}_{kj} = (1-\delta_{j4})\left\{-\frac{1}{2\pi}\left(\delta_{kj}\frac{\partial}{\partial\eta}\frac{1}{\lvert y-z\rvert}\ \eta_k\frac{\partial}{\partial x_j}\frac{1}{\lvert y-z\rvert}\right) + 2\mu\sum_{i=1}^{3}U_{ik}(y-z)D_j(\partial_y,\eta)\eta_j\right\} + \delta_{j4}\sum_{i=1}^{3}\eta_i U_{ik}(y-z)$.

Table II (continued)

The three-dimensional dynamic boundary value problem of unsymmetrical elasticity.	The systems of two-dimensional singular integral equations (S.I.E).	List of quantities contained in the corresponding systems of S.I.E.
$(I^{\pm}_{\omega})$: $\forall y \in S$: $u^{\pm}(y,t) = f(y,t)$ $\omega^{\pm}(y,t) = g(y,t)$, f,g are given functions on S .	$\mp\varphi(z) + \int_S [T(\partial_y,\eta)\Psi(y-z,\sigma)]\varphi(y)d_yS = f(z)$ $\forall$ z,y$\in$S .	$\Psi(x,\sigma) = \begin{Vmatrix} \Psi^{(1)}(x,\sigma) & \Psi^{(2)}(x,\sigma) \\ \Psi^{(3)}(x,\sigma) & \Psi^{(4)}(x,\sigma) \end{Vmatrix}$, $\Psi^{(\ell)}(x,\sigma) = \Vert \Psi^{(\ell)}_{kj}(x,\sigma) \Vert_{3\times 3}$ $\ell = \overline{1,4}$, $\Psi^{(1)}_{kj}(x,\sigma) = \sum_{\ell=1}^{4}\left[\delta_{kj}\alpha_\ell + \beta_\ell \frac{\partial^2}{\partial x_k \partial x_j}\right]\frac{\exp(ik_\ell\vert x\vert)}{\vert x\vert}$,
$(II^{\pm}_{\omega})$: $\forall y \in S$: $\left\{T^{(1)}(\partial_y,\eta)u(y,t) + T^{(2)}(\partial_y,\eta)\omega(y,t)\right\}^{\pm} = f(y,t)$ $\left\{T^{(4)}(\partial_y,\eta)\omega(y,t)\right\}^{\pm} = g(y,t)$.	$\pm\psi(z) + \int_S [T(\partial_y,\eta)\Psi(y-z,\sigma)]\psi(y)d_yS = f(z)$ $\forall$ z,y$\in$S .	$\Psi^{(2)}_{kj}(x,\sigma) = \Psi^{(3)}_{kj}(x,\sigma) = \frac{2\alpha}{\mu+\alpha}\sum_{\ell=1}^{4}\sum_{p=1}^{3}\varepsilon_\ell \varepsilon_{kjp}\frac{\partial}{\partial x_p}\frac{\exp(ik_\ell\vert x\vert)}{\vert x\vert}$ $\Psi^{(4)}_{kj}(x,\sigma) = \sum_{\ell=1}^{4}\left[\delta_j\gamma_\ell + \delta_\ell \frac{\partial^2}{\partial x_k \partial x_j}\right]\frac{\exp(ik_\ell\vert x\vert)}{\vert x\vert}$, $\alpha_\ell = \frac{(-1)^\ell(\sigma_2^2-k_\ell^2)(\delta_{3\ell}+\delta_{4\ell})}{2\pi(\mu+\alpha)(k_3^2-k_4^2)}$, $\beta_\ell = -\frac{\delta_{1\ell}}{2\pi\rho\sigma^2} + \frac{\alpha_\ell}{k_1^2}$, $\sum_{\ell=1}^{4}\beta_\ell = 0$, $\gamma_\ell = \frac{(-1)^\ell(\sigma_1^2-k_\ell^2)(\delta_{3\ell}+\delta_{4\ell})}{2\pi(\beta+\upsilon)(k_3^2-k_4^2)}$, $\delta_\ell = -\frac{\delta_{2\ell}}{2\pi(J\sigma^2-4\alpha)} + \frac{\gamma_\ell}{k_\ell^2}$, $\sum_{\ell=1}^{4}\delta_\ell = 0$, $\varepsilon_\ell = \frac{(-1)^\ell(\delta_{3\ell}+\delta_{4\ell})}{2\pi(\beta+\upsilon)(k_3^2-k_4^2)}$, $\sum_{\ell=1}^{4}\varepsilon_\ell = 0$, $k_1^2 = \frac{\rho\sigma^2}{\lambda+2\mu}$, $k_2^2 = (J\sigma^2-4\alpha)/(\varepsilon+2\upsilon)$, $k_3^2+k_4^2 = \sigma_1^2+\sigma_2^2+4\alpha^2/(\mu+\alpha)(\beta+\upsilon)$ $k_3^2k_4^2 = \sigma_1^2\sigma_2^2$.

Table II (continued)

The three-dimensional thermoelastic boundary value problem.	The systems of two-dimensional singular integral equations (S.I.E).	List of quantities contained in the corresponding systems of S.I.E.
$(I^{\pm}):\ u^{\pm} = f\ ,\ \vartheta^{\pm} = g\ .$	$\mp\psi(z) + -\!\!\!\!\!\!\int_S [R(\partial_y,\eta)\Phi(z-y)]\psi(y)d_yS = F(z)$ $\forall\ z,y\in S\ .$	$\Phi(z-y) \equiv \lVert \Phi_{kj}(z-y) \rVert_{4\times 4}\ ,$
$(II^{\pm}):\ \{P(\partial_y,\eta)U\}^{\pm} = f\ ,$ $\left\{\frac{\partial\vartheta}{\partial\eta}\right\}^{\pm} = g\ .$	$\pm\varphi(z) + -\!\!\!\!\!\!\int_S [R(\partial_z,\nu)\Phi(z-y)]\varphi(y)d_yS = F(z)$ $\forall\ z,y\in S\ .$	$\Phi_{kj}(z-y) = (1-\delta_{k4})(1-\delta_{j4})U_{kj}(z-y) +$ $+\ \frac{\gamma\delta_{j4}(1-\delta_{k4})(z_k-y_k)}{4\pi(\lambda+2\mu)\lvert z_k-y_k\rvert} + \frac{\delta_{k4}\delta_{j4}}{2\pi}\frac{1}{\lvert z-y\rvert}\ ,$
$(III^{\pm}):\ u^{\pm} = f\ \ \left\{\frac{\partial\vartheta}{\partial\eta}\right\} = g\ .$	$\mp\lambda(z) + -\!\!\!\!\!\!\int_S \{Q(\partial_z,\nu)[P(\partial_y,\eta)\Phi(z-y)]\lambda(y)\ d_yS =$ $= F(z)$ $\forall\ z,y\in S\ .$	$P_{kj} = (1-\delta_{k4})(1-\delta_{j4})\left[\delta_{kj}\mu\frac{\partial\cdot}{\partial\eta} + \lambda\eta_k\frac{\partial\cdot}{\partial x_j} +\right.$ $\left.\mu\eta_j(x)\frac{\partial\cdot}{\partial x_k}\right] - \delta_{k4}\delta_{j4}\ .$
$(IV^{\pm}):\ \{P(\partial_y,\eta)U\}^{\pm} = f\ ,\ \vartheta^{\pm} = g\ .$	$\pm\mu(z) + -\!\!\!\!\!\!\int_S \{P(\partial_z,\nu)[Q(\partial_y,\eta)\Phi(z-y)]\mu(y)d_yS = F(z)$ $\forall\ z,y\in S\ .$	....

Π Ε Ρ Ι Λ Η Ψ Ι Σ

Εἰς τὸ παρὸν ἄρθρον μορφώνονται ἀρχικῶς, τῇ βοηθείᾳ τοῦ τελεστικοῦ λογισμοῦ, τὰ συστήματα διαφορικῶν ἐξισώσεων τὰ ἐπιλύοντα τὸ τριδιάστατον συνοριακὸν πρόβλημα τυχόντος ἐγκλείσματος D (τοῦ ὁποίου ἡ ἐπιφάνεια θεωρεῖται ὡς λεία ἐπιφάνεια Lyapounov) εἰς τὴν γενικωτάτην του μορφήν. Συγκεκριμένως, παρέχεται κατὰ τρόπον συστηματικόν, (εἰς τὸν πίνακα I), ἡ μορφὴ τῶν διαφορικῶν αὐτῶν ἐξισώσεων διὰ τὰς περιπτώσεις στατικῶν, δυναμικῶν, στατικῶν - θερμοελαστικῶν καὶ δυναμικῶν - θερμοελαστικῶν τρισδιαστάτων συνοριακῶν προβλημάτων τῆς συμμέτρου ἢ ἀσυμμέτρου ἐλαστικότητος (ἐλαστικότητος τύπου COSSERAT).

Παρατηρεῖται ὅτι ἡ μορφὴ τῶν ἐξισώσεων αὐτῶν παραμένει ἀναλλοίωτος (ἐξίσωσις (8)) ἀνεξαρτήτως τοῦ ἐπιλυομένου προβλήματος. Ἡ παρατήρησις αὐτὴ ἀποτελεῖ τὴν συγκεκριμένην ἔκφρασιν ἀορίστων μέχρι τώρα ἰσχυρισμῶν [**1** ἕως **29**] περὶ τῆς δυνατότητος ἑνοποιημένης ἀντιμετωπίσεως τῶν τρισδιαστάτων συνοριακῶν προβλημάτων (τύπων $I^{\pm}$, $II^{\pm}$, $III^{\pm}$, $IV^{\pm}$) ποὺ ἀφοροῦν λείαν ἐπιφάνεια S τυχόντος ἐγκλείσματος D.

Δεδομένου ὅτι ἡ ὑπὸ κλειστὴν μορφὴν ἐπίλυσις τῶν συστημάτων τῶν διαφορικῶν ἐξισώσεων ποὺ προτείνονται δὲν κατέστη πρὸς τὸ παρὸν ἐφικτή, ἡ ἐπίλυσίς των ἀφορᾷ τὴν ἐφαρμογὴν δοκιμασμένων ἀριθμητικῶν μεθόδων.

Τὰ κύρια χαρακτηριστικὰ τυχούσης ἀριθμητικῆς μεθόδου, ἀνεξαρτήτως τοῦ ἐπιλυομένου προβλήματος, εἶναι : i) τὸ μέγεθος τοῦ πολυπλόκου τῆς μεθόδου, ii) ἡ ταχύτης συγκλίσεως τῶν ἀποτελεσμάτων της, iii) ἡ οἰκονομικὴ ἐφαρμογή της διὰ τῶν διατιθεμένων ἠλεκτρονικῶν ὑπολογιστῶν.

Ἡ γενικωτέρα μέθοδος ποὺ ὑφίσταται μέχρι σήμερον εἶναι ἡ μέθοδος τῶν π ε π ε ρ α σ μ έ ν ω ν σ τ ο ι χ ε ί ω ν. Ἡ κατάστρωσις τοῦ πρὸς ἐπίλυσιν γραμμικοῦ συστήματος ἐξισώσεων πρὸς τὸ ὁποῖον εἶναι ἀριθμητικῶς ἰσοδύναμοι οἱ ἐξισώσεις τοῦ πίνακος I, δὲν συναντᾶ θεωρητικὰς δυσχερείας. Ἐν τούτοις ἡ ἐπίλυσις τοῦ συστήματος αὐτοῦ, λόγῳ τῆς μεγάλης διαστάσεως τῶν πρὸς ἀντιστροφὴν μητρῴων, καθίσταται δυσχερεστάτη, ἄκρως ἀντιοικονομικὴ πολλὲς φορὲς δὲ λόγῳ περιορισμοῦ τῶν θέσεων μνήμης τοῦ διατιθεμένου ἠλεκτρονικοῦ διερευνητοῦ καὶ ἀνέφικτος. Τοῦτο ὀφείλεται ἀφ᾽ ἑνὸς μὲν εἰς τὸ γεγονὸς ὅτι διαμερίζεται ὁλόκληρος ὁ ὄγκος τοῦ τριδιαστάτου ἐγκλείσματος D εἰς στοιχειώδεις περιοχάς, ἀφ᾽ ἑτέρου δὲ εἰς τὸν μεγάλον ἀριθμὸν τῶν ἐνδιαμέσων καὶ ἄνευ σημασίας ἀποτελεσμάτων ποὺ παρέχονται διὰ τῆς μεθόδου τῶν πεπερασμένων στοιχείων. Πρόσθετος δυσκολία διὰ τὸν ἐρευνητὴν εἶναι καὶ ἡ προετοιμασία τοῦ τεραστίου ἀριθμοῦ δεδομένων ποὺ προηγοῦνται τοῦ προγράμματος τῶν πεπερασμένων στοιχείων.

Ἤδη, κατὰ τὰς τελευταίας δεκαετίας ἀνεπτύχθησαν μέθοδοι ἀναγωγῆς τῆς ἐπιλύσεως τῶν προτεινομένων διαφορικῶν ἐξισώσεων, εἰς τὴν ἐπίλυσιν τῶν ἀντιστοίχων συστημάτων ἰ δ ι ο μ ό ρ φ ω ν δ ι δ ι α σ τ ά τ ω ν ὁ λ ο κ λ η ρ ω τ ι κ ῶ ν ἐ ξ ι σ ώ σ ε ω ν. Ἡ εὕρεσις τῆς μορφῆς τῶν ἰδιομόρφων αὐτῶν ἐξισώσεων, ποὺ πραγματοποιεῖται τῇ βοηθείᾳ τῆς θεωρίας τῶν ἰ δ ι ο μ ό ρ φ ω ν ὁ λ ο κ λ η ρ ω τ ι κ ῶ ν τ ε λ ε σ τ ῶ ν, εἶναι δυσχερεστάτη.

Εἰς τὸ παρὸν ἄρθρον παρέχονται αἱ μορφαὶ τῶν συστημάτων τῶν ἰδιομόρφων διδιαστάτων ὁλοκληρωτικῶν ἐξισώσεων (πίναξ II) ποὺ ἐπιλύουν τὰ ὑπ' ἀριθμ. ($I^{\pm}$, $II^{\pm}$, $III^{\pm}$, $IV^{\pm}$) συνοριακὰ προβλήματα δι' ἑκάστην τῶν περιπτώσεων τοῦ πίνακος I.

Ἡ προτεινομένη ἐπίλυσις τῶν ἐξισώσεων τοῦ πίνακος II ὑπερτερεῖ σημαντικῶς τῆς μεθόδου τῶν πεπερασμένων στοιχείων ὡς πρὸς τὴν ταχύτητα συγκλίσεως τῶν ἀποτελεσμάτων καὶ τὴν οἰκονομικὴν ἐπίλυσιν τῶν προτεινομένων γραμμικῶν συστημάτων. Τοῦτο δὲ διότι διαμερίζεται ἡ ἐπιφάνεια S τοῦ τριδιαστάτου ἐγκλείσματος καὶ ὄχι ὁλόκληρος ὁ ὄγκος του D, ὅπως συνέβαινε εἰς τὴν μέθοδον τῶν πεπερασμένων στοιχείων. Τοιουτοτρόπως ἡ διάστασις τοῦ ἐπιλυομένου προβλήματος ἐλαττοῦται κατὰ μονάδα, τὸ δὲ μέγεθος τῶν πρὸς ἀντιστροφὴν μητρῴων καθίσταται σημαντικῶς μικρότερον.

Ἡ προτεινομένη ὅμως μέθοδος εἶναι θεωρητικῶς κατὰ πολὺ δυσχερεστέρα τῆς μεθόδου τῶν πεπερασμένων στοιχείων.

Δύναται, κατὰ συνέπειαν, νὰ λεχθῇ ὅτι ἡ πρώτη μέθοδος μεταφράζει τὴν δυσχέρειαν ἐπιλύσεως τοῦ τριδιαστάτου προβλήματος καθ' ὁλοκληρίαν εἰς ὑπολογιστικὴν τοιαύτην, ἐνῶ ἡ δευτέρα ποὺ ἀφορᾷ τὴν ἐφαρμογὴν τῆς θεωρίας τῶν ἰδιομόρφων τελεστῶν τὴν μετατρέπει κυρίως εἰς δυσχέρειαν ἀναλυτικῆς φύσεως.

Παρατηρεῖται καὶ πάλιν, ὅτι ἀνεξαρτήτως τοῦ ἐπιλυομένου προβλήματος, ἡ μορφολογία τῶν προτεινομένων ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων παραμένει ἀναλλοίωτος. Ὅσον πολυπλοκώτερον γίνεται τὸ πρὸς ἐπίλυσιν πρόβλημα, τόσον περισσότερον πολυπλόκου μορφῆς καθίστανται οἱ πυρῆνες τῶν προτεινομένων ἐξισώσεων (πίναξ II). Ἡ παρατήρησις αὐτὴ ἐπιτρέπει νὰ διατυπωθῇ ἡ ἄποψις τῆς ἑνοποιήσεως ὅλων τῶν προβλημάτων ποὺ ἀνεφέρθησαν εἰς τὸν πίνακα I εἰς ἓν γενικώτατον πρόβλημα μὲ ὡρισμένην διαφορικὴν ἐξίσωσιν καὶ καθωρισμένην πρὸς ἐπίλυσιν ἰδιόμορφον ὁλοκληρωτικὴν ἐξίσωσιν.

Τέλος, προσφάτως ἀνεπτύχθησαν μέθοδοι ἀμέσου ἐπιλύσεως τῶν διδιαστάτων ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων ὑπὸ τοῦ συγγραφέως καὶ τῶν συνεργατῶν του. Βασίζονται αὐταὶ εἰς τὴν διατύπωσιν ἀριθμητικῶν τύπων GAUSS διὰ τὸν ὑπολογισμὸν τῶν ὑπεισερχομένων εἰς αὐτὰς διδιαστάτων ἰδιομόρφων ὁλοκληρωμάτων [**30** ἕως **34**]. Οὕτως, ὁ ἀνεπιθύμητος διαμερισμὸς τῆς ἐπιφανείας S καταργεῖται, ἡ δὲ διάστασις τῶν πρὸς ἐπίλυσιν μητρῴων καθίσταται εἰσέτι μικροτέρα. Ἄμεσον ἀποτέλεσμα εἶναι ἡ ἀκρίβεια ἀποτελεσμάτων καὶ ἡ οἰκονομικὴ ὑπολογιστικὴ ἐπεξεργασία τῶν σχετικῶν προγραμμάτων.

Αἱ ἄμεσοι μέθοδοι ἔχουν ἰδιαιτέραν σημασίαν εἰς τὴν ἀντιμετώπισιν τῶν τριδιαστάτων προβλημάτων ρωγμῶν, ὅπου ἡ ἐφαρμογὴ ἄλλων μεθόδων, μὲ ἱκανοποιητικὴν ἀκρίβειαν, καθίσταται ἐκ τῶν πραγμάτων ἀδύνατος.

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