ΜΑΘΗΜΑΤΙΚΑ.— **Duality and Functional Representations of Certain Complete Algebras,** *by George F. Nassopoulos**. 'Ανεκοινώθη ὑπὸ τοῦ 'Ακαδημαϊκοῦ κ. Φιλ. Βασιλείου.

1. INTRODUCTION

The Representation Theory of commutative algebras in their natural setting of function algebras, is certainly based on Gel'fand's pioneering work on complete algebras, and on those of Mazur and Naĭmark. Since then, there has been formulated a number of relevant Theorems (e.g., in **[1]**, **[4]**, **[8]**, **[10]**, **[12]** and **[13]** to mention a few), in an attempt to generalize and/or to enlarge the original results. However, the treatment eludes to be always completely satisfactory in particular, of the real algebras. For in this case, instead of the natural spectrum of these algebras, a «carrier space» of the two-dimensional representations of them is directly considered, with implication the representation of real algebras to take, in general, place into complex function algebras (cf. **[5]** and **[2]**, p. 85). The ultimate reason of this unpleasant asymmetry lies in a deficient utilization of the «complexification procedure».

The objective of this paper is twofold: On the one hand, to present dense and faithful representations of suitable (real and complex) algebras, in particular of group algebras, improving thus considerably the classical ones. This is achieved by inspecting more closely and studying thoroughly the refined notion of an involutive, or a C^*-complexification of a given *real commutative (Banach) algebra,* and of course, that of the respective natural spectra of the algebras involved. On the other hand, to treat of in the same spirit the noble features of the duality and principally, to point it out explicitly in the real case. Following recent training, these are made by establishing convenient adjointnesses in an initial stage and then firm equivalences of appropriately determined subcategories.

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* ΓΕΩΡΓΙΟΥ Φ. ΝΑΣΟΠΟΥΛΟΥ, **Δυϊκότης καὶ συναρτησιακαὶ παραστάσεις ὡρισμένων πλήρων ἀλγεβρῶν.**

Among other consequences, another insight regarding to the strucature of commutative C*-algebras is thus revealed, and a new realiztion for the enveloping C*-algebra is offered, having a remarkable application in group algebras. Furthermore, several significant results on and around C*-algebras pass now to the corresponding real ones, attaining to give an answer to certain open questions on them. Finally, a generalization to the context of compactly generated topological algebras is outlined.

It is worth mentioning, that the application of category theoretical methods not only allows, as usual, to proceed to a concise exposition of the subject, but mostly leads to the proper setting by illuminating and clarifying the existent subtle distinctions. Moreover, the direct and elementary proof for characterizing intrinsically the real algebras corresponding to C*-ones, in essence real function algebras, simplifies essentially Arens' argumentation in obtaining representations of suitable such algebras [**1**], an argumentation which is on the other hand, independent of the one given herein.

For categorical concepts we refer to MacLane [**9**], and for basic material concerning complete normed algebras to Bonsall and Duncan [**2**]. More details with the proofs of the results obtained so far, as well as further applications along these lines will appear elsewhere.

2. NOTATION AND TERMINOLOGY

Concerning the categories we are dealing with in the sequel we adopt the following: $Alg_{\mathbb{F}}$ stands for the category of all (linear associative) algebras over $\mathbb{F}$ and $\mathbb{F}$-algebra morphisms, the scalar field $\mathbb{F}$ specified to be either $\mathbb{R}$ or $\mathbb{C}$. The addition of the suffix C indicates the restriction to the (full) subcategory of all commutative algebras, while the index 1 is used for that of all unital ones and the unit-respecting morphisms. The presence of the letter N or B means now that the category is the one of all normed, or all Banach algebras, and the continuous morphisms, respectively. The star symbol refers further to the corresponding subcategory of that in question, consisting of all involutive algebras and the *involution-preserving morphisms*. Especially, A denotes the full subcategory of $CBAlg^*_C$ of all C*-algebras, and Γ that one of $CBAlg_R$ consisting of all algebras satisfying the R-property below.

On the other hand, the *complexification functor* $P: NAlg_R \longrightarrow NAlg_C$ is one of those which draw our attention to them. Given a real normed algebra E, P(E) is, of course [5], the vector space direct sum E⊕E, the multiplication being that of the complex numbers. It is further provided with an algebra-norm, via the Minkowski functional p of a suitably chosen subset $V \subset E \oplus E$ (i.e., the absolutely convex hull of $U \times \{0\}$, U being the open unit ball of E), in such a way that, for all $x, y \in E$, p satisfies the three conditions:

$$(1)\ m \leqslant p(x,y) \leqslant 2m, \quad (2)\ p(x,0) = \|x\|, \text{ and } (3)\ p(x,-y) = p(x,y),$$

with $m := max(\|x\|, \|y\|)$. Note that the algebra P(E) is commutative, unital, or even complete respectively if, and only if, this is the case for the algebra E. Likewise, for any morphism h of real algebras, P(h) is the consistent with the additional structure linear map h⊕h and in fact, the effect of P is norm-preserving.

Denoting now by $U: NAlg_C \longrightarrow NAlg_R$ the evident *underlying functor*, assigning to each complex algebra the underlying real one, the canonical embedding $E \longrightarrow UP(E)$ is a universal morphism, so that we get:

P r o p o s i t i o n. *The complexification functor P forms a left adjoint to the underlying functor U.*

3. INVOLUTIVE COMPLEXIFICATIONS

From now on, we confine ourselves to the case of *commutative algebras*. The main reason to do so, aims at making the natural vector space involution, which is defined on P(E) as the complex conjugation, compatible with the algebra structure of it. This having already been established we denote henceforth, the resulting in the present context functor by $P^*: CNAlg_R \longrightarrow CNAlg^*_C$, and we refer to it as the *involutive complexification functor*, although the complexification procedure remains seemingly invariant. It is the manipulation of this (different!) functor which turns out to be so fruitful[1].

To begin with, let us observe that the whole picture described in section 2 is now radically altered. For in this replacement: a) The new codomain category differs widely from the preceding one, because of the presence of an algebra involution requiring the *restriction of the morphisms* to those which preserve it. b) A second backwards functor, say the *Hermitian functor* $H: CNAlg_{C}^{*} \longrightarrow CNAlg_{R}$, appears in its own right. It assigns namely, to each complex (unital, resp. complete) involutive algebra A, the real (unital, resp. complete) algebra $H(A)$ consisting of all Hermitian elements of A, and to each morphism of involutive algebras, the obvious restriction of it. c) The inherent canonical map $E \longrightarrow HP^{*}(E)$ offers in this case a universal (iso)morphism from the real algebra E to the functor H, so that the involutive complexification functor P^{*} is actually, a left adjoint to the Hermitian functor H *and not to the associated here underlying functor.*

The new point of view exhibits the deep significance of the involutive complexifications as opposed to the customary ones. In fact, we are now in a position to deduce the next result, the second part of which by applying the Open Mapping Theorem to the counit ε of the adjointness just formulated.

Equivalence Theorem. *The involutive complexification functor P^{*} and the Hermitian functor H provide an adjoint equivalence between the categories of each of the four pairs:* $(CAlg_{R}, CAlg_{C}^{*})$, $(C_{1}Alg_{R}, C_{1}Alg_{C}^{*})$, $(CBAlg_{R}, CBAlg_{C}^{*})$ *and* $(C_{1}BAlg_{R}, C_{1}BAlg_{C}^{*})$, *respectively.*

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1. By the very definition of a functor this is a triple, say (A, P, B), with A the domain, B the codomain category, and P the specified map of the morphism classes. Nevertheless, it is sometimes convenient to use the same symbol for denoting (different) functors defined of course similarly as maps, but between several pairs of categories, whenever this abuse of notation is immaterial and does not cause confusion as for instance, in the cited Equivalence Theorem. However, this is by no means the case for the considered functors P and P^{*}, the distinction of which is, in fact, crucial.

4. SPECTRA AND REPRESENTATIONS

Regarding to the concept of the spectrum of an algebra, it should be observed that it is naturally realized by the *set of morphisms* of the given algebra onto *«the simplest one of the same type»,* topologized appropriately. Hence, the spectrum of a real algebra E consists plainly of all the *real-valued characters* (i. e., non-zero, multiplicative (and continuous) linear forms), whereas the (Hermitian) spectrum of a complex involutive algebra A includes precisely the *involution-respecting complex-valued characters,* which actually correspond to the individual structure of these algebras. Although from the categorical point of view the empty set, being the initial object, is not excluded, however it is interesting to know when the spectra considered are non empty. Certainly, this is the case for any (non-trivial subalgebra of a) function algebra, while the field $\mathbb{C}$ considered as a real two-dimensional algebra E provides a counterexample. Moreover, this question turns out to be equivalent to that of the existence of a nontrivial enveloping C*-algebra for the algebra in examination (see also § 5).

These having so, we further remark that the spectrum just named for an algebra A in $CBAlg_{\mathbb{C}}^{*}$ is a closed, in general proper, subspace of the ordinary spectrum of the underlying A Banach algebra and in fact, the two kinds of spectra *coincide whenever* A *is in the subcategory* $\mathcal{A}$. Thus, the said spectrum is still a locally compact Hausdorff space and in particular, compact if, and only if, the algebra A is unital. Giving now to the spectrum of any algebra E in $CBAlg_{\mathrm{R}}$ the corresponding Gel'fand topology, it is readily seen that the latter is homeomorphic to that of $P^{*}(\mathrm{E})$, by means of the adjunction isomorphism established in Equivalence Theorem. Let us therefore, denote by $\Omega(\mathrm{E})$ either spectrum.

Under these circumstances, it becomes now clear that the right range of the Gel'fand map of a given (real) algebra E, say Φ: $\mathrm{E} \longrightarrow H(C_{\infty}(\Omega(\mathrm{E}), \mathbb{C}))$, is the «real part» of the associate C*-algebra of all complex-valued continuous functions on $\Omega(\mathrm{E})$ vanishing at infinity. Likewise, its extension $\overline{\Phi}: P^{*}(\mathrm{E}) \longrightarrow C_{\infty}(\Omega(\mathrm{E}), \mathbb{C})$ provided by the universal property of involutive complexifications offers again, the very Gel'fand map of the algebra $P^{*}(\mathrm{E})$, as it is a non-expansive *-morphism. The point is now that, by the Stone - Weierstrass Approximation Theorem, the so visualized Gel'fand maps Φ and $\overline{\Phi}$ have always a dense image.

Finally, saying that an algebra E in $CBAlg_R$ (resp. A in $CBAlg_C^*$) is *functionally semi - simple* (resp. **-semi - simple)*, if the intersection all of the kernels of its characters equals to the zero ideal, the functor P^* sends functionally semi - simple algebras on *- semi - simple ones, and the functor H reversely, since both of them being equivalences preserve and reflect monomorphisms (and epimorphisms too). All told, we deduce the following strengthened form of the classical result of Gel'fand [**5**], giving at the same time the proper setting for the case of real algebras (compare with the situation appearing also in [**2**], p. 85).

Representation Theorem. *Let* E *be an algebra in* $CBAlg_R$ *(resp.* A *an algebra in* $CBAlg_C^*$*) with spectrum* Ω(E) *(resp.* Ω(A)*). Then* E *(resp.* A*) has a representation onto a dense subalgebra of* $C_\infty(\Omega(E), \mathbb{R})$ *(resp. of* $C_\infty(\Omega(A), \mathbb{C})$*), which is faithful if, and only if,* E *is functionally semi - simple (resp.* A *is *- semi - simple).*

By specializing now to an algebra A in the subcategory A, in which case the Gel'fand map is of course an isometry, one obtains the classical isometric representation ([**6**], [**13**]) of A onto the function algebra $C_\infty(\Omega(A), \mathbb{C})$. On the other hand, the appeal to Theorem is mush more effective for investigating the nature of group algebras. Given a locally compact commutative group G, the group algebra $L^1(G)$ of it is an object in the category $CBAlg_C^*$ with *Hermitian* spectrum homeomorphic to the character group $\widehat{G}$ of G, as every one of the algebra - characters is involution - preserving. Besides, $L^1(G)$ being an A^*- algebra is *- semi - simple, so that we conclude the next of special bearing

Corollary. *The group algebra* $L^1(G)$ *admits a dense and faithful representation on the function algebra* $C_\infty(\widehat{G}, \mathbb{C})$.

The stated results are, in particular, valid for unital algebras.

5. DUALITY THEORY

Categorically speaking, duality is expressed by means of an equivalence between convenient categories. More accurately, of the one to the opposite (: dual) of the other Certainly, this is also the case for the Gel'fand - Naĭmark duality (see [8], or [12], and references there). Our attempt is here firstly to enlarge a bit the categorical framework for this duality.

Let L stand for the category of all locally compact Hausdorff spaces and their continuous maps, and K for the full reflective subcategory of all compact spaces. Then there are defined two functors : The *spectral space functor* $\boldsymbol{\Omega} : \boldsymbol{CBAlg}^*_{\mathbb{C}} \longrightarrow \boldsymbol{L}^{\mathrm{op}}$ assigning to each algebra A its *(Hermitian)* spectrum $\boldsymbol{\Omega}(\mathrm{A})$ as previously, and the *function algebra functor* $C_\infty(-, \mathbb{C}) : L^{\mathrm{op}} \longrightarrow \boldsymbol{CBAlg}^*_{\mathbb{C}}$ sending every space X on the indicated C*-algebra of continuous functions and actually, there is no problem with the evident by means of the functional composition, correspondence on the morphisms. Besides, these data are connected in the sense of the following Theorem, the last part of which being well-known (ibid. ref.), is merely recorded for the sake of completeness.

Complex Duality Theorem. *The spectral space functor* $\boldsymbol{\Omega}$ *is left adjoint to the function algebra functor* $C_\infty(-, \mathbb{C})$, *the defining the adjointness universal unit provided by the associated Gel'fand map.*

In particular, these functors establish a dual equivalence of the subcategory A *of all C*-algebras to the category* L *and by further restriction, of the subcategory* A_1 *of unital ones to the subcategory* K, *respectively.*

As a consequence, one gets an other «more constructive realization» for the enveloping C*-algebra $E(\mathrm{A})$ of a given algebra A in $\boldsymbol{CBAlg}^*_{\mathbb{C}}$ [3]. Indeed, this is provided by the pair $(\overline{\Phi}, C_\infty(\boldsymbol{\Omega}(\mathrm{A}), \mathbb{C}))$ since the Gel'fand map $\overline{\Phi} : \mathrm{A} \longrightarrow C_\infty(\boldsymbol{\Omega}(\mathrm{A}), \mathbb{C})$ is, in addition, an epimorphism (Representation Theorem). The given description makes now straightforward the fact that «the (Hermitian) spectrum of the algebra A is homeomorphic to that of the enveloping C*-algebra $E(\mathrm{A})$», and vice versa. Put differently : A is a full epireflective subcategory of $\boldsymbol{CBAlg}^*_{\mathbb{C}}$, the reflector being the *enveloping C*-algebra functor* $E \cong C_\infty(\boldsymbol{\Omega}(-), \mathbb{C})$.

Hence, the function algebra $C_\infty(G, \mathbb{C})$ in Corollary before, is precisely the group C*-algebra of G.

We can now derive further information for the structure of (commutative) C*-algebras. To this end, consider the composite adjointness of the enveloping one with that of the involutive complexification, and call the resulting then algebra $EP^*(\mathrm{E})$, the *C*-complexification* of the given algebra E in $CBAlg_{\mathrm{R}}$. As the restriction of the equivalence H to the full subcategory $\mathcal{A}$ remains of course, a full and faithful functor, the composite counit $\bar{\varepsilon}_{\mathrm{A}} : EP^*H(\mathrm{A}) \longrightarrow \mathrm{A}$ is still an isomorphism, but now in $\mathcal{A}$. That is, more explicitly, we have:

S t r u c t u r e T h e o r e m. *Every commutative C*-algebra* A *is isometrically *-isomorphic with the C*-complexification* $EP^*H(\mathrm{A})$ *of the real part of itself.*

To supplement categorically the preceding result, it should also be remarked that, by ([**9**], p. 91, Thm 1), the subcategory $\mathcal{A}$ is essentially equivalent to the full reflective subcategory, say Γ, of $CBAlg_{\mathrm{R}}$ consisting of all algebras E for which there exists some algebra A in $\mathcal{A}$ such that $\mathrm{E} \cong H(\mathrm{A})$ within an isometric isomorphism, and that this subcategory Γ is plainly, the largest one. This being the case, we are now led in a natural way to the following:

I n t r i n s i c C h a r a c t e r i z a t i o n T h e o r e m. *For any algebra* E *in* $CBAlg_{\mathrm{R}}$ *the following two statements are equivalent:*

(1) The algebra E *belongs to the subcategory* Γ.

(2) The algebra E *fulfills the R-property: For all* x, y *in* E *the inequality* $\|x\|^2 \leqslant \|x^2 + y^2\|$ *holds true.*

Sketch of the proof. That (1) implies (2) rests upon standard properties of the order structure of the algebra $H(\mathrm{A})$. For the converse observe, the R-property entrains a very simple and familiar description for the C*-complexification $EP^*(\mathrm{E})$ of the algebra E: This is just the algebra $\mathrm{E}\oplus\mathrm{E}$ but renormed now with the well-defined, complete, algebra-norm q, given (for all $(x, y) \in \mathrm{E}\oplus\mathrm{E}$) by $q(x, y) := \|x^2+y^2\|^{1/2}$ and thus, satisfying the C*-property. In other words, $EP^*(\mathrm{E})$ is that which we are looking for. ∎

In this concern, it is quite clear that Γ is a proper (non-void) subcategory of $CBAlg_R$. Especially, for any space X in L the function algebra $C_\infty(X, \mathbb{R})$, in particulal $\mathbb{R}$ itself, belongs to Γ, but not the real algebra $\mathbb{C}$. On the other hand, denoting by $Q : \Gamma \longrightarrow A$ the restriction of the C*-complexification functor EP^* to the indicated subcategory, one obtains by composing the above stated two equivalences, the next complete form of duality for real algebras, which in connection with the above considerations extends and reinforces previous relevant results of [1], [5] and [8].

Real Duality Theorem. *The composite functors ΩQ and $HC_\infty(-, \mathbb{C})$ provide a dual equivalence between the categories Γ and L and by restriction, between the subcategories Γ_1 and K as well.*

More specifically, the Theorem means in essence, that:

a) Every space X in L is *homeomorphic* to the corresponding spectrum $\Omega_R(C_\infty(X, \mathbb{R}))$, and

b) Any algebra E in $CBAlg_R$ is in the subcategory Γ if, and only if, it is *isometrically isomorphic with the function algebra* $C_\infty(\Omega_R(E), \mathbb{R})$, where the spectral space functor $\Omega_R : CBAlg_R \longrightarrow L^{op}$ defined here on real algebras too, is naturally isomorphic to the composite functor ΩEP^*.

The second result can also be extended to the case of «compactly generated topological algebras» with the indispensable modifications on the notation. Indeed, the same style of argument together with the main result of [4] shows, that the «Kelleyfication» of (the underlying locally m-convex algebra to) a real commutative multinormed algebra $(E, \{p\})$ which is further complete, unital and possesses the R-property, in the sense that each of the algebra-semi-norms p satisfies it, is *topologically isomorphic onto* the function algebra $C(\Omega(E), \mathbb{R})$ with respect to the natural Kelley topology of the latter, where of course, the spectrum $\Omega(E)$ is now a suitable k-space.

6. FUNDAMENTALS OF R-ALGEBRAS

We conclude with a brief discussion on several properties of R-algebras (: real algebras satisfying the R-property before), which illustrate their significance. They are the real analoga of well-known properties of C*- algebras to which otherwise, are more or less immediately reduced, in virtue of the preceding Theorems. They also provide an answer to certain open questions on real algebras.

(i) Divisibility. The functor Q respects divisibilily, so that every division algebra D in Γ_1 is isometrically isomorphic to $\mathbb{R}$ (compare with [**5**] and [**11**]).

(ii) Automatic continuity [**14**]. Any algebra - morphism $h: B \longrightarrow E$ with B in $CBAlg_R$ and E in Γ, especially a character $\varphi: B \longrightarrow \mathbb{R}$, is necessarily contractive, as the equality $p(x,o) = q(x,o)$ holds true for all x in E. In the same vein, every positive linear functional $\omega: E \longrightarrow \mathbb{R}$ i.e., one satisfying $\omega(x^2) \geqslant 0$ for all $x \in E$, is continuous, whenever E is in Γ, because of the relevant preservation property of Q.

(iii) Norm uniqueness. On a given algebra E in $CAlg_R$ there exists at most one norm turning E into an algebra of Γ. For the algebra - monomorphisms in Γ are exactly the isometric ones, as this is also the case for those of A.

(iv) Categorical initiality. For each algebra E in Γ the unique norm is expressed by the formula $\|x\| = sup\{|\varphi(x)| : \varphi \in \Omega_R(E)\}$ for all $x \in E$.

(v) Semi - simplicity. All algebras in Γ are semi - simple, in full agreement with ad *(i)*.

(vi) Nonexistence of derivations. The functor Q preserves properly derivations, so that according to [**15**], there are no non - zero derivations on any algebra E of Γ. In particular, this is also true of $\mathbb{R}$.

(vii) Ordering. Every algebra E in Γ possesses a natural order structure, that of a real function algebra.

(viii) Singleness. There is precisely one endomorphism of $\mathbb{R}$ in Γ_1 the identity one, since its spectrum is the singleton.

(ix) Enveloping R - algebra. The functors HEP^* and $C_\infty(\Omega_R(-), \mathbb{R})$ are naturally isomorphic and realize the reflector in the subcategory Γ of $CBAlg_R$, assigning thus to each algebra of the latter its enveloping R - algebra.

(x) Algebricity. The category Γ_1 is weakly algebraic over *Set*, as this is also true of its skeleton of all real function algebras [**7**], or of A_1, etc.

ΠΕΡΙΛΗΨΙΣ

Ἡ Θεωρία Παραστάσεως μεταθετικῶν ἀλγεβρῶν εἰς τὸ φυσικὸν δι᾽ αὐτὰς πλαίσιον τῶν ἀλγεβρῶν συναρτήσεων ἑδράζεται, ὡς γνωστόν, εἰς τὴν πρωτοποριακὴν ἐργασίαν τοῦ Gel'fand ἐπὶ τῶν πλήρων normées ἀλγεβρῶν, ὡς καὶ εἰς ἐκείνας τῶν Mazur καὶ Naĭmark. Αἱ πραγματοποιηθεῖσαι κατὰ τὴν διαρρεύσασαν ἔκτοτε 40ετίαν βελτιώσεις καὶ γενικεύσεις δὲν ἀπήλλαξαν ἐν τούτοις αὐτὴν ἀπὸ ὡρισμένα πρωτογενῆ ἀσθενῆ σημεῖα. Πράγματι τοῦτο συμβαίνει κατ᾽ ἐξοχὴν διὰ τὰς πραγματικὰς ἀλγέβρας, ἡ παράστασις τῶν ὁποίων λαμβάνει γενικῶς χώραν εἰς ἀλγέβρας μιγαδικῶν συναρτήσεων. Τὸ βαθύτερον αἴτιον τῆς προδήλου ἀντιφάσεως ἔγκειται εἰς μίαν περιωρισμένην ἀξιοποίησιν τῆς ἐννοίας τῆς μιγαδοποιήσεως, ἡ ὁποία ἤγαγεν μὲν εἰς τὴν ἀπ᾽ εὐθείας θεώρησιν ἑνὸς χώρου «δισδιαστάτων παραστάσεων», ἠγνόησεν δὲ τὸ φυσικὸν (μὴ κενὸν) φάσμα τῶν πραγματικῶν ἀλγεβρῶν.

Ἡ παροῦσα μελέτη ἀποσκοπεῖ εἰς τὴν ἀνάταξιν τοῦ ὅλου προβλήματος καὶ τὴν ἐξ αὐτοῦ παρουσίασιν ἑνὸς θεωρήματος πιστῆς καὶ πυκνῆς παραστάσεως καταλλήλων (πραγματικῶν καὶ μιγαδικῶν) ἀλγεβρῶν, ἰδιαιτέρως ἀλγεβρῶν ὁμάδος, βελτιώνουσα σημαντικῶς τὰ κλασσικὰ ἀποτελέσματα. Τοῦτο κατορθοῦται διὰ τῆς χρήσεως συγχρόνων μεθόδων τῆς Θεωρίας τῶν Κατηγοριῶν, διὰ τῶν ὁποίων ἀπεικονίζονται ἐναργέστερον καὶ ἐπισημαίνονται ἐπιτυχέστερον αἱ ὑφιστάμεναι λεπταὶ διαφοραί. Κατ᾽ αὐτὸν τὸν τρόπον καθίσταται δυνατὴ ἡ ἐνδελεχὴς μελέτη τῆς ἐκλεπτυσμένης ἐννοίας μιᾶς ἐνελικτικῆς, ἢ μιᾶς C^*- μιγαδοποιήσεως δοθείσης πραγματικῆς μεταθετικῆς ἀλγέβρας Banach, ὡς ἐπίσης καὶ τῶν φυσικῶν φασμάτων τῶν ὑπεισερχομένων ἀλγεβρῶν.

Ἀφ᾽ ἑτέρου σπουδάζεται μὲ τὸ ἴδιον πνεῦμα ἡ συμφυὴς δυϊκότης, ἰσχυροποιοῦνται δὲ καὶ ἐπεκτείνονται γνωστὰ σχετικὰ ἀποτελέσματα, τόσον εἰς τὴν μιγαδικήν, ὅσον κυρίως εἰς τὴν πραγματικὴν περίπτωσιν, εἰς τὴν ὁποίαν ἡ διϋκότης καταδεικνύεται κατηγορηματικῶς καὶ εἰς πλήρη μορφήν, διὰ τοῦ ἐγγενοῦς χαρακτηρισμοῦ τῶν ἀναλόγων πρὸς τὰς C^*- ἀλγέβρας, πραγματικῶν ἀλγεβρῶν.

Ὡς συνέπειαι τῶν προηγουμένων ἀποκαλύπτεται μία ἄλλη ἀντίληψις ἀναφορικῶς μὲ τὴν δομὴν τῶν μεταθετικῶν C^*- ἀλγεβρῶν καὶ προσφέρεται μία νέα ὑλοποίησις διὰ τὴν περιβάλλουσαν C^*- ἄλγεβραν, ἡ ὁποία ἔχει ἀξιοσημείωτον ἐφαρμογὴν εἰς τὰς ἀλγέβρας ὁμάδος. Περαιτέρω ποικίλα σημαντικὰ ἀποτελέσματα τῆς Θεωρίας τῶν C^*- ἀλγεβρῶν μεταφέρονται πλέον εἰς τὰς ἀντιστοίχους πραγματικὰς ἀλγέβρας, ἐπιτυγχανομένου κατ᾽ αὐτὸν τὸν τρόπον νὰ δοθῇ μία ἀπάντησις εἰς ὡρισμένα ἐκκρεμῆ ἐρωτήματα ἐπὶ τῶν ἀλγεβρῶν αὐτῶν. Ἐξ ἄλλου, σκιαγραφεῖται μία γενίκευσις εἰς τὸ πλαίσιον τῶν «συμπαγῶς παραγομένων τοπολογικῶν ἀλγεβρῶν».

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