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ΜΑΘΗΜΑΤΙΚΑ. — **On the complex path-independent integrals used in plane elasticity problems,** *by P. S. Theocaris and G. Tsamasphyros**.

ABSTRACT

A general method to construct path-independent integrals (P. I. I.) for the case of a plane body weakened by a number of collinear cracks is presented in this communication. The path-independent integrals are classified in three classes, which satisfy some well-defined restrictions. Using this classification a table is prepared containing a number of path-independent integrals related to the solution of the elastic problem of a single crack. The extension of these results to the case of several collinear cracks, or an infinite number of periodic cracks is also indicated. Besides, it is further indicated how to proceed in the case of a crack between dissimilar media and in the case of a star shaped symmetric crack. It was also derived that the form of the proposed P. I. I. integrals is more simple than the already known integrals. This fact will facilitate their use in conjunction with finite element or experimental methods

A number of illustrative examples is also presented.

INTRODUCTION

The concept of path-independent integrals (P. I. I.) was proved [**1** - **8**] to be an efficient tool for the evaluation of stress-intensity factors (S. I. Fs) associated with notches and cracks. The method based on these integrals bypassed a detailed solution of the corresponding boundary-

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value problem and yielded directly the value of the respective S. I. F. Path-independent integrals were also used in many circumstances in connection with the finite-element method (see for example ref. [**4**]) for the calculation of S. I. Fs. The same integrals were also used for a closed-form calculation of S.I.Fs [**2, 5, 7, 8, 12**]. This was done by shifting the path of integration to two «convenient» positions, e. g. to infinity and along the lips of the crack. By equating the values of the two integrals the value of the respective S. I. F. may be readily derived since the integral was path-independent. The analytic calculation of these integrals was obtained by using Cauchy's residue theorem.

Generally for the establishment of the path-independence of an integral the Green-Gauss theorem and the Cauchy equation of equilibrium (or motion) was used. But, in the case of plane problems, it is also possible to proceed by using Muskhelishvili's complex potentials together with the Cauchy theorem concerning analytic functions. A first tentative for the demonstration of some complex P.I.I. is made recently [**8**].

In the present paper a general procedure for the construction of path-independent integrals in plane elasticity is given. Another generalization is that we are concerned with the problem of a finite or infinite plate with not only a single crack but a finite number of collinear cracks.

Using this general formulation the integrands of the P. I. I. are classified into three classes. Each class satisfies some well-defined restriction. In this way it is possible to construct an important number of path-independent integrals. However, we limited ourselves in this paper to give the path-independent integrals valued only for the case of a single crack. In the table containing the above path-independent integrals we give also their values, if the path is shrunk to the crack. The extension of these results to the case of several collinear cracks is also indicated.

The above results are generalized for the case of a crack between dissimilar media, as well as either for the case of a star-shaped symmetric crack, or for the case of an infinite row of collinear cracks.

It is also interesting to note that the proposed integrals have the advantage when compared to the already known integrals that their form is simpler and, therefore, more suited for a combined use with the finite-element method or other experimental methods.

2. DEFINITION OF THE PROBLEM

We suppose that the crack, or cracks, have their axis coinciding with the Ox-axis of the complex z-plane and they are defined by the intervals (a_{2k-1}, a_{2k}), $k = 1, 2, \ldots, n$ with:

$$-\infty < a_1 < a_2 < \ldots\ldots < a_{2n-1} < a_{2n} < \infty$$

Following Muskhelishvili [9] we can define the holomorphic function $\omega(z)$ by:

$$\omega(z) = z\bar{\varphi}'(z) + \bar{\psi}(z) \tag{1}$$

from which we derive:

$$\sigma_{yy} - i\sigma_{xy} = \Phi(z) + \Omega(\bar{z}) + (z - \bar{z})\,\Phi'(z) \tag{2}$$

where:

$$\Phi(z) = \varphi'(z), \qquad \Psi(z) = \psi'(z) \tag{3a}$$

$$\Omega(z) = \omega'(z) = \bar{\Phi}(z) + z\bar{\Phi}'(z) + \bar{\Psi}(z) \tag{3b}$$

We suppose further that a force (X, Y) acts at the point $C(c, 0)$ of the upper crack lip of any crack (a_{2k-1}, a_{2k}). On the other hand, let N_1, N_2 be the principal stresses at infinity and α the angle subtended by the direction of N_1 and the Ox-axis. Then, according to the refs. [9] and [13] we have the following asymptotic behavior:

(i) At the left crack tip, a_m $(m = 2j - 1)$:

$$\begin{aligned} \Phi(z) \sim \Omega(z) &\sim \frac{1}{2i}\left(K_{I_m} - iK_{II_m}\right)\left[2\pi(z - a_m)\right]^{-1/2} \\ \bar{\Omega}(z) &\sim -\frac{1}{2i}\left(K_{I_m} + iK_{II_m}\right)\left[2\pi(z - a_m)\right]^{-1/2} \end{aligned} \tag{4}$$

(ii) At the right crack tip, a_l $(l = 2k)$:

$$\begin{aligned} \Phi(z) \sim \Omega(z) &\sim \frac{1}{2}\left(K_{I_m} - iK_{II_m}\right)\left[2\pi(z - a_m)\right]^{-1/2} \\ \bar{\Omega}(z) &\sim \frac{1}{2}\left(K_{I_m} + iK_{II_m}\right)\left[2\pi(z - a_m)\right]^{-1/2} \end{aligned} \tag{5}$$

where in both cases $(K_{I_m} - iK_{II_m})$ is the complex S. I. F. of the corresponding m-tip of the crack considered.

(iii) At the point C:

$$\Phi(z) \sim \begin{cases} -\dfrac{X+iY}{2\pi(z-c)} & \text{for } Im\, z>0 \\ 0 & \text{for } Im\, z<0 \end{cases} \tag{6a}$$

$$\Omega(z) \sim \begin{cases} 0 & \text{for } Im\, z>0 \\ \dfrac{X+iY}{2\pi(z-c)} & \text{for } Im\, z<0 \end{cases} \tag{6b}$$

$$\bar{\Omega}(z) \sim \begin{cases} \dfrac{X+iY}{2\pi(z-c)} & \text{for } Im\, z>0 \\ 0 & \text{for } Im\, z<0 \end{cases} \tag{6c}$$

(iv) At infinity:

$$\Phi(z) \sim \frac{(N_1+N_2)}{4} - \frac{X+iY}{2\pi(1+\varkappa)}\,\frac{1}{z} \tag{7a}$$

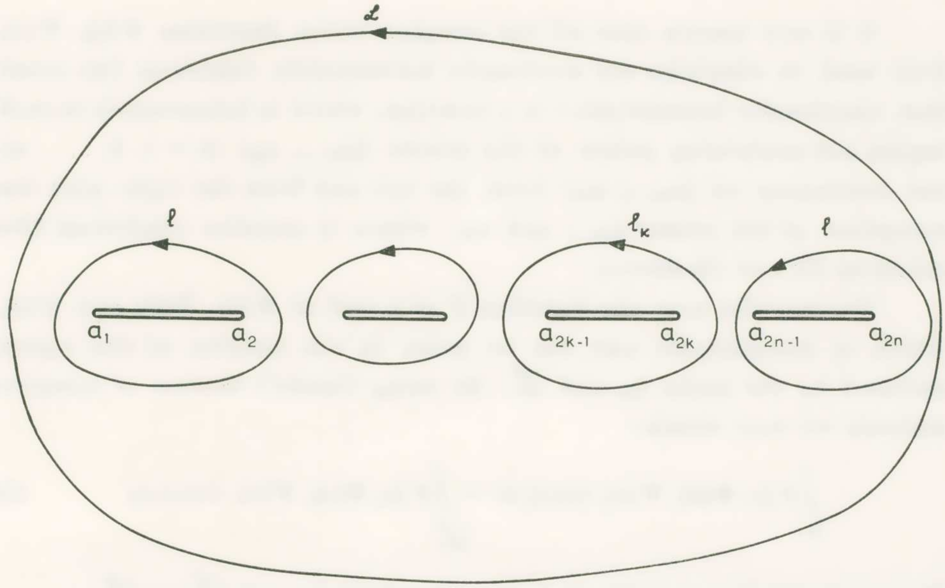
$$\Omega(z) \sim \frac{N_1+N_2}{4} - \frac{1}{2}(N_1-N_2)\,e^{2i\alpha} + \frac{\varkappa(X+iY)}{2\pi(1+\varkappa)}\,\frac{1}{z} \tag{7b}$$

Now, let l_k ($k = 1, 2, \ldots, n$) represent a smooth Jordan path surrounding only the crack (a_{2k-1}, a_{2k}) in the counterclockwise sense (Fig. 1), all other cracks lying outside of l_k. The curves l_k shall not mutually intersect. We denote by L the union of all these paths:

$$L = \bigcup_{k=1}^{n} l_k \tag{8}$$

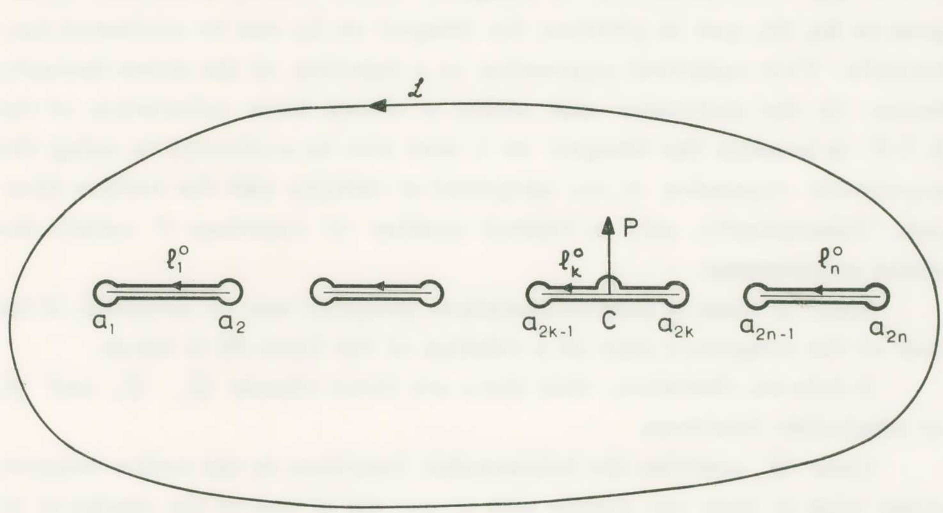
The Jordan path $\mathscr{L}$ taken in the counterclockwise sense and containing all the cracks in its interior, may be considered as a degenerate case of the path L. We denote by $\mathscr{L}_\infty$ the infinite path, which may be considered as a particular case of the path $\mathscr{L}$.

By shrinking the path L in such a way to coincide everywhere with the lips of the cracks except on the tips, where the path is composed of circles of radius $\varepsilon \to 0$ centered at a_m (Fig. 2), we obtain another degenerate path, the path $L_0 = \bigcup_{k=1}^{n} l_k^0$. It is to be noted that, if a force acts at the point C then the path L_0 contains a semi-circular identation (Fig. 2).



$\mathcal{L}$
ℓ
ℓ_κ
ℓ
a_1
a_2
a_{2k-1}
a_{2k}
a_{2n-1}
a_{2n}

Fig. 1.



$\mathcal{L}$
P
ℓ_1°
ℓ_κ°
ℓ_n°
a_1
a_2
a_{2k-1}
C
a_{2k}
a_{2n-1}
a_{2n}

Fig. 2.

3. THE PATH-INDEPENDENT INTEGRALS

It is well known that all the complex stress functions $\Phi(z)$, $\Psi(z)$, $\Omega(z)$ used in elasticity are sectionally holomorphic functions [we recall that «sectionally holomorphic» is a function, which is holomorphic in each region not containing points of the cracks (a_{2k-1}, a_{2k}) $(k = 1, 2, \ldots, n)$, but continuous on (a_{2k-1}, a_{2k}) from the left and from the right with the exception of the points a_{2k-1} and a_{2k}, where it satisfies conditions like relations (3) and (4) above].

We consider now any function F of z and of $\Phi(z)$, $\Psi(z)$ and $\Omega(z)$, which is holomorphic and has no poles in the interior of the region enclosed by the paths L_0 and $\mathscr{L}$. By using *Cauchy's theorem* of complex analysis we may obtain:

$$\int_{L} F[z, \Phi(z), \Psi(z), \Omega(z)]\, dz = \int_{\mathscr{L}} F[z, \Phi(z), \Psi(z), \Omega(z)]\, dz \qquad (9)$$

The same relation is valid, if L is replaced by L_0 and $\mathscr{L}$ by $\mathscr{L}_\infty$.

This result, which is not surprising, gives the more general form of a path-independent integral. But, it is to be noted that the term P. I. I. has been attributed to integrals which satisfy a relation analogous to Eq. (9), and in addition the integral on L_0 can be evaluated analytically. This analytical expression is a function of the stress-intensity factor. In the particular case where a closed form calculation of the S. I. F. is possible the integral on L may also be evaluated by using the asymptotic expansion of the integrand at infinity and the residue theorem. Consequently, only a limited number of functions F satisfy the above requirement.

Another class of path-independent integrals may be obtained, if the real or the imaginary part of a relation of the form (9) is taken.

It follows, therefore, that there are three classes $\mathscr{F}_1$, $\mathscr{F}_2$ and $\mathscr{F}_3$ of admissible functions.

Class $\mathscr{F}_1$ contains the holomorphic functions in the entire complex plane with at least one simple pole at one tip of one of the cracks or by sectionally holomorphic functions with known discontinuity across the cracks and possessing also at least one simple pole at one tip of one of the cracks.

Class $\mathcal{F}_2$ contains the holomorphic functions, which have either pure real, or pure imaginary values at the crack lips and in addition they possess at least one simple pole at one tip of one of the cracks. If F is real we take the imaginary part of relation (9) and vice-versa. In this class of functions the corresponding integral along the upper or

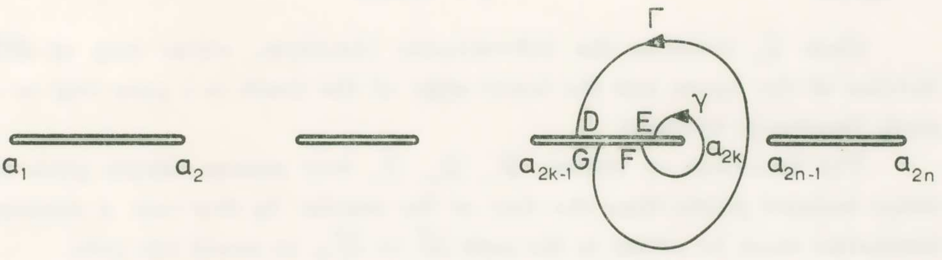


Fig. 3.

lower lip of any crack is zero. Thus, the path L may be replaced by the open path γ and the path $\mathcal{L}$ by the open path Γ (Fig. 3). These paths do not intersect one another and terminate at the same crack.

Another restruction is that these paths must not contain in their interior any other crack. Indeed, if :

$$C = \Gamma \cup L^+_{DE} \cup (-\gamma) \cup L^-_{FG}$$

where L^+_{DE}, L^-_{FG} are segments of the upper and lower edges of one of the cracks respectively (the positive sense being indicated in Fig. 3), then F is holomorphic inside and on C, and besides, it is real or pure imaginary on L^+_{DE} and L^-_{FG}. Thus taking into consideration Cauchy's residue theorem we have respectively :

$$Im \int_{\Gamma \cup L^+_{DE} \cup L^-_{FG}} F\,[z, \Phi(z), \Psi(z), \Omega(z)]\,dz = Im \int_{\gamma} F\,[z, \Phi(z), \Psi(z), \Omega(z)]\,dz \tag{10}$$

or :

$$Re \int_{\Gamma \cup L^+_{DE} \cup L^-_{FG}} F\,[z, \Phi(z), \Psi(z), \Omega(z)]\,dz = Re \int_{\gamma} F\,[z, \Phi(z), \Psi(z), \Omega(z)]\,dz \tag{11}$$

But, as F in relation (10) is real along the crack line and in relation (11) it is pure imaginary the integrals along L_{DE}^{+} and L_{FG}^{-} vanish and relations (10) and (11) take the form:

$$\begin{Bmatrix} Im \\ or \\ Re \end{Bmatrix} \left\{ \int_{\Gamma} F[z, \Phi(z), \Psi(z), \Omega(z)]\, dz \right\} = \begin{matrix} Im \\ or \\ Re \end{matrix} \left\{ \int_{\gamma} F[z, \Phi(z), \Psi(z), \Omega(z)]\, dz \right. \quad (13)$$

Class $\mathcal{F}_3$ contains the holomorphic functions, whose sum of difference at the upper and the lower edge of the crack is a pure real or a pure imaginary quantity.

The functions of classes $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ may possess simple poles at other isolated points than the tips of the cracks. In this case a circular identation must be added to the path $\mathcal{L}$ or $\mathcal{L}_\infty$ to avoid the pole.

The integrals of the elements of the set $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ along L (or Γ) constitute the set $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ respectively of the path-independent integral.

The first class $\mathcal{F}_1$ is a combination of the following functions:

$$f_1(z) = \varphi(z) + \omega(z) \quad (14)$$

$$f_2(z) = \frac{d}{dz} f_1(z) = \Phi(z) + \Omega(z) \quad (15)$$

The second class $\mathcal{F}_1$ is a combination of the functions:

$$f_3(z) = \varphi(z) + \varpi(z) = \varphi(z) + z\varphi'(z) + \psi(z) \quad (16)$$

$$f_4(z) = \varpi(z) - \varphi(z) = z\varphi'(z) + \psi(z) - \varphi(z) \quad (17)$$

$$f_5(z) = \frac{d}{dz} f_3(z) = \Phi(z) + \bar{\Omega}(z) = 2\Phi(z) + z\Phi'(z) + \Psi(z) \quad (18)$$

$$f_6(z) = \frac{d}{dz} f_4(z) = \bar{\Omega}(z) - \Phi(z) = z\Phi'(z) + \Psi(z) \quad (19)$$

$$f_7(z) = f_5^2(z) - f_6^2(z) = 4\Phi(z)\bar{\Omega}(z) \quad (20)$$

$$f_8(z) = \Phi(z) + i\bar{\Omega}(z) \quad (21)$$

$$f_9(z) = \Phi(z) - i\Omega(z) \quad (22)$$

Indeed, if the crack edges are unloaded we have from relation (2) that:

$$\overline{\varphi(x)} + x\varphi'(x) + \psi(x) = 0 \tag{23}$$

$$\Phi(x) + \overline{\Phi(x)} + x\Phi'(x) + \Psi(x) = 0 \tag{24}$$

for $x \in (a_{2k-1}, a_{2k}), \; k = 1, 2, 3, \ldots$

Using the last relations we obtain:

$$f_3(x) = \varphi(x) - \overline{\varphi(x)} \qquad x \in (a_{2k-1}, a_{2k}) \tag{25}$$

$$f_4(x) = -\left[\varphi(x) + \overline{\varphi(x)}\right] \qquad x \in (a_{2k-1}, a_{2k}) \tag{26}$$

$$f_5(x) = \Phi(x) - \overline{\Phi(x)} \qquad x \in (a_{2k-1}, a_{2k}) \tag{27}$$

$$f_6(x) = -\left[\Phi(x) + \overline{\Phi(x)}\right] \qquad x \in (a_{2k-1}, a_{2k}) \tag{28}$$

$$f_7(x) = -4\Phi(x)\overline{\Phi(x)} \qquad x \in (a_{2k-1}, a_{2k}) \tag{29}$$

$$f_8^2(x) = f_5(x)\, f_6(x) + \frac{if_7(X)}{2} \qquad x \in (a_{2k-1}, a_{2k}) \tag{30}$$

$$f_9^2(x) = f_5(x)\, f_6(x) - \frac{if_7(X)}{2} \qquad x \in (a_{2k-1}, a_{2k}) \tag{31}$$

It follows that f_3, f_5, f_8^2, f_9^2 have pure imaginary values and f_4, f_6, f_7 have pure real values for $x \in (a_{2k-1}, a_{2k})$.

The third class $\mathcal{F}_3$ contains a combination of the following functions:

$$f_{10,11}(z) = \left[\beta\Phi(z) + \gamma\Omega(z)\right]^2 \pm \left[\bar{\beta}\bar{\Phi}(z) + \bar{\gamma}\bar{\Omega}(z)\right]^2 \tag{32}$$

$$f_{12,13}(z) = \left[\beta\Phi(z) + \gamma\bar{\Omega}(z)\right]^2 \pm \left[\bar{\beta}\bar{\Phi}(z) + \bar{\gamma}\Omega(z)\right]^2 \tag{33}$$

$$f_{14,15}(z) = \left[\beta\Phi(z) \pm \bar{\gamma}\bar{\Phi}(z)\right]^2 + \left[\beta\Omega(z) \pm \bar{\gamma}\bar{\Omega}(z)\right]^2 \tag{34}$$

where β, γ are complex numbers.

Using the above ideas it is easy to construct a number of path-independent integrals. For reasons of simplicity we consider here only the case of a single crack of length 2a instead of an array of collinear parallel cracks. The path-independent integrals corresponding to such a case are presented in the table I. In the same table we give also the

TABLE I

Number of P.I.I.	Class of P.I.I.	Explicit form of the path independent integral	Path	Residue calculation along the circular identation surrounding the crack tips
(1)	(2)	(3)	(4)	(5)
T_1	$\mathcal{F}_1$	$\int f_2^2(z)dz$	L	$-i(K_{I_1}-iK_{II_1})^2-i(K_{I_2}-iK_{II_2})^2$
T_2	$\mathcal{F}_1$	$\int (z^2-a^2)^{-1/2} f_2(z)dz$	L	$i\left(\frac{\pi}{a}\right)^{1/2}[(K_{I_1}-iK_{II_1})+$ $+(K_{I_2}-iK_{II_2})]$
T_3	$\mathcal{F}_1$	$\int \left(\frac{z+a}{z-a}\right)^{1/2} f_2(z)dz$	L	$2i(K_{I_2}-iK_{II_2})\sqrt{a\pi}$
T_4	$\mathcal{F}_1$	$\int (z+a)f_2^2(z)dz$	L	$2ia(K_{I_2}-iK_{II_2})$
T_5	$\mathcal{F}_1$	$\int (z-c)f_2^2(z)dz$	L	$i(a-c)(K_{I_2}-iK_{II_2})^2+$ $+i(a+c)(K_{I_1}-iK_{II_1})^2$
T_6	$\mathcal{F}_1$	$\int (z-c)(z^2-a^2)^{-1/2} f_2(z)dz$	L	$i\sqrt{\frac{\pi}{a}}\,[(a-c)(K_{I_2}-iK_{II_2})-$ $-(a+c)\,(K_{I_1}-iK_{II_1})]$
T_7	$\mathcal{F}_1$	$\int (z-c)\left(\frac{z+a}{z-a}\right)^{1/2} f_2(z)dz$	L	$i\sqrt{\frac{\pi}{a}}\,(a-c)(K_{I_2}-iK_{II_2})$
T_8	$\mathcal{F}_1$	$\int (z-a)^{-3/2} f_1(z)dz$	L	$i\sqrt{\frac{\pi}{a}}\,(K_{I_2}-iK_{II_2})$
T_9	$\mathcal{F}_1$	$\int \left(\frac{z+a}{z-a}\right)^{1/2} \frac{f_1(z)}{z-a}dz$	L	$2i\sqrt{\pi a}(K_{I_2}-iK_{II_2})$
T_{10}	$\mathcal{F}_1$	$\int \frac{f_1(z)}{(z^2-a^2)^{3/2}}\,dz$	L	$\frac{i}{2a}\sqrt{\frac{\pi}{a}}\,[(K_{I_2}-iK_{II_2})+$ $+i(K_{I_1}-iK_{II_1})]$
T_{11}	$\mathcal{F}_1$	$\int \frac{f_1^2(z)}{(z-a)^2}\,dz$	L	$i(K_{I_2}-iK_{II_2})$
T_{12}	$\mathcal{F}_1$	$\int \frac{f_1^2(z)}{(z^2-a^2)^2}\,dz$	L	$\frac{i}{4a^2}[(K_{I_2}-iK_{II_2})^2-$ $-(K_{I_1}-iK_{II_1})^2]$
T_{13}	$\mathcal{F}_1$	$\int \frac{(z-c)}{(z^2-a^2)^2}\,f_1^2(z)dz$	L	$i/4a^2[(a-c)(K_{I_2}-iK_{II_2})^2+$ $+(a+c)(K_{I_1}-iK_{II_1})^2]$
T_{14}	$\mathcal{F}_2$	$Im\int f_5^2(z)dz$	L Γ	$K_{I_2}^2-K_{I_1}^2$ $K_{I_2}^2$

Table I (continued)

(1)	(2)	(3)	(4)	(5)
T_{15}	$\mathfrak{F}_2$	$Im\int f_6^2(z)dz$	L Γ	$-(K_{II_2}^2-K_{II_1}^2)$ $-K_{II_2}^2$
T_{16}	$\mathfrak{F}_2$	$Re\int f_5(z)f_6(z)dz$	L Γ	$K_{I_1}K_{II_1}-K_{I_2}K_{II_2}$ $-K_{I_2}K_{II_2}$
T_{17}	$\mathfrak{F}_2$	$Im\int f_7(z)dz$	L Γ	$-(K_{I_1}^2+K_{II_1}^2)+(K_{I_2}^2+K_{II_2}^2)$ $K_{I_2}^2+K_{II_2}^2$
T_{18}	$\mathfrak{F}_2$	$Re\int f_8^2(z)dz$	L Γ	$\frac{1}{2}(K_{I_1}-K_{II_1})^2-\frac{1}{2}(K_{I_2}-K_{II_2})^2$ $-\frac{1}{2}(K_{I_2}-K_{II_2})^2$
T_{19}	$\mathfrak{F}_2$	$Re\int f_9^2(z)dz$	L Γ	$-\frac{1}{2}(K_{I_1}+K_{II_1})^2+\frac{1}{2}(K_{I_2}+K_{II_2})^2$ $\frac{1}{2}(K_{I_2}+K_{II_2})^2$
T_{20}	$\mathfrak{F}_2$	$Im\int\left(\frac{z+a}{z-a}\right)^{\frac{1}{2}}f_5(z)dz$	L or Γ	$2K_{I_2}\sqrt{\pi a}$
T_{21}	$\mathfrak{F}_2$	$Im\int(z-a)^{-\frac{1}{2}}f_5(z)dz$	Γ	$K_{I_2}\sqrt{2\pi}$
T_{22}	$\mathfrak{F}_2$	$Im\int(z^2-a^2)^{-\frac{1}{2}}f_5(z)dz$	L Γ	$\sqrt{\frac{\pi}{a}}(K_{I_1}+K_{I_2})$ $K_{I_2}\sqrt{\frac{\pi}{a}}$
T_{23}	$\mathfrak{F}_2$	$Re\int\left(\frac{z+a}{z-a}\right)^{\frac{1}{2}}f_6(z)dz$	L or Γ	$-2K_{II_2}\sqrt{\pi a}$
T_{24}	$\mathfrak{F}_2$	$Re\int(z-a)^{-\frac{1}{2}}f_6(z)dz$	Γ	$-K_{II_2}\sqrt{2\pi}$
T_{25}	$\mathfrak{F}_2$	$Re\int(z^2-a^2)^{-\frac{1}{2}}f_6(z)dz$	L Γ	$-(K_{II_1}+K_{II_2})\sqrt{\frac{\pi}{a}}$ $-K_{II_2}\sqrt{\frac{\pi}{a}}$
T_{26}	$\mathfrak{F}_2$	$Im\int\{f_3(z)/(z-a)\}^2dz$	L or Γ	$4K_{I_2}^2$
T_{27}	$\mathfrak{F}_2$	$Im\int\{f_3(z)/(z^2-a^2)\}^2dz$	L Γ	$(-K_{I_1}^2+K_{I_2}^2)/a^2$ $K_{I_2}^2/a^2$
T_{28}	$\mathfrak{F}_2$	$Im\int\{f_4(z)/(z-a)\}^2dz$	L or Γ	$-4K_{II_2}^2$
T_{29}	$\mathfrak{F}_2$	$Im\int\{f_4(z)/(z^2-a^2)\}^2dz$	L Γ	$(K_{II_1}^2-K_{II_2}^2)/a^2$ $-K_{II_2}^2/a^2$

Table I (continued)

(1)	(2)	(3)	(4)	(5)
T_{30}	$\mathfrak{F}_2$	$Im\int \frac{f_3(z)}{z-a}\left(\frac{z+a}{z-a}\right)^{\frac{1}{2}} dz$	L or Γ	$4K_{I_2}\sqrt{a\pi}$
T_{31}	$\mathfrak{F}_2$	$Im\int f_3(z)(z-a)^{-3/2}dz$	Γ	$2K_{I_2}\sqrt{2\pi}$
T_{32}	$\mathfrak{F}_2$	$Im\int f_3(z)(z^2-a^2)^{-3/2}dz$	L Γ	$(K_{I_1}+K_{I_2})/a\sqrt{\frac{\pi}{a}}$ $K_{I_2}/a\sqrt{\pi/a}$
T_{33}	$\mathfrak{F}_2$	$Re\int \frac{f_4(z)}{z-a}\left(\frac{z+a}{z-a}\right)^{\frac{1}{2}} dz$	L or Γ	$-4K_{II_2}\sqrt{a\pi}$
T_{34}	$\mathfrak{F}_2$	$Re\int f_4(z)(z-a)^{-3/2}dz$	Γ	$-2\sqrt{2\pi}K_{II_2}$
T_{35}	$\mathfrak{F}_2$	$Re\int f_4(z)(z^2-a^2)^{-3/2}dz$	L Γ	$-\frac{1}{a}(K_{II_1}+K_{II_2})\sqrt{\frac{\pi}{a}}$ $-\sqrt{\pi/a}K_{II_2}/a$
T_{36}	$\mathfrak{F}_2$	$Im\int (z-c)f_5^2(z)dz$	L Γ	$(a-c)K_{I_2}^2+(a+c)K_{I_1}^2$ $(a-c)K_{I_2}^2$
T_{37}	$\mathfrak{F}_2$	$Im\int (z-c)f_6^2(z)dz$	L Γ	$-\{(a-c)K_{II_2}^2+(a+c)K_{II_1}^2\}$ $-(a-c)K_{II_2}^2$
T_{38}	$\mathfrak{F}_2$	$Re\int (z-c)f_5(z)f_6(z)dz$	L Γ	$-\{(a+c)K_{I_1}K_{II_1}+(a-c)K_{I_2}K_{II_2}\}$ $(a-c)K_{I_2}K_{II_2}$
T_{39}	$\mathfrak{F}_2$	$Im\int (z-c)f_7(z)dz$	L Γ	$(a+c)(K_{I_1}^2+K_{II_1}^2)+(a-c)(K_{I_2}^2+K_{II_2}^2)$ $(a-c)(K_{I_2}^2+K_{II_2}^2)$
T_{40}	$\mathfrak{F}_2$	$Re\int (z-c)f_8^2(z)dz$	L Γ	$-\frac{1}{2}[(a+c)(K_{I_1}-K_{II_1})^2+$ $+(a-c)(K_{I_2}-K_{II_2})^2]$ $-1/2(a-c)(K_{I_2}-K_{II_2})^2$
T_{41}	$\mathfrak{F}_2$	$Re\int (z-c)f_9^2(z)dz$	L Γ	$1/2[(a+c)(K_{I_1}+K_{II_1})^2+$ $(a-c)(K_{I_2}+K_{II_2})^2]$ $1/2(a-c)(K_{I_2}+K_{II_2})^2$
T_{42}	$\mathfrak{F}_2$	$Im\int\left(\frac{z+a}{z-a}\right)^{\frac{1}{2}}(z-c)f_5(z)dz$	L or Γ	$2(a-c)K_{I_2}\sqrt{\pi a}$
T_{43}	$\mathfrak{F}_2$	$Im\int (z-a)^{-\frac{1}{2}}(z-c)f_5(z)dz$	Γ	$(a-c)\sqrt{\pi a}\,K_{I_2}$
T_{44}	$\mathfrak{F}_2$	$Im\int (z^2-a^2)^{-\frac{1}{2}}(z-c)f_5(z)dz$	L Γ	$\sqrt{\frac{\pi}{a}}[(a-c)K_{I_2}-(a+c)K_{I_1}]$ $\sqrt{\frac{\pi}{a}}(a-c)K_{I_2}$

Table I (continued)

(1)	(2)	(3)	(4)	(5)
T_{45}	$\mathfrak{F}_2$	$Re\int\left(\frac{z+a}{z-a}\right)^{\frac{1}{2}}(z-c)f_6(z)dz$	L or Γ	$-2\sqrt{\pi a}(a-c)K_{II_2}$
T_{46}	$\mathfrak{F}_2$	$Re\int(z-a)^{-\frac{1}{2}}(z-c)f_6(z)dz$	Γ	$-(a-c)K_{II_2}\sqrt{2\pi}$
T_{47}	$\mathfrak{F}_2$	$Re\int(z^2-a^2)^{-\frac{1}{2}}(z-c)f_6(z)dz$	L Γ	$\{(a+c)K_{II_1}-(a-c)K_{II2}\}\sqrt{\frac{\pi}{a}}$ $-(a-c)K_{II_2}\sqrt{\frac{\pi}{a}}$
T_{48}	$\mathfrak{F}_2$	$Im\int\{f_3(z)/(z-a)\}^2(z-c)dz$	L or Γ	$4(a-c)K_{I_2}^2$
T_{49}	$\mathfrak{F}_2$	$Im\int\{f_3(z)/(z^2-a^2)\}^2(z-c)dz$	L Γ	$\frac{a+c}{a^2}K_{I_1}^2+\frac{a-c}{a^2}K_{I_2}^2$ $\frac{a-c}{a^2}K_{I_2}^2$
T_{50}	$\mathfrak{F}_2$	$Im\int\{f_4(z)/(z-a)\}^2(z-c)dz$	L or Γ	$-4(a-c)K_{II_2}^2$
T_{51}	$\mathfrak{F}_2$	$Im\int\{f_4(z)/(z^2-a^2)\}^2(z-c)dz$	L Γ	$-\{(a+c)K_{II_1}^2+(a-c)K_{II2}^2\}/a^2$ $-(a-c)K_{II_2}^2/a^2$
T_{52}	$\mathfrak{F}_2$	$Im\int\frac{f_3(z)}{z-a}(z-c)\left(\frac{z+a}{z-a}\right)^{\frac{1}{2}}dz$	L or Γ	$4(a-c)K_{I_2}\sqrt{a\pi}$
T_{53}	$\mathfrak{F}_2$	$Im\int f_3(z)(z-c)(z-a)^{-3/2}dz$	Γ	$2(a-c)K_{I_2}\sqrt{2\pi}$
T_{54}	$\mathfrak{F}_2$	$Im\int f_3(z)(z-c)(z^2-a^2)^{-3/2}dz$	L Γ	$\{-(a+c)K_{I_1}+(a-c)K_{I_2}\}/a\sqrt{\frac{\pi}{a}}$ $\frac{a-c}{a}K_{I_2}\sqrt{\frac{\pi}{a}}$
T_{55}	$\mathfrak{F}_2$	$Re\int f_4(z)\frac{z-c}{z-a}\left(\frac{z+a}{z-a}\right)^{\frac{1}{2}}dz$	L or Γ	$-4(a-c)K_{II_2}\sqrt{a\pi}$
T_{56}	$\mathfrak{F}_2$	$Re\int f_4(z)(z-c)(z-a)^{-3/2}dz$	Γ	$-2\sqrt{2\pi}(a-c)K_{II_2}$
T_{57}	$\mathfrak{F}_2$	$Re\int f_4(z)(z-c)(z^2-a^2)^{-3/2}dz$	L Γ	$\frac{1}{a}[(a+c)K_{II_1}-(a-c)K_{II2}]\sqrt{\frac{\pi}{a}}$ $-\frac{a-c}{a}\sqrt{\frac{\pi}{a}}K_{II2}$
T_{58}	$\mathfrak{F}_3$	$Im\int f_{10}(z)dz$	L	$-Im\{(\beta^2+\gamma^2)[-(K_{I1}-iK_{II1})^2+$ $+(K_{I2}-iK_{II2})^2]\}$
T_{59}	$\mathfrak{F}_3$	$Re\int f_{11}(z)dz$	L	$Re\{(\beta^2+\gamma^2)[-(K_{I1}-iK_{II1})^2+$ $+(K_{I2}-iK_{II2})^2]\}$

Table I (continued)

(1)	(2)	(3)	(4)	(5)
T_{60}	$\mathfrak{F}_3$	$Im\int f_{12}(z)dz$	L	$Im\{[(\beta+\gamma)K_{I_1}-i(\beta-\gamma)K_{II_1}]^2-$ $-[(\beta+\gamma)K_{I_2}-i(\beta-\gamma)K_{II_2}]^2\}$
T_{61}	$\mathfrak{F}_3$	$Re\int f_{13}(z)dz$	L	$-Re\{[(\beta+\gamma)K_{I_1}-i(\beta-\gamma)K_{II_1}]^2-$ $-[(\beta+\gamma)K_{I_2}-i(\beta-\gamma)K_{II_2}]^2\}$
T_{62}	$\mathfrak{F}_3$	$Im\int f_{14}(z)dz$	L	$Im\{[(\beta+\bar{\gamma})K_{I_1}-i(\beta-\bar{\gamma})K_{II_2}]^2-$ $-[(\beta+\bar{\gamma})K_{I_2}+i(\beta-\bar{\gamma})K_{II_2}]^2\}$
T_{63}	$\mathfrak{F}_3$	$Re\int f_{15}(z)dz$	L	$-Re\{[(\beta-\bar{\gamma})K_{I_1}-i(\beta+\bar{\gamma})K_{II_1}]^2-$ $-[(\beta-\bar{\gamma})K_{I_2}+i(\beta+\bar{\gamma})K_{II_2}]^2\}$

values of each integral along the circular identations centered at a and $-a$, if the path is shrunk to the crack. This evaluation is made by using Cauchy's residue theorem.

4. DISCUSSION

From Table I, one can observe that the integrands of the integrals T_5 to T_7, T_{13} and T_{36} to T_{57} may be obtained respectively from the integrands of the integrals T_1 to T_3, T_{12}, T_{14} to T_{35} by a multiplication of the last integrand by the factor $(z-c)$. Another number of integrands may be obtained by a multiplication by $(z-a)$, $(z+a)$, $(z-a)^{1/2}$, $(z+a)^{1/2}$ etc. Consequently it is obvious that table I is not complete and we can construct a great number of other elements by multiplying the integrands of the integrals presented in this table by a factor, which does not alter the nature of the integrand.

Thus, the integrands of class $\mathcal{J}_2$ may be multiplied by any function, holomorphic. or sectionaly holomorphic, but continuous on $(-a, a)$

(i. e possessing branch points at the crack tips), which has real values in the interval (—a, a). In contrast the integrands of classes $\mathcal{F}_1$ and $\mathcal{F}_3$ may be multiplied only by holomorphic functions, which may take any value in the interval (—a, a). It should be noted that, if a closed form solution is desired the integrand must not grow faster than $1/z$ at infinity.

In our previous paper [10] we have demonstrated that the complex potentials Φ, Ψ, φ, and ψ along a line L are given by:

$$\Phi(t) = q'(t), \qquad (t \in L) \tag{35}$$

$$\Psi(t) = \left\{\sigma_n(t) - i\sigma_t(t) - \overline{q'(t)}\right\} \frac{\overline{dt}}{ds} / \frac{dt}{ds} - \bar{t}q''(t), \qquad (t \in L) \tag{36}$$

$$\varphi(t) = q(t), \qquad (t \in L) \tag{37}$$

$$\psi(t) = \int_{t_0}^{t} \left[\sigma_n(\tau) - i\sigma_t(\tau)\right] d\tau - \overline{q(t)} - \bar{t}q'(t), \qquad (t \in L) \tag{38}$$

where:

i) σ_n, σ_t are respectively the components of normal and tangential stresses along L.

ii) the last integral is taken along L from an arbitrary point $t_0 \in L$ and

iii) $q(t)$, $q'(t)$ are defined by:

$$q(t) = \left\{\int_{t_0}^{t} \left[\sigma_n(\tau) + i\sigma_t(\tau)\right] d\tau - 2\mu\left[u(t) + iv(t)\right]\right\} / (\varkappa + 1) \qquad (t \in L) \tag{39}$$

$$q'(t) = \left\{\sigma_n(t) + i\sigma_t(t) - 2\mu\left(\frac{\partial u}{\partial t} + i\frac{\partial v}{\partial t}\right)\right\} / (\varkappa + 1) \qquad (t \in L) \tag{40}$$

with [9]:

$$\varkappa = \begin{cases} 3 - 4\nu & \text{for plane strain} \\ \dfrac{3 - \nu}{1 + \nu} & \text{for plane stresses} \end{cases} \tag{41}$$

and ν Poisson's ratio.

If the last relations are substituted in the integrands of Table I, we obtain new expressions of the path-independent integrals in function of the normal and tangential stresses and also of the displacements along the line of the path considered. It is obvious that the expressions obtained are more simple and more convenient, when compared to the expressions of the integrands of the classical path independent integrals J, L, M. Thus, a number of the proposed integrals are more suited for an application in conjuction with finite element method or classical experimental (i. e caustics, photoelasticity etc).

Another observation is that some of the prososed integrals are better suited for the one or the other loading condition. Thus, the integrands containing a factor $(z-c)$ are more suited for cracks loaded by a concentrated force at the point C.

Although this statement is not immediately obvious, some of the prososed integrals coincide with the well known J, L, M integrals. Indeed, the integral T_{16}, taking also into consideration relation (20) and (3b), may be written in the form:

$$T_{16} = Im\left[4\int_{\Gamma}[\Phi^2(z) + z\Phi'(z)\Phi(z) + \Psi(z)\Phi(z)]\,dz\right] \tag{42}$$

After an integration by parts we obtain:

$$\begin{aligned} T_{16} &= Im\left[4\int_{\Gamma}[\Psi(z) - z\Phi'(z)]\,\Phi(z)\,dz + 4\left[z\Phi^2(z)\right]_D^G\right] = \\ &= Im\left[2\int_{\Gamma}[\Phi(z) + 2\Psi(z)]\,\Phi(z)\,dz + 2\left[z\Phi^2(z)\right]_D^G\right] \end{aligned} \tag{43}$$

But the last expression of T_{16} coincides [6] with the J-integral divided by Young's modulus E.

Let now consider the integral T_{37} which, by taking into consideration relations (20) and (3b), may be written as:

$$T_{37} = Im\left[4\int_{\Gamma}[2\Phi^2(z) + z^2\Phi'(z)\Phi(z) + z\Phi(z)\Psi(z)]\,dz\right] \tag{44}$$

or, after an integration by barts, it is valid that:

$$T_{37} = Im\left[4\int_{\Gamma} z\Phi(z)\Psi(z)\,dz + 2\left[z^2\Phi^2(z)\right]_D^G\right] \tag{45}$$

And the last expression of T_{37} coincides [6] with M-integral divided by E.

5. APPLICATIONS TO THEORETICAL PROBLEMS

In this section the above developed theory was applied to some particular examples in order to show the extreme shortness of the calculations, as well as the possibility to construct a large number of path independent integrals. The examples considered concern a single crack of lenght 2a. In particular we study:

i) A crack whose upper edge is loaded by one compressive concentratet load P acting at a distance c from the midpoint of the crack.

ii) A crack loaded at the point C by two opposite compressive concentrated loads of intensity P.

iii) A crack, whose upper edge is loaded at the origin by a couple M.

iv) A crack loaded at its opposite lips by the same surface tractions:

$$p(x) = \sigma_{xx}(x) - i\sigma_{xy}(x)$$

5.1. **A crack whose one lip is submitted to a concentrated compressive load P.**

For the solution of this probem it is possible to use a number of P. I. I. given in table I or a number of integrals which do not figure in this table. Thus we can use the integrals T_3, T_5, T_{20}, T_{23} or the following ones:

$$T_{64} = Im \int_{L or \Gamma} (z-c)(z+a) f_5^2(z)\, dz \tag{46}$$

$$T_{65} = Im \int_{L or \Gamma} (z+a)(z-c) f_6^2(z)\, dz \tag{47}$$

$$T_{66} = Im \int_{L or \Gamma} (z+a)(z-c) f_7^2(z)\, dz \tag{48}$$

$$T_{67} = Re \int_{L or \Gamma} (z+a)(z-c) f_8^2(z)\, dz \tag{49}$$

The integrals T_3, T_5 give the complex stress intensity factor, the integrals T_{20}, T_{64} the crack opening S. I. F. (K_{I_2}), whereas the integrals T_{23}, T_{65} the shear S. I. F. (K_{II_2}) at the right crack-tip. Finally, the integrals T_{66} and T_{67} give a combination of K_{I_2} and K_{II_2}.

We consider the integral T_3. By shrinking the contour into the crack boundary, as shown in Fig. 2, we can evaluate the integral, if the asymptotic expansions (4) to (6) and Cauchy's residue treorem are taken into conideration.

Thus, the integral along L_0 becomes (see also table I)

$$T_1 = 2i\,(K_{I_2} - iK_{II_2})\,\sqrt{a\pi} - iP\left(\frac{a+c}{a-c}\right)^{1/2}. \tag{50}$$

Because of relations (7) the integral along $\mathscr{L}_\infty$ gives :

$$T_1 = i\left(\frac{\varkappa-1}{\varkappa+1}\right)P \tag{51}$$

and consequently :

$$K_I = \frac{P}{2\sqrt{\pi a}}\left(\frac{a+c}{a-c}\right)^{1/2}, \qquad K_{II} = -\left(\frac{\varkappa-1}{\varkappa+1}\right)\frac{P}{2\sqrt{\pi a}} \tag{52}$$

The last results are identical to those obtained by the closed-form solution of the problem. It is easy to obtain the same results by using the other proposed P. I. Is and the asymptotic expansions (4) to (7).

5. 2. A crack with opposite compressive concentrated loads P.

For the solution of this problem we can use anyone of the integrals T_3, T_5, T_{19}, T_{20}, T_{36}, T_{37}, T_{64} and T_{66}, T_{67}.

We consider, for example, the integral T_{67}. A simple residue calculation, using asymptotics expressions (6), (7), gives :

$$\frac{P^2}{\pi}(c-a) - K_I\,(a+c)\,a = 0 \tag{53}$$

and consequently :

$$K_I = P\left(\frac{1}{\pi a}\,\frac{a+c}{a-c}\right)^{1/2} \tag{54}$$

The same result may also be obtained by using the following integrals which do not appear in table I:

$$T_{68} = Re\int_L (z+a)\,(z-c)\,f_9^2(z)\,dz \tag{55}$$

$$T_{69} = Re\int_L (z+a)\,(z-c)\,f_7(z)\,dz \tag{56}$$

5.3. **A crack loaded with a couple M applied at the origin.**

Problems with couples applied at any point C of the plane domain may be solved by considering the P. I. I. integral T_6. Similarly, we can use the integrals arising from a multiplication of the integrand of the previous integrals T_3, T_5, T_{20}, T_{23}, T_{64} to T_{67} by $(z-c)$.

In our particular problem the couple M is applied at the origin $(c=0)$ and thus the asymptotic behavior of the complex potentials at this point and at infinity is:

$$\Phi(z) \sim \frac{iM}{2\pi z^2} \tag{57}$$

$$\Psi(z) \sim \frac{iM}{2\pi z^2} \tag{58}$$

Thus, we have:

$$T_{12} = \left[\frac{\pi a i}{\sqrt{\pi a}}\left[(K_{I_2} - iK_{II_2}) - (K_{I_1} - iK_{II_1})\right] - \frac{iM}{a}\right] = 0 \tag{59}$$

But

$$K_{I_2} = -K_{I_1} \quad \text{and} \quad K_{II_1} = K_{II_2} = 0 \tag{60}$$

and consequently (59) gives:

$$K_{I_2} = \frac{M}{2a\sqrt{\pi\alpha}} \tag{61}$$

This result coincides with the theoretical result obtained by Erdogan [**13**].

It is worthwhile noting that this is the first application of a path-independent integral for a closed form calculation of the stress intensity factor a cracked body loaded by a couple.

5.4. **A crack loaded by an arbitrary surface traction p(x).**

The integral T_3 is more convenient for the calculation of the complex stress intensity factor. In this case, the integral along the edges of the crack is different from zero. Thus the integral along L_0 gives:

$$T_3 = 2i(K_{I_2} - iK_{II_2})\sqrt{a\pi} - i\int_{-a}^{a}\left(\frac{a+x}{a-x}\right)^{1/2}[\Phi^+(x) + \Omega^+(x) + \Phi^-(x) + \Omega^-(x)]dx \tag{62}$$

But taking into consideration relation (2) and the boundary, conditions, we have:

$$\begin{aligned} \Phi^{+}(x) + \Omega^{-}(x) &= p(x) \\ \Phi^{-}(x) + \Omega^{+}(x) &= p(x) \end{aligned} \tag{63}$$

Consequently, from relation (52) we find:

$$2i\,(K_{I_2} - iK_{II_2})\,\sqrt{a\pi} - 2i\int_{-a}^{a}\left(\frac{a+x}{a-x}\right)^{1/2} p(x)\,dx = 0 \tag{64}$$

This result is derived by taking into consideration that the integral along $\mathscr{L}_\infty$ is zero. Then, it follows that:

$$K_{I_2} - iK_{II_2} = \frac{1}{\sqrt{\pi a}}\int_{-a}^{a}\left(\frac{a+x}{a-x}\right)^{1/2} p(x)\,dx \tag{65}$$

If a separate calculation of the opening-mode stress intensity factor K_{I_2} and the shear stress intensity factor K_{II_2} is desired, the integrals T_{20} and T_{23} may be used.

Let as use the integral T_{20}. Taking into consideration that in this case, the right hand side of relation (24) is equal to p(x), we have:

$$f_5(x) = p(x) + \Phi(x) - \overline{\Phi(x)} \qquad x \in (-a, a) \tag{66}$$

Thus, by taking into consideration that the integral along is zero, we obtain:

$$Im\left[2iK_{I_2}\sqrt{\pi a} - 2i\int_{-a}^{a}\left(\frac{a+x}{a-x}\right)^{1/2} p(x)\,dx\right] = 0 \tag{67}$$

or:

$$K_{I_2} = \frac{1}{\sqrt{\pi a}}\int_{-a}^{a}\left(\frac{a+x}{a-x}\right)^{1/2} p_1(x)\,dx \tag{60}$$

where $p_1(x) = Re\,p(x)$.

6. FURTHER APPLICATIONS TO COMPLICATED PROBLEMS

6. 1. **A crack between dissimilar media.**

We consider now two dissimilar elastic half-planes S_1 and S_2 bonded together along the Ox-axis except over the interval $(-a, a)$ where there is a flat-line crack. We suppose that the upper half-plane S_1 is characterized by the elastic coefficients μ_1 and $\varkappa_1$ and the lower half-plane S_2 by the elastic coefficients μ_2 and $\varkappa_2$ and, further, that both lips of the single crack on $y = 0$, $|x| \leqslant a$ are subject to the same load (concentrated force, couple, or distributed surface tractions). This problem was studied previously several times in papers by Rice and Sih, England, Erdogan, Willis, Theocaris etc. (An extensive bibliography on this subject is given in the paper by Theocaris [**11**]).

From these investigations or directly from the analysis it follows $\Phi_1(z) + \Omega_1(z)$ and $\Phi_2(z) + \Omega_2(z)$ are sectionally holomorphic functions and continuous along the bonded interface. Thus, for the solution of such problems the functions $f_1(z)$ and $f_2(z) = \Phi_1(z) + \Omega_1(z) = \Phi_2(z) + \Omega_2(z)$ may be used. In particular, the function $f_2(z)$ presents the behavior for $|x| < a$:

$$f_2^+(x) + \alpha f_2^-(x) = \gamma p(x) \tag{69}$$

where the parameters α and γ stand for:

$$\alpha = \frac{\mu_1 + \mu_2 \varkappa_1}{\mu_2 + \mu_1 \varkappa_2} \qquad \gamma = \frac{(\varkappa_2 + 1)\mu_2 + (\varkappa_1 + 1)\mu_2}{(\mu_1 + \mu_2 \varkappa_1)\ (\mu_2 + \mu_1 \varkappa_2)} \tag{70}$$

and $p(x)$ is the given surface tractions ($p(x)$ is a generalized function). Moreover, $f_2(z)$ presents in the vicinity of the crack tips the asymptotic behavior:

$$\begin{aligned} f_2(z) &\sim c_1(z-a)^{-1/2 - i\beta} \qquad \text{for} \quad z \to a \\ f_2(z) &\sim c_2(z+a)^{-1/2 + i\beta} \qquad \text{for} \quad z \to -a \end{aligned} \tag{71}$$

$$\beta = \frac{1}{2\pi} \ln \alpha \tag{72}$$

where c_1 and c_2 are complex constants.

It follows, then, that the function :

$$f_{c2}(z) = \left(\frac{z+a}{z-a}\right)^{-i\beta} f_2(z) \tag{73}$$

has the same behavior as the function $f_2(z)$ of the previous paragraphs. Thus, the complex stress intensity factor $K^{(2)}$ at the right-hand tip of the crack is given by :

$$K^{(2)} = \lim_{x \to a+} \left[(x^2 - a^2)^{1/2} f_{c2}(z) \right] \tag{74}$$

Another consequence of the previous remark is that the integrals valid for the function $f_2(z)$ remain also valid for the function $f_{c2}(z)$. In particular for this problem we can use among other P. I. I. and the following :

$$T_{c1} = \int_L \left(\frac{z+a}{z-a}\right)^{1/2} f_{c2}(z)\, dz \tag{75}$$

$$T_{c2} = \int_L (z^2 - a^2)^{-1/2} f_{c2}(z)\, dz \tag{76}$$

$$T_{c3} = \int_L f_{c2}^2(z)\, dz \tag{77}$$

$$T_{c4} = \int_L (z+a)\, f_{c2}^2(z)\, dz \tag{78}$$

$$T_{c5} = \int_L (z-c)\, f_{c2}^2(z)\, dz \tag{79}$$

$$T_{c6} = \int_L (z-c)\,(z+a)\, f_{c2}^2(z)\, dz \tag{80}$$

$$T_{c7} = \int_L (z-c) \left(\frac{z+a}{z-a}\right)^{1/2} f_{c2}(z)\, dz \tag{81}$$

$$T_{c8} = \int_L (z-c)\,(z^2 - a^2)^{-1/2} f_{c2}(z)\, dz \tag{82}$$

If the crack is loaded only by concentrated forces, then path-independent integrals like T_{c_1}, T_{c_2}, T_{c_5} and T_{c_6} may be used. If the crack is loaded only by couples then P.I.Is like T_{c_7}, T_{c_8} may be used. If, finally the crack is loaded by continuously distributed surface tractions p(x), then integrals like T_{c_1}, T_{c_2}, T_{c_7}, T_{c_8} may be used.

We consider now the continuously distributed loading condition and the integral T_{c_1}. Taking into consideration relations (6) and (69) to (73) and using the residue theorem we obtain:

$$(\varkappa_1 \mu_2 + \mu_1) \gamma \int_{-a}^{a} \left(\frac{x+a}{x-a} \right)^{1/2 - i\beta} p(x)\, dx + 2\pi i K^{(2)} = 0 \tag{83}$$

and consequently:

$$K^{(2)} = \frac{\gamma}{2\pi i} (\varkappa_1 \mu_2 + \mu_1) \int_{-a}^{a} \left(\frac{x+a}{x-a} \right)^{1/2 - i\beta} p(x)\, dx \tag{84}$$

A particular case of this investigation was studed recently in a note by Ioakimidis [12]. Specifically he derived an integral analogous to T_{c_1} and he calculated the stress intensity factor in the case where a pair of opposite compressive concentrated loads acts at a point along the crack. But, contrary to his assertion, it may be shown that it is not possible to extend the above results for the case of cracks along circular interfaces.

6. 2. **A number of collinear cracks.**

Up to now a single crack was considered. But the same process may be applied to the problem of a number of collinear cracks. Special attention is to be made to integrals whose integrand contains factors like $X_1(z) = (z-a)^{\pm 1/2} (z+a)^{\pm 1/2}$, $(z+a)$, $(z-c)$. All these integrals are to be reconsidered.

Thus $X_1(z)$ is to be replaced by $X_n(z) = (z-a_2)^{\pm 1/2} \ldots (z-a_n)^{\pm 1/2}$. It is obvious that, if in the previous expression of $X_n(z)$ the factor $(z-a_l)$ has a negative exponent, then the P.I.I. has a pole at the crack tip a_l. Consequently, if the P. I. I. is evaluated along L_0, the result would a function of the stress intensity factor at the tip a_l. The factors $(z \pm a)$ and $(z-c)$ may be modified in the same way. Thus, for example,

$(z-a)$ may be replaced by a product containing one or more factors of the form $(z-a_l)$ $(l=1, 2, \ldots, n)$. Here it is worth noting that, if a closed-form solution is desired, then the integrand must not grow faster than $1/z$ at infinity. Consequently, the sum of the exponents of the function $X_n(z)$ must be less than, or equal to zero, (or to one, if the crack is loaded only by couples), and only one factor, either $(z-a_l)$, or $(z-c_k)$, is allowed.

6.3. A periodic array of collinear craks of length 2a spaced at a constant interval d.

By taking into consideration the formula:

$$z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2\pi^2}\right)=\sin z \tag{85}$$

the function $X_n(z)$ (described in the previous paragraph) for $n\to\infty$ takes the form:

$$X_{\infty}(z)=\left[\sin\frac{\pi(z+a)}{d}\,/\,\sin\frac{\pi(z-a)}{d}\right]^{1/2} \tag{86}$$

In the same way the terms $(z-c)$ are to be substituted by terms of the form $\sin\pi(z-c)/d$ etc. Thus, we can extend the integrals T_3, T_5, T_{20}, T_{23} and T_{64} to T_{67} [relations (46) to (49)] as follows:

$$T_{p_1}=\int_L X_{\infty}(z)\,f_2(z)\,dz \tag{87}$$

$$T_{p_2}=\int\sin\frac{\pi(z-c)}{d}\,f_2^2(z)\,dz \tag{88}$$

$$T_{p_3}=Im\int_{L\,or\,\Gamma} X_{\infty}(z)\,f_2(z)\,dz \tag{89}$$

$$T_{p_4}=Im\int_{L\,or\,\Gamma}\sin\frac{\pi(z-c)}{d}\,\sin\frac{\pi(z+a)}{d}\,f_5^2(z)\,dz \tag{90}$$

$$T_{p_5}=Im\int_{L\,or\,\Gamma} X_{\infty}(z)\,f_6(z)\,dz \tag{91}$$

$$T_{p_6} = Im\int_{L\,or\,\Gamma} \sin\frac{\pi(z-c)}{d}\sin\frac{\pi(z+a)}{d}\, f_6^2(z)\, dz \tag{92}$$

$$T_{p_7} = Im\int_{L\,or\,\Gamma} \sin\frac{\pi(z+a)}{d}\sin\frac{\pi(z-c)}{d}\, f_7(z)\, dz \tag{93}$$

$$T_{p_8} = Im\int_{L\,or\,\Gamma} \sin\frac{\pi(z+a)}{d}\sin\frac{\pi(z-c)}{d}\, f_8^2(z)\, dz \tag{94}$$

In order to illustrate this case we suppose that the upper and the lower lips of the cracks are loaded regularly by the same surface distribution p(x). We suppose also that p(x) is a periodic function with a period d. If the integral T_{p_1} is considered and by taking into consideration relation (6) we obtain:

$$-(K_{I_2} - iK_{II_2})\sqrt{\frac{d}{2}\sin\frac{2\pi a}{d}} + \int_{-a}^{a} X_\infty(x)\, p(x)\, dx = 0 \tag{95}$$

and consequently:

$$K_{I_2} - iK_{II_2} = \left(\frac{2}{d\sin\frac{2\pi a}{d}}\right)^{1/2}\int_{-a}^{a} X_\infty(x)\, p(x)\, dx \tag{96}$$

The same result may be obtained by using Bueckner's investigation [**14**].

6.4. **Cracks with cyclic symmetry.**

The only problem with cyclic symmetry considered is the problem of a star-shaped crack with n-equal arms. If we denote by $\alpha = 2\pi/n$ the angle between these arms we have:

$$\Psi(\varepsilon z) = \bar{\varepsilon}^2\,\Psi(z) \tag{97}$$

$$\Phi(\varepsilon z) = \Phi(z) \tag{98}$$

where:

$$\varepsilon = e^{i\alpha} \tag{99}$$

Thus, if the integral T_{16} is considered we have:

$$T_{16} = Im\int_L z\Phi(z)\,\Psi(z)\, dz = Im\int_L \varepsilon z\Phi(\varepsilon z)\,\Psi(\varepsilon z)\, d(z) \tag{100}$$

and consequently this integral is suited for this particular problem.

ΠΕΡΙΛΗΨΙΣ

Αἱ μέθοδοι ποὺ στηρίζονται εἰς τὴν χρῆσιν τῶν ἀνεξάρτητων τοῦ δρόμου ὁλοκληρωμάτων, ἔχουν ἀποδειχθῆ ἰδιαίτερα ἀποτελεσματικαὶ διὰ τὸν ὑπολογισμὸν τοῦ συντελεστοῦ ἐντάσεως τάσεων. Μὲ τὰς μεθόδους αὐτὰς εἶναι δυνατὸς ὁ ἄμεσος ὑπολογισμὸς τοῦ συντελεστοῦ ἐντάσεως τάσεων, χωρὶς νὰ προηγηθῇ λεπτομερὴς λύσις τοῦ ἀντιστοίχου προβλήματος ὁριακῶν τιμῶν. Τὰ ἀνεξάρτητα τοῦ δρόμου ὁλοκληρώματα ἔχουν ἐφαρμοσθῆ ἀπὸ πολλοὺς ἐρευνητὰς σὲ συνδυασμὸ μὲ τὴν μέθοδον τῶν πεπερασμένων στοιχείων. Τὰ ἴδια ὁλοκληρώματα ἔχουν ἐφαρμοσθῆ καὶ αὐτοτελῶς διὰ τὸν ἄμεσον ἀναλυτικὸν ὑπολογισμὸν τοῦ συντελεστοῦ ἐντάσεως τῶν τάσεων εἰς ἐλαστικὰ προβλήματα ρωγμῶν.

Τὸ πρῶτον ἀνεξάρτητον τοῦ δρόμου ὁλοκλήρωμα, τὸ γνωστὸν καὶ ὡς J-ὁλοκλήρωμα προετάθη ἀπὸ τὸν J. Rice [**1**] τὸ 1968 διὰ νὰ ἀκολουθήση ἐν συνεχείᾳ τὸ I-ὁλοκλήρωμα ποὺ δὲν εἶναι τίποτε ἄλλο ἀπὸ τὸ συμπληρωματικὸν ἀντίστοιχον τοῦ προηγουμένου.

Οἱ Knowles καὶ Sternberg [**2**] ἀπέδειξαν τὴν ὕπαρξιν τῶν L- καὶ M-ὁλοκληρωμάτων. Ὁ Carlsson [**3**] ἔδωσε γενικὸν τρόπον κατασκευῆς ὁλοκληρωμάτων ἀνεξαρτήτων τοῦ δρόμου καὶ κατεσκεύασε τρία ὁλοκληρώματα, τὰ ὁποῖα ἀποτελοῦν γενίκευσιν τῶν ἤδη γνωστῶν J, L, M-ὁλοκληρωμάτων. Κατεσκεύασε ἐπίσης καὶ τὰ συμπληρωματικὰ ἀντίστοιχα τῶν προηγουμένων. Τέλος, μετὰ ἀπὸ γραμμικὸν συνδυασμὸν τῶν ἀντιστοίχων ὁλοκληρωμάτων ἐπέτυχε τρία νέα ὁλοκληρώματα.

Ἡ ἀπόδειξις ὅλων αὐτῶν τῶν ὁλοκληρωμάτων ἐβασίσθη εἰς τὸ θεώρημα τῶν Green-Gauss καὶ εἰς τὴν διαφορικὴν ἐξίσωσιν κινήσεως ἢ ἰσορροπίας τῶν σωμάτων.

Εἶναι ὅμως δυνατὸν εἰς τὴν περίπτωσιν ἐπιπέδων προβλημάτων ἐλαστικότητος νὰ χρησιμοποιηθοῦν καὶ τὰ μιγαδικὰ δυναμικὰ τοῦ Maskhelishvili ἐν συνδυασμῷ μὲ τὸ θεώρημα Cauchy περὶ ἀναλυτικῶν συναρτήσεων. Πρώτη ἐφαρμογὴ τῆς ἀρχῆς αὐτῆς ἔγινε ἀπὸ τὸν ὁμιλοῦντα καὶ τοὺς συνεργάτας του [**8**], [**15**].

Εἰς τὴν παροῦσαν ἐργασίαν ἀποδεικνύεται γενικὴ μέθοδος διὰ τὴν κατασκευὴν ἀνεξαρτήτων τοῦ δρόμου ὁλοκληρωμάτων, ἡ ὁποία ἰσχύει διὰ τυχαίαν διάταξιν καμπύλων ρωγμῶν. Εἰδικώτερα ἐφαρμογὴ εἶναι δυνατὸν νὰ γίνῃ εἰς τὴν περίπτωσιν n-συγγραμμικῶν εὐθυγράμμων ρωγμῶν.

Περὶ τὴν τυχοῦσαν k-ρωγμήν, ἡ ὁποία ὁρίζεται ἀπὸ τὸ διάστημα (a_{2k-1}, a_{2k}), φέρομεν τὸν ὁμαλὸν κατὰ Jordan καμπύλον δρόμον l_k (Σχ. 1) εἰς

τρόπον ὥστε ὁ l_k νὰ μὴ διαπερᾷ ἄλλην ρωγμήν. Ἔστω ὅτι οἱ δρόμοι αὐτοὶ $\{l_k\}_{k=1}^{n}$ δὲν τέμνονται μεταξύ τους καὶ τοὺς συμβολίζομεν μὲ τὴν ἔκφρασιν:

$$L = \bigcup_{k=1}^{n} l_k \tag{E. 1}$$

Ἐξ ἄλλου, ὁ ὁμαλὸς κατὰ Jordan δρόμος ὁ ὁποῖος περιβάλλει ὅλας τὰς ρωγμὰς συμβολίζεται μὲ $\mathscr{L}$. Συμβολίζομεν ἐπίσης μὲ $\{l_k^0\}_{k=1}^{n}$ (Σχ. 2) τοὺς δρόμους ποὺ προκύπτουν ὅταν οἱ ἀντίστοιχοι δρόμοι $\{l_k\}_{k=1}^{n}$ συμπέσουν μὲ τὰ χείλη τῆς ρωγμῆς ἐκτὸς ἀπὸ τὰ ἄκρα της a_{2k-1} καὶ a_{2k}, ὅπου ὁ δρόμος δὲν συμπίπτει μὲ αὐτά, ἀλλὰ ἀποτελεῖται ἀπὸ δύο κύκλους μὲ κέντρα τὰ σημεῖα a_{2k-1} καὶ a_{2k} ἀντιστοίχως καὶ ἀκτῖνα $\varepsilon \to 0$. Συμβολίζομεν τέλος μὲ L_0 τὴν ἑνότητα:

$$L_0 = \bigcup_{k=1}^{n} l_k^0 \tag{E. 2}$$

Ἐὰν F εἶναι συνάρτησις τῶν z, Φ, Ψ, Ω, ὁλόμορφος εἰς τὸ χωρίον ποὺ περικλείεται ἀπὸ τὰς καμπύλας L καὶ $\mathscr{L}$, τότε δι' ἐφαρμογῆς τοῦ θεωρήματος τοῦ Cauchy περὶ ἀναλυτικῶν συναρτήσεων, ἔχομεν:

$$\int_L F[z, \Phi(z), \Psi(z), \Omega(z)]\,dz = \int_{\mathscr{L}} F[z, \Phi(z), \Psi(z), \Omega(z)]\,dz \tag{E. 3}$$

Εἰς τὴν ἀνωτέρω σχέσιν τὸ ὅριον L τῆς ὁλοκληρώσεως μπορεῖ νὰ ἀντικατασταθῆ ἀπὸ τὸ L_0, ἀρκεῖ ἡ συνάρτησις F νὰ παραμένη ὁλομορφικὴ εἰς τὸ χωρίον ποὺ ὁρίζεται ἀπὸ τὰς καμπύλας L_0 καὶ $\mathscr{L}$.

Ἡ γνωστὴ σχέσις (Ε. 3) καθὼς καὶ αἱ σχέσεις ποὺ προκύπτουν ὅταν λάβωμεν τὸ πραγματικὸ ἢ φανταστικὸ μέρος τῆς (Ε. 3) ἀποτελοῦν τὴν γενικευμένην ἔκφρασιν τοῦ ἀνεξαρτήτου τοῦ δρόμου ὁλοκληρώματος. Συνήθως ὅμως ὁ ὅρος «ἀνεξάρτητον τοῦ δρόμου ὁλοκλήρωμα» ἀποδίδεται εἰς ὁλοκληρώματα τὰ ὁποῖα ἱκανοποιοῦν σχέσιν τινά, ὅπως ἡ (Ε. 3), ἀλλὰ ἐπὶ πλέον καὶ ἡ τιμή των εἶναι γνωστὴ ἀναλυτικῶς. Ἡ τιμὴ αὐτὴ εἶναι συνάρτησις τῶν συντελεστῶν ἐντάσεως τάσεων εἰς τὰ ἄκρα τῆς ρωγμῆς. Ἀπὸ τὸ γεγονὸς αὐτὸ προκύπτει καὶ ἡ χρησιμότης τῶν ὁλοκληρωμάτων αὐτῶν. Ὁρίζομεν εἰς τὴν ἐργασίαν αὐτὴν τρεῖς τάξεις συναρτήσεων τὰς $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$.

Ἡ πρώτη τάξις $\mathscr{F}_1$ περιλαμβάνει τὰς συναρτήσεις τὰς τμηματικῶς ὁλομόρφους μὲ γνωστὴν ἀσυνέχειαν κατὰ μῆκος τῆς ρωγμῆς. Ἡ τάξις $\mathscr{F}_2$ περιλαμβάνει τὰς ὁλομόρφους συναρτήσεις, αἱ ὁποῖαι λαμβάνουν μόνον πραγματικὰς ἢ φανταστικὰς τιμὰς εἰς τὰ χείλη τῆς ρωγμῆς. Ἡ τρίτη τάξις $\mathscr{F}_3$ περιλαμβάνει τὰς ὁλομόρφους συναρτήσεις, τὸ ἄθροισμα ἢ ἡ διαφορὰ τῶν τιμῶν τῶν ὁποίων εἰς τὰ χείλη τῆς ρωγμῆς εἶναι καθαρὸς πραγματικὸς ἢ φανταστικὸς ἀριθμός. Βεβαίως διὰ νὰ εἶναι δυνατὸς ὁ ὑπολογισμὸς τῶν ὁλοκληρωμάτων αὐτῶν κατὰ μῆκος τοῦ δρόμου L_0 εἶναι ἀναγκαῖον ὅπως ὅλαι αἱ συναρτήσεις διαθέτουν ἕνα ἁπλοῦν πόλον εἰς ἓν τουλάχιστον ἄκρον a_m μιᾶς ἀπὸ τὰς ρωγμάς.

Αἱ ἀπαιτήσεις αὐταὶ ἱκανοποιοῦνται διὰ τὴν πρώτην τάξιν $\mathcal{F}_1$ ὑπὸ τῶν συναρτήσεων :

$$f_1(z) = \varphi(z) + \omega(z) \qquad (E. 4)$$

$$f_2(z) = \frac{d}{dz} f_1(z) = \Phi(z) + \Omega(z) \qquad (E. 5)$$

ὅπου :

$$\Omega(z) = \omega'(z) = \overline{\Phi}(z) + z\overline{\Phi}'(z) + \overline{\Psi}(z) \qquad (E. 6)$$

$$\Phi(z) = \varphi'(z) \qquad (E. 7)$$

Διὰ τὴν δευτέραν τάξιν $\mathcal{F}_2$ οἱ κάτωθι συναρτήσεις ἱκανοποιοῦν τὰς τεθείσας συνθηκας :

$$f_3(z) = \varphi(z) + \overline{\omega}(z) = \varphi(z) + z\varphi'(z) + \psi(z) \qquad (E. 8)$$

$$f_4(z) = \overline{\omega}(z) - \varphi(z) = z\varphi'(z) + \psi(z) - \varphi(z) \qquad (E. 9)$$

$$f_5(z) = \frac{d}{dz} f_3(z) = \Phi(z) + \overline{\Omega}(z) = z\Phi(z) + z\Phi'(z) + \Psi(z) \quad (E. 10)$$

$$f_6(z) = \frac{d}{dz} f_4(z) = \overline{\Omega}(z) - \Phi(z) = z\Phi'(z) + \Psi(z) \qquad (E. 11)$$

$$f_7(z) = f_5^2(z) - f_6^2(z) = 4\Phi(z)\overline{\Omega}(z) \qquad (E. 12)$$

$$f_8(z) = \Phi(z) + i\overline{\Omega}(z) \qquad (E. 13)$$

$$f_9(z) = \Phi(z) - i\overline{\Omega}(z) \qquad (E. 14)$$

Ἐνῶ, διὰ τὴν τρίτην τάξιν $\mathcal{F}_3$ αἱ ἑξῆς συναρτήσεις πληροῦν τὰς ἀναφερθείσας συνθήκας :

$$f_{10,11}(z) = [\beta\Phi(z) + \gamma\Omega(z)]^2 \pm [\bar{\beta}\overline{\Phi}(z) + \bar{\gamma}\overline{\Omega}(z)]^2 \qquad (E. 15)$$

$$f_{12,13}(z) = [\beta\Phi(z) + \gamma\overline{\Omega}(z)]^2 \pm [\bar{\beta}\overline{\Phi}(z) + \bar{\gamma}\Omega(z)]^2 \qquad (E. 16)$$

$$f_{14,15}(z) = [\beta\Phi(z) \pm \bar{\gamma}\overline{\Phi}(z)]^2 + [\beta\Omega(z) \pm \bar{\gamma}\overline{\Omega}(z)]^2 \qquad (E. 17)$$

ὅπου β καὶ γ εἶναι μιγαδικοὶ ἀριθμοί.

Χρησιμοποιώντας τὰς συναρτήσεις $f_1(z)$ ἕως $f_{15}(z)$ εἶναι δυνατὸν νὰ σχηματίσωμεν μέγαν ἀριθμὸν ἀνεξαρτήτων τοῦ δρόμου ὁλοκληρωμάτων. Διὰ λόγους ἁπλότητος ἐξετάζομεν μόνον τὴν περίπτωσιν μιᾶς εὐθυγράμμου ρωγμῆς μήκους 2α. Εἰς τὸν πίνακα Ι δίδομεν μερικὰ ἀπὸ τὰ πλέον οὐσιώδη ὁλοκληρώματα διὰ τὴν ἁπλῆν εὐθύγραμμον ρωγμήν.

Παρὰ τὸν μεγάλον ἀριθμὸν τῶν ὁλοκληρωμάτων τῶν περιλαμβανομένων εἰς τὸν πίνακα Ι εἶναι δυνατὸν νὰ κατασκευάσωμεν καὶ ἄλλα, ἀρκεῖ νὰ πολλαπλασιάσωμεν τὰς ὁλοκληρωτέας ποσότητας ποὺ ἐμφανίζονται εἰς τὸν πίνακα Ι μὲ ὅρους τῆς μορφῆς $(z-a)$, $(z+a)$, $(z-c)$, ἢ μὲ ἄλλας καταλλήλους ὁλομόρφους συναρτήσεις.

Τὰ ὁλοκληρώματα ποὺ περιλαμβάνονται εἰς τὸν πίνακα τοῦτον ἐκφράζονται μὲ τὴν βοήθειαν τῶν μιγαδικῶν συναρτήσεων Φ, Ψ καὶ Ω. Εἰς τὰς ἐφαρμογὰς εἶναι ἀπαραίτητον τὰ ὁλοκληρώματα αὐτὰ νὰ ἐκφράζωνται συναρτήσει τῶν τάσεων καὶ μετατοπίσεων τοῦ προβλήματος. Τοῦτο εἶναι δυνατὸν ἐὰν λάβωμεν ὑπ' ὄψιν τὰς ἐκφράσεις τῶν μιγαδικῶν δυναμικῶν ἐπὶ τυχούσης καμπύλης L, τὰς ὁποίας ἔχομεν ἤδη ἀποδείξει εἰς προηγούμενον ἄρθρον [**10**]. Ὅταν γίνουν αἱ πράξεις αὗται αἱ προκύπτουσαι ἐκφράσεις διὰ μερικὰ ἀπὸ τὰ προτεινόμενα ὁλοκληρώματα εἶναι πολὺ ἁπλούστεραι καὶ πλέον κατάλληλοι διὰ τὰς ἐφαρμογάς.

Ἀποδεικνύεται, τέλος, ἡ σύμπτωσις μερικῶν ὁλοκληρωμάτων μὲ τὰ ἤδη προταθέντα J καὶ L ὁλοκληρώματα.

Τὰ ὁλοκληρώματα αὐτὰ ἐφαρμόζονται εἰς τέσσαρα θεωρητικὰ προβλήματα. Ἐκ τῶν ἐφαρμογῶν αὐτῶν καθίσταται φανερὸν ὅτι τὰ προτεινόμενα ὁλοκληρώματα προσφέρουν ἐκτὸς ἀπὸ τὰ ἄλλα πλεονεκτήματα καὶ ἁπλούστερον καὶ σύντομον τρόπον διὰ τὸν ἀναλυτικὸν ὑπολογισμὸν τοῦ συντελεστοῦ ἐντάσεως τάσεων εἰς προβλήματα ρωγμῶν. Τὰ προβλήματα τὰ ὁποῖα ἐξητάσθησαν ἀφοροῦν ἁπλῆν εὐθύγραμμον ρωγμὴν φορτιζομένην κατὰ ἕνα τῶν κατωτέρω τρόπων :

i) Διὰ συγκεντρωμένου φορτίου P εἰς τὸ ἄνω χεῖλος τῆς ρωγμῆς.

ii) Διὰ δύο ἀντιθέτων δυνάμεων P εἰς τὰ χείλη τῆς ρωγμῆς.

iii) Διὰ ροπῆς M ἐφαρμοζομένης εἰς τὸ μέσον τοῦ ἄνω χείλους τῆς ρωγμῆς, καὶ

iv) Διὰ κατανεμημένου φορτίου $p(x)$ ἐφαρμοζομένου εἰς τὰ χείλη τῆς ρωγμῆς.

Ἀξιοσημείωτον τυγχάνει ὅτι διὰ πρώτην φορὰν ἐφαρμόζεται εἰς τὴν περίπτωσιν τῶν δύο τελευταίων φορτίσεων ἀνεξάρτητον τοῦ δρόμου ὁλοκλήρωμα διὰ τὸν ἄμεσον ὑπολογισμὸν τοῦ συντελεστοῦ ἐντάσεως τάσεων.

Ἐν συνεχείᾳ ἐπεκτείνονται τὰ προηγούμενα ἀποτελέσματα εἰς μερικὰ συνθετώτερα προβλήματα ἐπιπέδου ἐλαστικότητος, ὅπως ἡ ρωγμὴ μεταξὺ δύο διαφορετικῶν μέσων, πλῆθος συγγραμμικῶν ρωγμῶν εἰς ἰσότροπον μέσον, συγγραμμικαὶ περιοδικαὶ ρωγμαὶ ὁμοίως εἰς ἰσότροπον μέσον καί, τέλος, κυκλικῶς συμμετρικαὶ ρωγμαὶ εἰς τὸ αὐτὸ μέσον. Διὰ τὰ προηγούμενα προβλήματα δίδονται μερικὰ ὁλοκληρώματα καθὼς καὶ ἡ γενικὴ μεθοδολογία διὰ τὴν κατασκευὴν περισσοτέρων καταλλήλων ὁλοκληρωμάτων. Εἰς τὴν περίπτωσιν τῶν δύο διαφορετικῶν μέσων, καθὼς καὶ εἰς τὴν περίπτωσιν τῶν περιοδικῶν ρωγμῶν, ἡ δυναμικότης τῆς εἰσαγομένης μεθόδου ἀποδεικνύεται τῇ βοηθείᾳ θεωρητικοῦ παραδείγματος. Ἐν συμπεράσματι καταλήγομεν εἰς τὸ γεγονὸς ὅτι ἡ προτεινομένη μέθοδος, ἐκτὸς τοῦ θεωρητικοῦ της ἐνδιαφέροντος, παρουσιάζει καὶ τὰ ἀκόλουθα πλεονεκτήματα:

i) Εἶναι δυνατὸν δι' αὐτῆς νὰ κατασκευασθοῦν ἀνεξάρτητα τοῦ δρόμου ὁλοκληρώματα διὰ τὴν περίπτωσιν φορτίσεως τῆς ρωγμῆς μὲ ζεῦγος δυνάμεων, πρόβλημα ποὺ δὲν ἦτο δυνατὸν νὰ ἀντιμετωπισθῇ μὲ τὰ ἤδη γνωστὰ ὁλοκληρώματα.

ii) Ἐπεξετάθη ἡ ἔννοια τῶν ἀνεξαρτήτων τοῦ δρόμου ὁλοκληρωμάτων εἰς τὴν περίπτωσιν τῶν περισσοτέρων συγγραμμικῶν ρωγμῶν, τῶν περιοδικῶν, καὶ τῶν ρωγμῶν μεταξὺ δύο μέσων. Ἡ ἐφαρμογὴ τῶν ὁλοκληρωμάτων αὐτῶν εἰς τοιαῦτα προβλήματα γίνεται ἐπίσης διὰ πρώτην φοράν.

iii) Ἐπετεύχθη ὁ ταχὺς ὑπολογισμὸς θεωρητικῶν προβλημάτων ἄνευ τῆς ἀνάγκης πλήρους λύσεως τοῦ προβλήματος.

iv) Ἐπιτυγχάνομεν ἁπλουστέρας ἐκφράσεις τῶν ὁλοκληρωτέων ποσοτήτων.

v) Αἱ ὁλοκληρωτέαι ποσότητες, δεδομένου ὅτι δὲν εἶναι τετραγωνικαὶ μορφαὶ ὡς πρὸς τὰς τάσεις, ἐπηρεάζονται ὀλιγώτερον ἀπὸ τυχὸν σφάλματα ὑπολογισμοῦ τῶν τάσεων, ὅταν τὰ ἀνεξάρτητα τοῦ δρόμου ὁλοκληρώματα συνδυάζονται μὲ ἄλλας ἀριθμητικὰς ἢ πειραματικὰς μεθόδους π. χ. μὲ τὴν μέθοδον τῶν πεπερασμένων στοιχείων, τὰς ψευδοκαυστικάς, τὴν φωτοελαστικότητα κ.λπ., ὁπότε αἱ τάσεις κατὰ μῆκος τοῦ δρόμου L ὑπολογίζονται ἀπὸ τὰς μεθόδους αὐτάς.

vi) Δυνάμεθα νὰ χρησιμοποιήσωμεν ὁλοκληρώματα τῶν ὁποίων ἡ ὁλοκληρωτέα ποσότης δύναται νὰ εἶναι ὁλόμορφος συνάρτησις ἡ ὁποία νὰ δίδη συνιστώσας τῶν τάσεων καὶ παραμορφώσεων, αἱ ὁποῖαι νὰ διαφέρουν κατὰ ἀνεξάρτητον μεταβλητὴν ἀπὸ τὰς τάσεις ἢ μετατοπίσεις τοῦ προβλήματος. Οὕτω, διὰ καταλλήλου ἐπιλογῆς τῆς τελευταίας συναρτήσεως εἶναι δυνατὸν νὰ ἐπιταχυνθῇ ἢ νὰ ἐπιβραδυνθῇ ἡ σύγκλισις μικτῶν θεωρητικῶν ἢ πειραματικῶν μεθόδων εἰς τὴν περίπτωσιν ὅπου τὰ ἀνεξάρτητα τοῦ δρόμου ὁλοκληρώματα ἐφαρμόζονται ἐν συνδυασμῷ μὲ πειραματικὰς ἢ ἄλλας ἀριθμητικὰς μεθόδους.

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