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ΜΑΘΗΜΑΤΙΚΑ.— **On the solution of systems of singular integral equations with variable coefficients and complex weight functions,** *by P. S. Theocaris and G. Tsamasfyros**.

## ABSTRACT

Simple numerical methods for solving singular integral equations of the first and second kind with variable coefficients are described. The methods are based on the generalized Gauss - Jacobi quadrature formula for singular integrals. A further extension of the first of the two methods described is presented for generalized Cauchy kernels.

## 1. INTRODUCTION

The solution of a large class of boundary-value problems in physics and engineering can be reduced to a system of singular-integral equations along a finite part of the real axis. Since any finine interval can converted to the interval [−1, 1] by means of a linear transformation, we assume the equation to be of the form:

$$\mathbf{a}(t)\,\boldsymbol{\omega}(t)+\frac{\mathbf{b}(t)}{\pi}\int_{-1}^{1}\frac{\boldsymbol{\omega}(x)}{x-t}\,dx+\int_{-1}^{1}\mathbf{k}(x,t)\,\boldsymbol{\omega}(x)\,dx=\mathbf{f}(t);\quad -1<t<1 \qquad (1.\ 1)$$

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where the first integral is to be considered as a principal-value integral. Moreover, $\boldsymbol{\omega} = (\omega_i)$, $(i = 1, \ldots, N)$, denotes the unknown (vector) function, the square matrices $\mathbf{a} = (a_{ij})$, $\mathbf{b} = (b_{ij})$, $(i, j = 1, \ldots, N)$, are considered known, with $\mathbf{a} \pm i\mathbf{b}$ non-singular in $[-1, 1]$, and the vector $\mathbf{f} = (f_i)$, $(i = 1, \ldots, N)$, consists of the known functions.

On the other hand, the elements $k_{ij}(x, t)$ on the square matrix $\mathbf{k}$ as well as the functions $a_{ij}$, $b_{ij}$, $f_i$ are assumed satisfying a Hölder condition in each of the variables x and t. The unknown functions $\omega_i(t)$ will always be sought in the class $H^*$ ($H^*$ is the class of functions which satisfy a Hölder condition in any closed interval of the open interval $(-1, 1)$ and have integrable singularities at the end points $t = \pm 1$ [**1**]).

Taking into consideration that any technique of solution of singular integral equations may be readily generalized to apply to a system of integral equations, it will be sufficient to consider the solution of the singular integral equations arising from Eq. (1. 1), if it is supposed that $N = 1$, i.e. $\mathbf{a}$, $\mathbf{b}$, $\boldsymbol{\omega}$, $\mathbf{f}$ are matrices with a single element. Thus, in the rest of the paper we use, instead of bold-type sumbols $\mathbf{a}$, $\mathbf{b}$, $\boldsymbol{\omega}$, $\mathbf{f}$ the symbols a, b, ω, f and thus Eq. (1. 1) takes the form:

$$a(t)\,\omega(t) + \frac{b(t)}{\pi}\int_{-1}^{1}\frac{\omega(x)}{x-t}\,dx + \int_{-1}^{1}k(x, t)\,\omega(x)\,dx = f(t); \quad -1<t<1 \qquad (1.\ 2)$$

The singular behavior of the functions ω around $x = \pm 1$ may be obtained from the dominant part of the integral equations (1. 2) by applying a method given in refs. [**1**] and [**2**]. It can be readily shown that the fundamental function w(z), which characterizes the singular behavior of ω, is given by:

$$w(z) = (1-z)^{\alpha}(1+z)^{\beta}\Lambda(z) \qquad (1.\ 3)$$

where:

$$\alpha = \frac{1}{2\pi i}\log\left[\frac{a(1)-ib(1)}{a(1)+ib(1)}\right] + \mu' \qquad (1.\ 4)$$

$$\beta = -\frac{1}{2\pi i}\log\left[\frac{a(-1)-ib(-1)}{a(-1)+ib(-1)}\right] + \mu'' \qquad (1.\ 5)$$

and $\Lambda$ is a non-vanishing in $(-1,1)$ Hölder-continuous function, $\mu'$ and $\mu''$ are integers, chosen in such a way that the behavior of the fundamental function w at points $x = \pm 1$ is compatible with the expected singular behavior of the unknown function $\omega(x)$ (i. e. either bounded, if $0 < \mathrm{Re}\alpha$, $\mathrm{Re}\beta < 1$, or infinite, but integrable, if $-1 < \mathrm{Re}\alpha$, $\mathrm{Re}\beta < 0$).

The constant $\varkappa$ is expressed by:

$$\varkappa = -(\mu' + \mu'') \tag{1. 6}$$

and it is known as the index of $w(x)$. If the index $\varkappa$ is greater than zero, then $\omega$, besides satisfying equation (1. 2), must also satisfy an additional condition, which is invariably of the form:

$$\int_{-1}^{1} \omega(x)\, dx = A \tag{1. 7}$$

where A is a constant.

In this way a system of singular integral equations is reduced to an equivalent system of Fredholm equations, whose solution may be obtained by numerical means. In order to avoid the unnecessary operations, it seems to be desirable to develop a direct approximate method, preserving the correct nature of singularities of the unknown function $\omega(x)$.

As yet only the case of constant values for a and b was considered. In ref. [**3**], a series expansion of $\omega(x)$ in Jacobi polynomials was used. With the aid of well-known orthogonality properties of the Jacobi polynomials, an infinite system of equations was obtained. Thus, the coefficients of the expansion were obtained approximately by truncation. For $a = 0$ a special quadrature formula was used in ref. [**3**] for the singular integral.

The general case with $a \neq 0$ has been considered in ref. [**4**] by using a cumbersome procedure to prove an integration formula for the singular integral. In ref. [**5**] it was proved that the quadrature formula used in [**3**] for the singular integrals was of a Gauss-type.

An extension of standard rules of numerical integration for regular integrals, to the case of Cauchy-type principal-value integrals with arbitrary integration intervals and weight functions was presented in [**6**]. In a publication of the first of the present authors [**8**], appeared for the first time in 1976 [**7**], it was given a general quadrature scheme to include an arbitrary number of preassigned nodes for the Cauchy-type integrals. In the same paper the integration rules of Gauss-Jacobi (open), Radau-Jacobi (semi-closed) and Lobatto-Jacobi (closed) were completely investigated. On the other hand, a number of theorems given in ref. [**7**] prescribed the condition of the applicability of the method.

These results were applied later-on to a series of papers [**9**] to [**12**]. The cases of complex values for $\alpha$ and $\beta$ were also considered in refs. [**13**, **14**]. Moreover, the convergence of the quadrature rule for singular integrals was proved [**15**] and, in a recent paper, the equivalence of the direct and indirect methods was demonstrated [**16**] and consequently the convergence of the solution was proved.

In the present paper the case of complex-weight functions is reconsidered. A generalized quadrature formula to include an arbitrary number of preassigned nodes is derived. For reasons of simplicity only the case where the preassigned node(s) is either at one limit, or at both limits of the integration interval, is considered here. Finally, based on this integration formula for the approximate evaluation of the singular integral, we propose two methods for the numerical solution of the singular integral equation (1. 2) with variable coefficients. But, in order to be in a measure to apply the method described in paragraph 4 of the paper, which is more accurate than the method described in paragraph 6, it is necessary to prove the existence of a sufficient number of roots of a complicated transcendental equation. The theorems, given in the text, allow to find a lower limit of the number of such roots.

## 2. THE QUADRATURE FORMULA

Let the singular integral $I(\omega)$ be defined by:

$$I(\omega) = \int_{-1}^{1} \frac{\omega(x)}{(x-t)} dx ; \quad -1<t<1 \tag{2. 1}$$

with :

$$\left.\begin{aligned} \omega(x) &= w(x)\,\varphi(x) \\ w(x) &= (1-x)^{\alpha}(1+x)^{\beta}\,L(x) \end{aligned}\right\} \tag{2. 2}$$

where $\varphi$ is an analytic function in a domain D containing the domain $[-1, 1]$ in its interior and L is free to be selected equal to either $\Lambda$ or to 1. If $\varphi$ is a Holder-continuous function ($\varphi \in H$), then $\omega$ is said to belong to $H^*$ ($\omega \in H^*$) [**1** § 77].

Let also the m preassigned nodes $\{y_k\}_{k=1}^{m}$ of the quadrature rule ($m \leqq 2$), if they exist, be chosen to coincide either with the one limit, or with both limits.

We denote now by:

$$W(x) = (1-x)^{\mu} (1+x)^{\nu} L(x) \tag{2. 3}$$

and we introduce the function $\Omega(n)$ where:

i) if $m = 2$ and $y_1 = -1$, $y_2 = 1$ then:

$$\mu = \alpha + 1, \quad \nu = \beta + 1 \quad \text{and} \quad \Omega(x) = (1 - x^2)$$

ii) if $m = 1$ and $y_1 = -1$ then:

$$\mu = \alpha, \quad \nu = \beta + 1 \quad \text{and} \quad \Omega(x) = (1 + x)$$

iii) if $m = 1$ and $y_1 = 1$ then:

$$\mu = \alpha + 1, \quad \nu = \beta \quad \text{and} \quad \Omega(x) = (1 - x)$$

iv) if $m = 0$ then:

$$\mu = \alpha, \quad \nu = \beta \quad \text{and} \quad \Omega(x) = 1.$$

We express now $\pi_n(x)$ the product:

$$\pi_n(x) = (x - x_1)(x - x_2) \ldots (x - x_n) \tag{2. 4}$$

which is a polynomial of degree n with simple roots at the real or complex points $\{x_j\}_{j=1}^{n}$. Furthermore, by Runge's theorem [**17**], we may approximate $\varphi(x)$ uniformly in D by a complex polynomial $P_{n+m}(x)$, which is of degree $(n+m)$, and satisfies the relations:

$$\left.\begin{aligned} \varphi(x_j) &= P_{n+m}(x_j), \quad j = 1, 2, \ldots, n; \quad \pi_n(x_j) = 0 \\ \varphi(y_k) &= P_{n+m}(y_k), \quad k = 1, \ldots, m \\ \varphi(t) &= P_{n+m}(t), \quad \text{for certain } t. \end{aligned}\right\} \tag{2. 5}$$

Then, by using the Lagrangian interpolation formula, we have:

$$\varphi(x) = P_{n+m}(x) + R_{n+m+1}(x) \tag{2. 6}$$

$$P_{n+m}(x) = \sum_{j=1}^{n} \frac{(x-t)\,\Omega(x)\,\pi_n(x)}{(x-x_j)(x_j-t)\,\Omega(x_j)\,\pi_n'(x_j)}\,\varphi(x_j) +$$

$$+\sum_{k=1}^{n} \frac{(x-t)\,\Omega(x)\,\pi_n(x)}{(x-y_k)(y_k-t)\,\Omega'(y_k)\,\pi_n(y_k)} \times \varphi(y_k) + \frac{\Omega(x)\,\pi_n(x)}{\Omega(t)\,\pi_n(t)}\,\varphi(t) \tag{2. 7}$$

In Eq. (2. 6) $R_{n+m+1}$ is the remainder of the interpolation formula, which is given by:

$$R_{n+m+1}(x) = \frac{(x-t)\,\Omega(x)\,\pi_n(x)}{2\pi i} \int_C \frac{\varphi(z)\,dz}{(z-x)\,(z-t)\,\Omega(z)\,\pi_n(z)} \tag{2. 8}$$

where C is a closed contour in the interior of D containing in its interior all points $\{x_j\}_{j=1}^{n}$ and $\{y_k\}_{k=1}^{m}$.

We multiply now (2. 6) by $w(x)/(x-t)$ and integrate from $-1$ to 1 to obtain:

$$\int_{-1}^{1} \frac{w(x)\,\varphi(x)}{x-t}\,dx = \sum_{j=1}^{n+m} \lambda_{j,n} \frac{\varphi(x_j)}{x_j-t} + q_n(t)\,\varphi(t) + E_{n+m+1}(t) \tag{2. 9}$$

with:

$$x_j = \begin{cases} x_j, & \text{if } j \leqq n \\ y_k, & \text{if } j = n+k \end{cases} \tag{2. 10}$$

$$\lambda_{j,n} = -2 \lim_{z \to x_j} \left[ \frac{(z-x_j)\,\psi_n(z)}{\Omega(z)\,\pi_n(z)} \right], \quad j = 1, 2, \ldots, (n+m) \tag{2. 11}$$

$$q_n(t) = -\frac{2\psi_n(t)}{\Omega(t)\,\pi_n(t)} \tag{2. 12}$$

$$E_{n+m+1}(t) = \frac{1}{\pi i} \int_C \frac{\varphi(z)\,\psi_n(z)\,dz}{(z-t)\,\Omega(z)\,\pi_n(z)} \tag{2. 13}$$

where $\psi_n(z)$ is the associated to $\pi_n(x)$ function defined by:

$$\psi_n(z) = \frac{1}{2}\int_{-1}^{1} \frac{W(x)\,\pi_n(x)}{z-x}\,dx\,; \quad z \notin [-1,1] \tag{2. 14}$$

and

$$\psi_n(t) = \frac{1}{2}\left[\psi_n(t+0i) + \psi_n(t-0i)\right]; \quad -1<t<1 \tag{2. 15}$$

Using the Plemelj formulas we have:

$$\psi_n(t) = \frac{1}{2}\int_{-1}^{1} \frac{W(x)\,\pi_n(x)}{t-x}\,dx\,; \quad -1<t<1 \tag{2. 16}$$

Using a Taylor-series expansion of $1/(z-x)$ and taking into consideration (2. 14) the error term may be written as [**6**]:

$$E_{n+m+1}(t) = \frac{1}{2\pi i}\sum_{k=1}^{\infty} M_{k-1}\int_{C} \frac{\varphi(z)}{z^k\,(z-t)\,\Omega(z)\,\pi_n(z)}\,dz \tag{2. 17}$$

where:

$$M_k = \int_{-1}^{1} W(x)\,\Lambda(x)\,x^k\,\pi_n(x)\,dx. \tag{2. 18}$$

But, following the theory developed in ref. [**18**] we may have:

$$\frac{1}{2\pi i}\int_{C} \frac{\varphi(z)}{z^k\,(z-t)\,\pi_n(z)}\,dz = \frac{\varphi^{(n+m+k)}(\xi_k)}{(n+m+k)!} \tag{2. 19}$$

where $\xi_k$ is a point of the interval $(-1,1)$ not coinciding with the points t, $x_j$ and $\varphi^{(p)}$ denotes the derivative of p order of the function $\varphi$.

If a limited number of terms is retained, we will have:

$$E_{n+m+1}(t) = \frac{M_0}{(n+m+1)!}\,\varphi^{(n+m+1)}(\xi). \tag{2. 20}$$

Thus, it follows that $E_{n+m+1} = 0$ for every $\varphi(x) \in \mathscr{P}_{n+m}$ where $\mathscr{P}_{n+m}$ is the class of polynomials of degree $\leqslant n+m$.

If as $\pi_n(x)$ is selected the Jacobi polynomial $P_n^{(\mu,\nu)}(x)$, and $L(x)=1$, we see that $\lambda_{j,n}$ are the weights of the classical Gauss-Jacobi integration [6] if $m=0$, or of the Radau-Jacobi if $m=1$, and of the Lobatto-Jabobi if $m=2$. In this case, taking into consideration the orthogonality relation, we conclude that the moments $M_k\,(k=0, 1, \ldots, (n-1))$ are zero and consequently the error is given by:

$$E_{n+m+1} = \sum_{k=n+1}^{\infty} \frac{M_{k-1}}{(n+m+k)!} \varphi^{(n+m+k)}(\xi_k). \tag{2. 21}$$

If a limited number of terms is retained, then:

$$E_{n+m+1} = \frac{M_n}{(2n+m+1)} \varphi^{(2n+m+1)}(\xi) \tag{2. 22}$$

and it follows that:

$$E_{n+m+1} = 0 \tag{2. 23}$$

for every $\varphi(x) \in \mathscr{P}_{2n+m-1}$.

## 3. REMARKS FOR THE NODES AND THE CONVERGENCE OF THE QUADRATURE RULE

The Jacobi polynomial $P_n^{(\mu,\nu)}(x)$ for real or complex values of $\mu, \nu$ satisfies the same differential equation, which is given by:

$$(1-x^2)\,y'' + [\nu - \mu - (\mu+\nu+2)\,x]\,y' + n\,(n+\mu+\nu+1)\,y = 0 \tag{3. 1}$$

Thus, $P_n^{(\mu,\nu)}(x)$ is given in explicit form by:

$$P_n^{(\mu,\nu)}(x) = \frac{\Gamma(n+\mu+1)}{\Gamma(\mu+1)\,\Gamma(n+1)}\,{}_2F_1\left(-n,\; n+\mu+\nu+1;\;\; \mu+1;\;\; \frac{1-x}{2}\right) =$$

$$= \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(n+\mu+\nu+1)\dots(n+\mu+\nu+k)\,(\mu+k+1)\dots(\mu+n)\left(\frac{x-1}{2}\right)^{k} \tag{3. 2}$$

where ${}_2F_1$ is the hypergeometric function.

The method of Liouville-Stekloff **[19]** may be applied just as in the case of real values of $\mu$, $\nu$ to obtain the following formula of Hilb's type:

$$\vartheta^{-1/2}\left(\sin\frac{\vartheta}{2}\right)^{\mu+1/2}\left(\cos\frac{\vartheta}{2}\right)^{\nu+1/2} P_n^{(\mu,\nu)}(\cos\vartheta) =$$

$$= 2^{-1/2} N^{-\mu}\frac{\Gamma(n+\mu+1)}{n!} J_\mu(N\vartheta) + \begin{cases} \vartheta^{1/2} O(n^{-3/2}), & \text{if } cn^{-1} \leqq \vartheta < \pi-\varepsilon \\ \vartheta^{\mu+2} O(n^{\mu}), & \text{if } 0 < \vartheta \leqq cn^{-1} \end{cases} \tag{3. 3}$$

and

$$N = n + (\mu+\nu+1)/2 \tag{3. 4}$$

where c and $\varepsilon$ are fixed positive numbers, $\mathrm{Re}\,\mu > -1$ and $J_\mu$ is the Bessel function of first kind of order $\mu$ **[20]** defined by:

$$J_\mu(z) = \sum_{k=0}^{\infty}\frac{(-1)^k (z/2)^{\mu+2k}}{k!\,\Gamma(k+\mu+1)}. \tag{3. 5}$$

From this formula we can furnish immediately a Mehler-Heine type formula (similar to the one given by Szegö **[19]**):

$$\left.\begin{aligned} &\left(\frac{z}{N}\right)^{-1/2}\left(\sin\frac{z}{2N}\right)^{\mu+1/2}\left(\cos\frac{z}{2N}\right)^{\nu+1/2} P_n^{(\mu,\nu)}\left(\cos\frac{z}{N}\right) = \\ &\qquad = 2^{-1/2} N^{-\mu}\frac{\Gamma(n+\mu+1)}{\Gamma(n+1)} J_\mu(z) + O(n^{-2}) \end{aligned}\right\} \tag{3. 6}$$

which holds uniformly in every bounded region of the complex z-plane.

Another direct proof to obtain this formula is elementary. In this procedure the following asymptotic expression is used **[21]**:

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = \left(\frac{2z+a+b-1}{2}\right)^{a-b}\left[1+0\left(\left(\frac{2z+a+b+1}{2}\right)^{-2}\right)\right], \qquad (3.\ 7)$$

$$|\arg(z+a)| \leqq \pi-\varepsilon, \quad \varepsilon>0$$

in connection with relations (3. 2) and (3. 5).

Using the following relation [**11**]:

$$P'^{(\mu,\nu)}_{n}(z) = \frac{1}{2}(n+\mu+\nu+1)\,P^{(\mu+1,\,\nu+1)}_{n-1}(z) \qquad (3.\ 8)$$

which remains valid for complex values of $\mu$, $\nu$, we obtain the following Hilb's type formula:

$$\vartheta^{-1/2}\left(\sin\frac{\vartheta}{2}\right)^{\mu+3/2}\left(\cos\frac{\vartheta}{2}\right)^{\nu+3/2} P'^{(\mu,\nu)}_{n}(z) = 2^{-3/2(n+\mu+\nu+1)}N^{-(\mu+1)} \times$$

$$\times\frac{\Gamma(n+\mu+1)}{n!}J_{\mu+1}(N\vartheta) + \begin{cases}\vartheta^{1/2}\;O(n^{-1/2}) & \text{if}\;\; cn^{-1}\leqq\vartheta<\pi-\varepsilon\\ \vartheta^{\mu+3}\;O(n^{\mu+2}) & \text{if}\;\; 0<\vartheta<cn^{-1}\end{cases} \qquad (3.\ 9)$$

with $z=\cos\vartheta$.

On the other hand, it is known that the associated function $\psi_n(t)$ is given by [**22**]:

$$\psi_n(t) = -\frac{\pi}{2}\cos\pi\mu W(t)\,P^{(\mu,\nu)}_{n}(t) + \frac{2^{\mu+\nu-1}\Gamma(\mu)\,\Gamma(n+\nu+1)}{\Gamma(n+\mu+\nu+1)}\times$$

$$\times\,{}_1F_2\left(n+1,\; -n-\mu-\nu;\; 1-\mu;\; \frac{1-t}{2}\right). \qquad (3.\ 10)$$

The second term of the right-hand side may be written as:

$$\frac{2^{\mu+\nu-1}\Gamma(\mu)\,\Gamma(n+\nu+1)}{\Gamma(n+\mu+\nu+1)}\,{}_1F_2\left(n+1,\; -n-\mu-\nu;\; 1-\mu;\; \frac{1-\cos z/N}{2}\right) =$$

$$= 2^{\mu+\nu-1}\,\Gamma(\mu)\,\Gamma(1-\mu)\left(\sin\frac{z}{2N}\right)^{\mu}\sum_{k=0}^{\infty}\frac{\Gamma(n+1+k)}{\Gamma(n+1)}\cdot\frac{\Gamma(n+\nu+1)}{\Gamma(n+\mu+\nu-k+1)}\times$$

$$\times(-1)^k\frac{(\sin z/2N)^{2k-\mu}}{k!\,\Gamma(k-\mu+1)}. \qquad (3.\ 11)$$

Taking into consideration relations (3.7) and (3.5), as well as the Euler reflection formula:

$$\Gamma(\mu)\,\Gamma(1-\mu) = \frac{\pi}{\sin \pi\mu} \tag{3.12}$$

we obtain:

$$\frac{2^{\mu+\nu-1}\,\Gamma(\mu)\,\Gamma(n+\nu+1)}{\Gamma(n+\mu+\nu+1)}\,{}_1F_2\left(n+1,\ -n-\mu-\nu;\ 1-\mu;\ \frac{1-\cos z/N}{2}\right) =$$

$$= \frac{2^{\mu+\nu-1}\pi}{\sin \pi\mu}\,(\sin z/2N)^{\mu}\,J_{-\mu}(z)\,. \tag{3.13}$$

Thus, by taking into consideration relation (3.6) we may write:

$$\psi_n\left(\cos\frac{z}{N}\right) =$$

$$= -2^{\mu+\nu-3/2}\,\pi\left(\sin\frac{z}{2N}\right)^{\mu-1/2}\left(\cos\frac{z}{2N}\right)^{\nu-1/2}\left(\frac{z}{n}\right)^{1/2} N^{-\mu}\,\frac{\Gamma(n+\mu+1)}{\Gamma(n+1)}\times$$

$$\times\left(\frac{\cos\mu\pi\,J_{\mu}(z)-J_{-\mu}(z)}{\sin\pi\mu}\right) \tag{3.14}$$

which may be written as follows (see [**20**]):

$$\psi_n\left(\cos\frac{z}{N}\right) =$$

$$= -2^{\mu+\nu-3/2}\,\pi\left(\sin\frac{z}{2N}\right)^{\mu-1/2}\left(\cos\frac{z}{2N}\right)^{\nu-1/2}\left(\frac{z}{N}\right)^{1/2} N^{-\mu}\,\frac{\Gamma(n+\mu+1)}{\Gamma(n+1)}\,Y_{\mu}(z) \tag{3.15}$$

where $Y_\mu$ is the Weber function. Now, using the well known asymptotic expansions [**20**]:

$$J_{\mu}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2}\left[\cos(z-\mu\pi/2-\pi/4)\,P(z,\mu)-\sin(z-\mu\pi/2-\pi/4)\times Q(z,\mu)\right] \tag{3.16}$$

$$Y_{\mu}(z) \sim \left(\frac{2}{\pi z}\right)^{1/2}\left[\sin(z-\mu\pi/2-\pi/4)\,P(z,\mu)+\cos(z-\mu\pi/2-\pi/4)\,Q(z,\mu)\right] \tag{3.17}$$

with

$$P(z,\mu) = \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(\mu+2m+1/2)}{(2m)!\,\Gamma(\mu-2m+1/2)} \frac{1}{(2z)^{2m}} \tag{3.18a}$$

and

$$Q(z,\mu) = \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(\mu+2m+3/2)}{(2m+1)!\,\Gamma(\mu-2m-1/2)} \frac{1}{(2z)^{2m+1}} \tag{3.18b}$$

we can obtain the same formulas for $P_n^{(\mu,\nu)}(z)$, $\psi_n$ and $P'^{(\mu,\nu)}_n(x_j)$, as the ones used in ref. [**15**], but valid for the case where $\mu$ and $\nu$ are only real numbers.

Let us examine now the equation $P_n^{(n,\nu)}(z) = 0$. It has n roots, whose values depend on $\mu$ and $\nu$; since $P_n^{(\mu,\nu)}$, as we expect from (3.2), is an analytic function of z and $\mu, \nu$, it follows that each root of the equation is an analytic function of $\mu$ and $\nu$. Thus, as $\mu$ or $\nu$ vary, the zeros of $P_n^{(\mu,\nu)}$ vary continuously. Then, it follows that the zeros of $P_n^{(\mu,\nu)}$ are derived from those of $P_n^{(\mu_1,\nu_1)}$ ($\mu_1 = \mathrm{Re}\mu$, $\nu_1 = \mathrm{Re}\nu$), or from those of Chebyshev polynomials by a process of continuous variation, as $\mu$ or $\nu$ vary.

On the other hand, differentianting (3.1) k-times we have:

$$(1-x^2)\,y^{(k+2)} + [\nu-\mu-(\mu+\nu+2k+2)\,x]\,y^{(k+1)} +$$

$$+ [n(n+\mu+\nu+1) - k(k+\mu+\nu+1)]\,y^{(k)} = 0 \tag{3.19}$$

We prove now that y does not vanish for $x = \pm 1$. In fact, if $y = 0$ for $x = \pm 1$, then from (3.1) we must have $y' = 0$, whence from Eq. (3.19) with $k = 1$ it may be readily derived that $y'' = 0$, and so on; that is $y = 0$. Therefore, each of the zeros of $P_n^{(\mu,\nu)}$ is different from $\pm 1$.

Now, using the well known Rouché's theorem [**10**] and the formulas (3.3) or (3.6) we obtain, for enough large values of n, a number of very interesting theorems on the zeros $\{x_j\}_{j=1}^{n}$ of $P_x^{(\mu,\nu)}$. These theorems are the following:

Theorem 3.1: *The number of zeros of* $P_n^{(\mu,\nu)}(\cos\vartheta)$ *between the imaginary axis and the line on which* $R(z) = \mathrm{Re}\,[(n+\mu/2+1/4)\pi/N]$, $(N = n+(\mu+\nu+1)/2)$ *is exactly* n *(*n *must be sufficiently large).*

Proof. : See Watson [20, § 15. 4] for the large zeros of $J_\mu$. Another proof may be obtained as an immediate consequence of the next theorem :

Theorem 3. 2 : *Let* $\{x_j\}_{j=1}^{n}$ *with* $\mathrm{Re}x_1 > \mathrm{Re}x_2 > \ldots > \mathrm{Re}x_n$ *be the zeros* $P_n^{(\mu,\nu)}$. *If we write* $x_l = \cos\vartheta_l$ , *then for a fixed* $l$, *it is valid that :*

$$\lim_{n\to\infty} N\vartheta_l = j_l \,; \quad N = n + (\mu + \nu + 1)/2 \tag{3. 20}$$

*where* $j_l$ *is the* $l^{th}$ *zero with positive real part of* $J_\mu(z)$.

Proof. : This is a consequence of the formula (3. 6). From (3. 20) (see Watson [20, § 14. 54]), we obtain the following asymptotic expansion for $\vartheta_l$ :

$$\vartheta_l = (l + \mu/2 - 1/4)\pi/N + \frac{1 - 4\mu^2}{8(l + \mu/2 - 1/4)\pi N} + \ldots \tag{3. 21}$$

Thus, we derive immediately the following three theorems :

Theorem 3. 3 : *Let* $0 < \mathrm{Re}\,\vartheta_1 < \mathrm{Re}\,\vartheta_2 < \ldots < \mathrm{Re}\,\vartheta_n < \pi$ *be the zeros of* $P_n^{(\mu,\nu)}(\cos\vartheta)$, *then :*

$$\vartheta_l = N^{-1}\{(l + \mu/2 - 1/4)\pi + 0(1)\} \tag{3. 22}$$

*with* $0(1)$ *being uniformly bounded for all values of* $l = 1, 2, , \ldots, n$ ; $n = 1, 2, 3, \ldots$

*Furthermore :*

$$P_n'^{(\mu,\nu)}(\cos\vartheta_l) = \frac{N^{1/2}(-1)^j(\sin\vartheta_l/2)^{-(2\alpha+3)/2}(\cos\vartheta/2)^{-(2\beta+3)/2}}{2(\pi)^{1/2}} \times$$
$$\times\,[1 + 0(N^{-1})] \tag{3. 23}$$

Theorem 3. 4 : *Let* $-1/2 \leqq \mathrm{Re}\mu \leqq 1/2$, $-1/2 \leqq \mathrm{Re}\nu \leqq 1/2$, *then :*

$$\mathrm{Re}(\vartheta_l - \vartheta_{l-1}) < \mathrm{Re}(\pi/N) \qquad l = 1, 2, \ldots, n+1 \tag{3. 24}$$

*Here we define* $\vartheta_0 = 0$ *and* $\vartheta_{n+1} = \pi$.

T h e o r e m 3. 5 : *Let* $-1/2 < \operatorname{Re}\mu \leqq 1/2$, $-1/2 \leqq \operatorname{Re}\nu \leqq 1/2$, *then :*

$$\operatorname{Re}\left\{\frac{l+(\mu+\nu-1)/2}{N}\right\}\pi \leqq \operatorname{Re}\left\{\frac{l+(\mu-1/2)/2}{N}\right\}\pi < \operatorname{Re}(\vartheta_l) < \operatorname{Re}\left(\frac{l}{N}\pi\right) \qquad (3.25)$$

It is obvious from relations (3. 21) or (3. 22) that, for large n's, the zeros of $P_n^{(\mu,\nu)}$ cannot have a large imaginary part; and so all the zeros of $P_n^{(\mu,\nu)}$ lie inside a rectangle whose left and right sides are given by theorem 3. 1 and upper and lower sides are parallel to the real axis at distances from it, which are bounded when $|\mu|$ and $|\nu|$ are bounded.

Using relations (3. 6), (3. 15), (3. 22) and (3. 23) we can also prove the convergence of the quadrature rule (2. 9) in the same way as in ref. [**15**].

## 4. METHOD OF SOLUTION OF SINGULAR INTEGRAL EQUATIONS

Using relations (1. 3), (1. 4), the quadrature formula for Cauchy-type integrals (2. 8), and the quadrature formula for bounded functions, equation (1. 2) may be reduced to the following functional equation:

$$F_n(t)\,\varphi(t) + \sum_{k=1}^{n+m}\left[\frac{b(t)}{\pi}\,\frac{\lambda_{k,n}}{x_k - t} + k(x_k, t)\,\lambda_{k,n}\right]\varphi(x_k) =$$

$$= f(t) + \varrho_n(t); \quad -1 < t < 1 \qquad (4.1)$$

where:

$$F_n(t) = a(t)\,w(t) + \frac{b(t)}{\pi}\,q_n(t) = \left[a(t)\,w(t)\,\pi_n(t) - \frac{2b(t)}{\pi}\,\psi_n(t)\right]/\pi_n(t). \qquad (4.2)$$

and:

i) $x_k, \lambda_{k,n}, q_n(t)$ are defined by the relations (2. 10), (2. 11), (2. 12) and

ii) $\varrho_n(t)$ is the remainder.

Taking into consideration the convergence of the described quadrature rule, $\varrho_n(t)$ can be made as small as required and hence may be neglected.

If the index $\varkappa$ is greater than zero, then a supplementary equation may be added to equation (4. 1), which results from the additional condition (1. 8):

$$\sum_{k=1}^{n+m} \lambda_{k,n}\, \varphi(x_k) = A. \tag{4. 3}$$

Obviously, if $a(t)$, $b(t)$, $k(x_k, t)$ and $f(t)$ are analytically extendable into the domain D containing $[-1,1]$ in its interior, then equation (4. 1) may be transformed to an equation valued for any t belonging to D.

Thus, by selecting as collocation points the $l$-zeros $\{t_r\}_{r=1}^{l}$ of $F_n$ (which are complex in general) we obtain from (4. 1) a system of $l$-linear algebraic equations in the $(n-m)$ unknows $\varphi(x_k)$. To these equations the $N\,(N=0$ or $1)$ equations (4. 3) must be added. If $l+N=n+m$, this system is solvable in the ordinary sense, while if $l+N>n+m$ it is solvable in the least square sense. Finally, if $l+N<n+m$ the system is insolvable.

If now any one of $a(t)$, $b(t)$, $k(x_k, t)$ and $f(t)$ are not analytically extendable into the domain D, it may be replaced by a polynomial approximation $\tilde{a}(t)$, $\tilde{b}(t)$, $\tilde{k}(x_k, t)$ and $\tilde{f}(t)$, which is analytically extendable. These approximations may be chosen in $[-1,1]$ as close as required to the original functions according to the following theorem of Gel'fond [**22**]:

T h e o r e m 4. 1: *Let a function* $f(x)$, *assigned on* $[-1,1]$, *have a derivative of order* m, *with* $f^{(m)}(x)\in H_\mu\,(0<\mu<1)$. *Then, for every natural* n *there exists a polynomial* $P_n(x)$ *of degree not higher than* n *for which it is valid that:*

$$|\, f^{(\nu)}(x) - P_n^{(\nu)}(x)\,| < \frac{A}{n^{m-\nu+\mu}} \qquad (\nu = 0, 1, \ldots, m;\quad -1 \leqq x \leqq 1) \tag{4. 4}$$

*where* A *is a constant (depending on* f *and* m*).*

But, in order to proceed in this manner, it is necessary to prove under what conditions $F_n$ has a sufficient number of roots $(l \geqq n+m-N)$. An answer to this question is given by the following theorems.

As a first step we introduce the notations:

$$\pi\chi(x) = \arctan\,[\,a(x)/b(x)\,] \tag{4. 5}$$

$$\psi_r = \arccos t_r \tag{4.6}$$

and we suppose that $L = 1$. Then, the following theorems hold :

T h e o r e m 4.2: *If* a, b *and* L *are constants, then :*

$$\psi_r = N^{-1}\{(\pi+\mu/2-1/4-\chi)\pi+0(1)\} \tag{4.7}$$

*with* $0(1)$ *being uniformly bounded for all values of* r.

P r o o f. : By Rouché's theorem, $F_n$ has the same number of zeros as the equation :

$$J_\mu(N\vartheta)\cos\pi\chi - Y_\mu(N\vartheta)\sin\pi\chi = 0 \tag{4.8}$$

Thus, (see Watson [**24**, § **54**]) we can obtain the following asymptotic expansion for the roots $\psi_r$ of $F_n$ :

$$\psi_r = (r+\mu/2-1/4-\chi)\pi/N + \frac{1-4\mu^2}{8(r+\mu/2-1/4-\chi)\pi N} + \ldots \tag{4.9}$$

This relation completes the proof.

From relation (4. 8) we can also prove theorems analogous to the theorems 3. 3 to 3. 5.

Now as an immediate consequence of the above we can arrive at the following corollary :

C o r o l l a r y 4. 1 : *If* $a(x)$ *and* $b(x)$ *are constants, then the number of roots* $t_r$ *of* $F_n(x)$ *in the strip* $-1<\mathrm{Re}\,t_r<1$ *is equal to* $(n-1)$.

T h e o r e m 4. 3 : *If in a contour surrounding* $[-1+\varepsilon,\ 1-\varepsilon]$, *where* $\varepsilon$ *is a small positive number, we have :*

$$|a(x)| > M(\delta)\,|b(x)| \tag{4.10}$$

*with :*

$$M(\delta) = \max|\tan[N\vartheta-(2\mu+1)\pi/4]| \quad \textit{for}\ \mathrm{Im}[N\vartheta-(2\mu+1)\pi/4)] \gtrless \delta \tag{4.11}$$

$\delta$ *being any positive number, then* $F_n(x)$ *has in the strip* $-1<\mathrm{Re}\,x<-1$ $(n+p)$ *roots, if* p *are the roots of* $a(x)$ *(*n *large enough).*

P r o o f : By Rouché's theorem, $F_n$ has the same number of roots as the equation :

$$J_\mu(N\vartheta)\,a(x) - Y_\mu(N\vartheta)\,b(x) = 0 \tag{4.12}$$

or the equation :

$$\tan[N\vartheta - (2\mu+1)\pi/4] = \frac{a(x)}{b(x)} \tag{4.13}$$

On the other hand, we have :

$$\tan z = \frac{2i}{1+e^{2iz}} - i\ ; \qquad z = x+iy \tag{4.14}$$

and for $y > \delta$ :

$$|1+e^{2iz}| \geqq 1 - |e^{2iz}| = 1 - e^{-2y} \geqq 1 - e^{-2\delta} \tag{4.15}$$

In this way we can see that $\tan[N\vartheta - (2\mu+1)\pi/4]$ is bounded in the half-plane $\mathrm{Im}[N\vartheta - (2\mu+1)\pi/4] \gtrless \delta$. Let now $M(\delta)$ be this upper bound. If we apply Rouché's theorem to equation (4.13), taking into consideration relation (4.10), we arrive to the desired result.

It is possible to obtain a more precise theorem if we choose $L = \Lambda$ and $b(t)$ is taken to be a polynomial $b_m$, say of degree m, defined by:

$$b_m(z) = \pm \prod_{i=1}^{\mu} (z-\beta_i)^{\alpha_i} \quad m = \sum_{i=1}^{\mu} \alpha_i\,,\ b_m(-1) \leqq 0\,.$$

This condition is not in reality very restrictive, because we may be able to multiply (1.2) by a non-vanishing in $(-1, 1)$ Hölder - continuous function g, such that gb is a polynomial $b_m$ of degree m (see [**23**]).

We denote now by :

$$W^{\pm}(t) = W(t \pm i0) \quad t \in (-1, 1) \tag{4.16}$$

the limiting value of W from the left and from the right. Thus, following Muskhelishvili [**1**] we will have :

$$W^{\pm}(t) = \{a(t) \pm ib(t)\}\,W(t)/r(t)\,, \quad r^2(t) = a^2(t) + b^2(t) \tag{4.17}$$

In order to be in a measure to demonstrate the theorem 4.5, which follows, it is necessary to remind the following lemma [**23**] :

L e m m a 4.4 : *Let* Q, R *be functions such that :*

(i) QX−R *is analytic in the deleted complex plane and zero at infinity.*

(ii) *on* $(-1,1)$ $Q^+(t) = Q^-(t)$, $R^+(t) = R^-(t)$ *and the functions* $aQz/r - R$, $bQz/r$ *are in* $H^*$. *Then, for* $-1 < t < 1$, *it is valid that :*

$$\frac{1}{\pi}\int_{-1}^{1} \frac{b(\tau) Q(t) z(\tau)}{r(\tau)(\tau - t)} dt = -\frac{a(t) Q(t) z(t)}{r(t)} + R(t) \qquad (4.19)$$

Theorem 4.5: *Let* $b = b_m$, $L = 1$ *and* $\varphi_n$ *is a polynomial of degree* $\leqq n$. *Let* $\Omega_{n-\varkappa-m}$ *denote that polynomial of degree* $n + \varkappa - m$ *such that* $\lim_{z\to\infty}\{\varphi_n(z) W(z)/b_m(z) - \Omega_{n-\pi-m}(z)\} = 0$. *Let* $R_{m-1}$ *be that polynomial of degree* $m-1$ *such that :*

$$R^{(j)}_{m-1}(\beta_i) = \left\{ \begin{array}{ll} \dfrac{d^j}{dz^j}[\varphi_n(z) W(z)]_{z=\beta_i} & \beta_i \notin [-1,1] \\ \dfrac{d^j}{dz^j}\left[\dfrac{a(t)\varphi_n(t) W(t)}{r(t)}\right]_{t=\beta_i} & \beta_i \in [-1,1] \end{array} \right\} \qquad (4.19)$$

*for* $j = 0(1)(\alpha_i - 1)$, $i = 1(1)\mu$. *Then, for* $-1 < t < 1$, *it is valid :*

$$\text{(i)} \quad a(t)\varphi_n(t) W(t) + \frac{b_m(t)}{\pi}\int_{-1}^{1}\frac{W(t)\varphi_n(\tau)}{\tau - t} d\tau =$$

$$= R_{m-1}(t) + \Omega_{\varkappa-n-m}(t) b_m(t) \qquad (4.20)$$

and

(ii) $F_n$ *has* $(n - \varkappa)$ *roots in the whole complex plane.*

Proof.: If in Lemma 4.5 we choose $Q = \varphi_n / b_m$ and $R = R_{m-1}/b_m + \Omega_{n-\varkappa-m}$, relation (4.20) follows by substitution. By choosing $\varphi_n = \pi_n$ in the last relation and taking into consideration (2.12) and (4.2) we see that :

$$\pi_n(t) F_n(t) = R_{m-1}(t) + \Omega_{n-\varkappa-m}(t) b_m(t) \qquad (4.21)$$

that is $\pi_n F_n$ is a polynomial of $(n - \varkappa)$ —degree. Taking into consideration (4.2) and the fact that a zero of $\pi_n(t)$ does not coincide with a zero of $\psi_n(t)$ we conclude that $F_n$ has $(n-\varkappa)$ roots.

In the case where $L = 1$ and $\mu, \nu$ real numbers we have proved, in a recent paper **[7]**, a number of more precise theorems and corollaries. For question of completeness of the paper we state them here without proof :

Theorem 4.6: *Let* a, b *be Hölder-continuous,* $\mu, \nu$ *real numbers and* $L = 1$. *The function* $F_{n'}$, *defined in (3. 2), has in the interval* $I_k = (x_k, x_{k+1})$, $k = 1, 2, \ldots n+m-1$:

i) *An odd number* $2l+1$, $(l = 0, 1, \ldots)$ *of real zeros, if* b(x) *has* 2p, $(p = 0, 1, 2, \ldots)$, *roots in* $I_k$;

ii) *An even number* $2l$, $(l = 0, 1, 2, \ldots)$, *of real zeros, if* b(x) *has an odd number of real roots* $z_j$. *In particular, it has at least two real zeros, if there is a* $z_j$ *for which:*

$$b(x_k)\, a(z_j) < 0\ , \quad z_j \in I_k . \tag{4. 22}$$

Theorem 4.7: *Let the preassigned nodes coincide with the limits* $\pm 1$ *and* W *be defined by* (2. 3) *with* $L = 1$. *Then, if* $\mu > 0$ (*or* $\nu > 0$), *theorem* 4. 6 *holds for the interval* $(x_n, 1)$ (*or* $(-1, x_1)$). *If* $\mu < 0$ (*or* $\nu < 0$) *function* $F_n$ *has in the interval* $(x_n, 1)$ (*or* $(-1, x_1)$):

i) *An even number* $2l$, $(l = 0, 1, 2, \ldots)$ *of real zeros, if* b(x) *has* 2p, $(p = 0, 1, 2, \ldots)$ *roots in the prescribed interval. In particular, it has at least two real zeros in* $(x_n, 1)$ (*or* $(-1, x_1)$), *if there is a* $z_j$ *for which it is valid respectively that:*

$$b(1)\, a(z_j) < 0\ ; \qquad z_j \in (x_n, 1) \tag{4. 23}$$

$$b(-1)\, a(z_j) < 0\ ; \quad z_j \in (-1, x_1 \tag{4. 24}$$

ii) *An odd number* $2l+1$, $(l = 0, 1, 2, \ldots)$, *of real zeros, if* b(x) *has* $2p+1$ $(p = 0, 1, 2, \ldots)$ *roots in* $(x_n, 1)$ (*or in* $(-1, x_1)$).

Corolary 4.8: *If* b *has no roots in* $(-1, 1)$ *and* $L = 1$, *then* $F_n$ *has at least:*

i) $(n-1)$ *zeros, if* $\mu < 0$ *and* $\nu < 0$

ii) n *zeros, if* $\mu \cdot \nu < 0$

iii) $(n+1)$ *zeros, if* $\mu > 0$ *and* $\nu > 0$

*All the zeros alternate with the zeros of* $P_n^{(\mu,\nu)}(x)$.

Corollary 4.9: *If* $\mu + \nu = k$, *where* k *is a negative* $(> -2)$ *or positive integer and* $\mu, \nu$ *complex numbers, then:*

$$F(x) = \frac{1}{\Omega(x) P_n^{(\mu,\nu)}(x)} \left\{ \left[ a(x) - b(x) \frac{a(1)}{b(1)} \right] W(x) P_n^{(\mu,\nu)} - \right.$$

$$\left. - 2^k \frac{\Gamma(\mu)\Gamma(1-\mu)}{\pi} P_{n+k}^{(-\mu,-\nu)} \right\} \tag{4. 25}$$

*and in the particular case where a, b are constants,* $F(x)$ *has* $(n+k)$ *roots, which coincide with the roots of* $P_{n+k}^{(-\mu,-\nu)}$ *and for* $k = 1, 0, 1$ *alternate with the zeros of* $P_n^{(\mu,\nu)}(x)$.

Last corollary may also be proved by using Theorem 4. 5.

## 5. ALTERNATE METHOD OF SOLUTION

We propose now another way to reduce the functional equation (4. 1) to a linear system of equations. If a n - point quadrature formula is used, we select at the beginning as collocation points $t_k$, the $(n+p)$ zeros of the polynomials $\pi_{n+p}(x)$ (where p is an integer $\neq 0$), subsequently, we use a $(n+p)$ - point quadrature formula and we select as collocation points $t_k$ the zeros of $\pi_n(x)$. We obtain by this process from the integral equation a linear system of $(2n+p)$ equations with $(2n+p+m)$ unknowns. To these equations the N ($N = 0$ or 1) equations (4. 3) may be also added. It is evident that, in order to obtain a number of equations equal to, or greater than, the unknowns, solely for $\varkappa > 0$, one and only one preassigned node is permitted.

It is obvious that the size of the final system of linear equations obtained by the previous process is approximately twice the size of the system obtained by the method described in section 4. The increase of the size of the system is not followed by any increase in the degree of accuracy. But, on the other hand, it is simpler to select as collocation points the roots of the polynomials $\pi_n(x)$ and $\pi_{n+p}(x)$, instead of the roots of $F(x)$. Moreover, the present method has the advantage to be applicable for any functions $a(x)$ and $b(x)$.

Consider now case where $\alpha_j$ or $\beta_j$ are complex :

$$\alpha_j = \alpha_j^1 + i\alpha_j^2 \tag{5. 1}$$

$$\beta_j = \beta_j^1 + i\beta_j^2 \tag{5.2}$$

In this case the polynomials $\pi_n(x)$, orthogonal with respect to $W(x)$ and the function $F(x)$, as it has already been proved, do not have real roots in the interval $(-1, 1)$. It is perhaps preferable for this case to consider a non-Gaussian quadrature formula. In this way, we are free to select arbitrarily the nodes $x_j$, $(j = 1, 2, \ldots n)$, where we interpolate $\varphi(x)$. If $\pi_n(x)$ is expressed by (2.4) and by using the same procedure as above (relations (2.5) to (2.8), see also [6]), we arrive at the same quadrature formula, as the one defined by (2.9) to (2.13). In this case $\psi_n(z)$ is not an a priori known function and it does not satisfy a recurrence formula, but it is possible to calculate it. The error of this integration formula is:

$$E_{n+m+1} = 0 \quad \text{for every } g(x) \in \mathscr{P}_{n+m+1}$$

Intuitively, we select as $\pi_n$ the polynomial, which is orthogonal with respect to the weight function:

$$W_{1j}(x) = (1-x)^{\alpha_j^1}(1+x)^{\beta_j^1}\,\Omega(x). \tag{5.3}$$

The methods described in this paper may be applied also to a system of generalized-Cauchy singular integral equations, where $\alpha_j$ and $\beta_j$ are not given by (1.4) and (1.5), but they are, in general, the roots of a transcendental equation. Such systems of integral equations appear, for example, in problems of the theory of elasticity, where angular points exist. Thus, we arrive at such systems of generalized-Cauchy integral equations in the cases of branched cracks, of contact problems with friction between bodies presenting angular points, of composite dissimilar materials meeting at angles [24] and so on.

Finally, the demonstration of the convergence of the method may be found in a recent paper by the authors [16].

## ΠΕΡΙΛΗΨΙΣ

Ἡ παροῦσα ἐργασία ἀναφέρεται εἰς τὴν ἀνάπτυξιν νέας μεθόδου ἀριθμητικῆς ἐπιλύσεως συστημάτων ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων μὲ μεταβλητοὺς συντελεστὰς καὶ μιγαδικὴν συνάρτησιν βάρους. Αἱ βάσεις τῆς χρησιμοποιουμένης μεθοδολογίας ἐτέθησαν ἤδη ἀπὸ τοῦ 1976 εἰς τὸ Ἐργαστήριον Ἀντοχῆς Ὑλικῶν τοῦ Ε.Μ.Π. καὶ ἐφηρμόσθησαν μὲ ἐπιτυχίαν εἰς τὴν ἐπίλυσιν μεγάλης ποικιλίας συνοριακῶν προβλημάτων τῆς ἐπιπέδου ἐλαστικότητος.

Τὸ πρὸς ἐπίλυσιν σύστημα ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων εἶναι γενικώτατον, εὑρίσκει δὲ ἐφαρμογὴν εἰς συγκεκριμένα προβλήματα τῆς πράξεως, ὅπως π.χ. τὸ πρόβλημα τῆς διακλαδιζομένης ρωγμῆς, ἢ τὸ πρόβλημα τῆς ἐπαφῆς δύο ἢ περισσοτέρων σωμάτων μεταξὺ τῶν ὁποίων ὑφίστανται δυνάμεις τριβῆς ἢ τὰ προβλήματα τῆς ἐπιπέδου Ἐλαστικότητος, ἀφορῶντα σώματα, τῶν ὁποίων τὸ σύνορον εἶναι σημειακῶς ἀσυνεχές, παρουσιάζει δηλαδὴ γωνιακὰ σημεῖα ἀσυνεχείας κ. ἄ.

Διὰ τὴν ἐπίλυσιν τοῦ συγκεκριμένου συστήματος ἐξισώσεων μορφώνεται πρῶτον κατάλληλος κανὼν ἀριθμητικῆς ὁλοκληρώσεως τῶν ἰδιομόρφων κατὰ Cauchy ὁλοκληρωμάτων τοῦ προβλήματος, διὰ τὸ τυχὸν σύστημα ὀρθογωνίων πολυωνύμων $\Pi_n$. Ἐξετάζεται ἐν συνεχείᾳ ἡ μορφὴ τῶν προτεινομένων τύπων διὰ τὴν εἰδικὴν περίπτωσιν κατὰ τὴν ὁποίαν ὡς σύστημα ὀρθογωνίων πολυωνύμων λαμβάνονται τὰ πολυώνυμα Jacobi $P_n^{(\alpha,\beta)}$ Αἱ προκύπτουσαι ἀναλυτικαὶ ἐκφράσεις τυγχάνουν γενικῆς ἐφαρμογῆς καὶ συμπεριλαμβάνουν ὡς εἰδικὰς περιπτώσεις τοὺς κανόνας ἀριθμητικῆς ὁλοκληρώσεως Gauss, Radau καὶ Labatto - Jacobi.

Ἐν συνεχείᾳ εὑρίσκονται αἱ ρίζαι $\{x_j\}_{j=1}^n$ τῶν πολυωνύμων Jacobi διὰ μιγαδικὰς τιμὰς τῶν δεικτῶν τους μ καὶ ν. Αἱ ρίζαι αὐταὶ χρησιμοποιοῦνται ὡς σημεῖα ἐφαρμογῆς (nodes) τῶν τύπων ἀριθμητικῆς ὁλοκληρώσεως Gauss - Jacobi. Ἀποδεικνύεται ὅτι αἱ ρίζαι αὐταὶ εἶναι ἀφ᾽ ἑνὸς μὲν ἀναλυτικαὶ συναρτήσεις τῶν δεικτῶν μ, ν, ἀφ᾽ ἑτέρου δὲ κεῖνται ἐντὸς καθωρισμένου τετραγωνικοῦ χωρίου, τοῦ ὁποίου ὑπολογίζονται αἱ γεωμετρικαὶ διαστάσεις.

Κατ᾽ ἀκολουθίαν προσδιορίζεται τὸ κατάλληλον σύστημα σημείων ταξιθεσίας, διὰ τὰ ὁποῖα εἶναι δυνατὴ ἡ ἐφαρμογὴ τῆς μεθόδου ἀριθμητικῆς ἐπιλύσεως ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων κατὰ Gauss - Jacobi.

Διὰ τῆς εἰσαγωγῆς τῆς ἐννοίας τῶν μιγαδικῶν σημείων ταξιθεσίας ἐπιτυγχάνεται ἀριθμητικὴ ἐπίλυσις τῶν συστημάτων ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων δι᾽ ἐφαρμογῆςτῶν τύπων ἀριθμητικῆς ὁλοκληρώσεως εἰς σημεῖα ποὺ κεῖνται ἐ κ τ ὸ ς τοῦ διαστήματος ὁλοκληρώσεως. Τοιουτοτρόπως, καθίστανται ἀκριβέστεροι οἱ ἀριθμητικοὶ ὑπολογισμοὶ εἰς περιοχὰς ἐξαιρετικῆς συγκεντρώσεως τάσεων, ὅπου καὶ ἀπαιτεῖται διὰ τοὺς ὑπολογισμοὺς μεγάλη πυκνότης σημείων ὁλοκληρώσεως.

Τέλος, γίνεται μελέτη τῆς συγκλίσεως τῶν προτεινομένων τύπων ἀριθμητικῆς ὁλοκληρώσεως καί, κατ' ἀκολουθίαν, δι' ἀποδείξεως ὅτι τὸ ὅριον τῆς συναρτήσεως τείνει εἰς τὸ μηδέν, καθὼς ὁ ἀριθμὸς τῶν σημείων ὁλοκληρώσεως αὐξάνει.

Τὰ πλεονεκτήματα τῆς προτεινομένης ἀριθμητικῆς μεθόδου ἐπιλύσεως συστημάτων ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων ἐν συγκρίσει μὲ τὰς ἤδη ὑπαρχούσας προσεγγιστικὰς μεθόδους ἐπιλύσεως τῶν ἰδίων συστημάτων εἶναι σημαντικά. Αὐτὸ ἀποδεικνύεται ἄλλωστε καὶ ἀπὸ τὸ γεγονὸς ὅτι ἡ μέθοδος αὐτὴ τείνει νὰ ἐκτοπίση τὰς προϋπαρχούσας μεθόδους εἰς διεθνῆ κλίμακα.

Μερικὰ ἐκ τῶν πλεονεκτημάτων ποὺ ἐμφανίζει ἡ ἐπίλυσις τοῦ ἀναφερομένου εἰς τὸ ἄρθρον συστήματος ἰδιομόρφων ὁλοκληρωτικῶν ἐξισώσεων εἶναι τὰ ἀκόλουθα:

1) Ἐπιτυγχάνεται περιορισμὸς τοῦ χρόνου, ὁ ὁποῖος ἀπαιτεῖται διὰ τὴν ἐπίλυσιν συγκεκριμένων συνοριακῶν προβλημάτων τῆς ἐλαστικότητος ἀπὸ τὸν ἠλεκτρονικὸν ὑπολογιστήν.

2) Γενικεύονται καὶ γίνονται ἀμέσως ἐφαρμόσιμοι αἱ μέθοδοι ἀριθμητικῆς ὁλοκληρώσεως κατὰ Gauss, Lobatto καὶ Radau - Jacobi.

3) Δίδεται ἡ δυνατότης νὰ καθορισθοῦν μὲ ἀκρίβειαν οἱ συντελεσταὶ συγκεντρώσεως τάσεων εἰς τὸ σημεῖον διακλαδώσεως διαδιδομένων ρωγμῶν. Εἰδικῶς τὸ πρόβλημα τοῦτο εἶναι δυσχερὲς καὶ δὲν ὑφίσταται μέχρι σήμερον ἱκανοποιητικὴ ἀριθμητικὴ ἢ πειραματικὴ λύσις του.

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