ΠΡΑΚΤΙΚΑ ΤΗΣ ΑΚΑΔΗΜΙΑΣ ΑΘΗΝΩΝ

ΣΥΝΕΔΡΙΑ ΤΗΣ 22ΑΣ ΜΑΪΟΥ 1997

ΠΡΟΕΔΡΙΑ ΝΙΚΟΛΑΟΥ ΜΑΤΣΑΝΙΩΤΗ

ΜΗΧΑΝΙΚΗ. — **Multilevel optimal design of cellular structures in biomaterials by the numerical homogenization method,** by Academician *Pericles S. Theocaris**.

Keywords: Numerical homogenization, Negative Poisson's ratio, Cellular Materials.

ABSTRACT

Multilevel iterative optimal design procedures, borrowed from the theory of structural optimization by means of homogenization, are used in this paper for the optimal material design of composite material structures. The method is quite general and includes materials with appropriate microstructure, which lead eventually to phenomenological overall negative Poisson's ratios. The benefits of optimal structural design gained by this approach, together with attempts to explain the task oriented microstructure of natural structures are investigated by means of a numerical example simulating human bones.

1. INTRODUCTION

This paper deals with the optimal material design problem for composites and cellular materials by using distributed, multilevel, iterative optimization techniques. This approach has recently gained on interest due to significant developments in the area of topology optimization, based on numerical

* ΠΕΡΙΚΛΕΟΥΣ Σ. ΘΕΟΧΑΡΗ, **Πολυβάθμιος ὑπολογισμὸς βελτιστοποιήσεως κυψελωτῶν κατασκευῶν προσομοιαζουσῶν πρὸς ἐμβίους δομάς, διὰ τῆς μεθόδου τῆς ἀριθμητικῆς ὁμογενοποιήσεως.**

homogenization techniques. The link with classical effective moduli theories and with cellular structures, extended also to materials with negative Poisson's ratios, is developed in some detail. By considering the solution procedure as a task-driven self-adaptation of the structure, a better insight in the adaptation and remodelling rules proposed for natural structures, mainly animal bones and trees, is gained. The feasibility of the proposed method and the benefits of using variable structure materials, as nature does, in structures and structural elements is studied by means of representative numerical examples.

The optimal material design problem is formulated in a general setting, by following recent developments in optimal shape and topology design by means of numerical homogenization [1]. The stiffest structure is obtained by maximizing the potential energy of the structure at equilibrium or, equivalently, by minimizing the complementary energy. The choice of the material at each point of the structure, or at each finite element in the respective discretized equivalent problem, is governed by the design variables of the optimal design problem.

From the formulation of the problem and since an equilibrium configuration makes the potential energy of the structure minimum, one expects that multilevel techniques will be most appropriate. In fact, a multilevel splitting of the problem into two subproblems is possible, where the first subproblem is the structural analysis problem itself, and the second subproblem tackles with the material adaptation, which is performed in a pointwise (or elementwise) sense, by means of small-scale, decomposed subproblems (adaptation rules). This approach has its origin in the optimality criteria methods for the solution of optimal layout problems [1, 2], and leads to local adaptation rules and repetitive solutions of classical structural analysis problems, which are readily implemented in general purpose structural analysis problems of open architecture.

The method has been applied to a series of porous, cellular materials [3]. Both analytic solutions and look-up tables, produced by numerical techniques, will be used to relate the design variables with the effective elastic moduli [4], in order to treat also cellular materials with negative Poisson's ratios. Thus, without going into details which would arise in the case of topology optimization problems, we will restrict our attention in this paper in practically realizable microstructural configurations, which include materials with negative Poisson's ratios.

2. FORMULATION OF THE OPTIMAL DESIGN PROBLEM

A distributed parameter, minimum compliance (or equivalently maximum stiffness) problem is assumed. Let the design vector be denoted by $[\rho(x), \alpha(x)]^T$, where $\rho(x)$ is the density of each point x of the structure and $\alpha(x)$ is the set of design variables that describe the microstructure of the material. For instance, in a composite material structure the fiber-volume fraction, the elastic constants of the matrix and the fiber, the mesophase variables etc, may be used as design variables in $\alpha(x)$. Analogously, for a cellular (foamy) structure $\alpha(x)$ describes the design parameters of the unit cell, as it will be discussed in detail with respect to concrete applications. Note also that in a discretized structure $[\rho(x), \alpha(x)]^T$ is defined elementwise.

In a displacement based formulation, the minimum compliance problem is written in the following form of a marginal function potential energy optimization problem [1]:

$$\max_{[\rho(x),\,\alpha(x)]\,\in\,A_{ad},x\in\Omega}\ \min_{u\in U_{ad}} \left\{ \overline{\Pi}(u,\rho,\alpha) = \Pi(e(u),\rho,\alpha) - l(u) \right\} \tag{1}$$

Here A_{ad} is the admissible set of the design variables which, on the assumption that a structure with maximum mass V is sought, reads:

$$A_{ad} = \left\{ \rho(x),\ \alpha(x),\ x\in\Omega \text{ such that} \int_\Omega \rho d\Omega \le V,\ 0 \le \rho \le 1 \right\} \tag{2}$$

The total potential energy $\overline{\Pi}(u,\rho,\alpha)$ is composed of the energy density $\Pi(e(u),\rho,\alpha)$, which in turn depends on the chosen design ρ, α and the loading potential term $l(u)$. Finally, the admissible displacements set U_{ad} depends on the nature of the analysed problem and on the boundary conditions.

For the case of linear elastic structures within the frame of a small displacement and deformation theory, problem (1) reads:

$$\max_{[\rho(x),\,\alpha(x)]\,\in\,A_{ad}}\ \min_{u\in U_{ad}} \left\{ \frac{1}{2} \int_\Omega E_{ijkl}(x,\rho(x),\alpha(x))\ \varepsilon_{ij}(u)\ \varepsilon_{kl}(u)\ d\Omega - l(u) \right\} \tag{3}$$

where $E_{ijkl}\,(x,\rho(x),\alpha(x))$ is the fourth-order elastic stiffness tensor.

The stress space problem reads analogously:

$$\min_{[\rho(x),\,\alpha(x)]\,\in\,A_{ad}}\ \min_{\sigma\in\Sigma_{ad}} \left\{ \Pi^c(\sigma,\rho,\alpha) \right\} \tag{4}$$

For the linear elastic problem considered here, it has the form:

$$\min_{[\rho(x),\,\alpha(x)]\,\in\,A_{ad}}\ \min_{\sigma\in\Sigma_{ad}} \left\{ \frac{1}{2}\int_\Omega C_{ijkl}(x,\rho(x),\alpha(x))\,\sigma_{ij}\,\sigma_{kl}\,d\Omega \right\} \tag{5}$$

Here Σ_{ad} is the set of the admissible stresses, which incorporates the stress equilibrium condition.

$$\mathrm{div}\sigma + p = 0$$

and the prescribed boundary tractions condition: $\sigma n = t$. Moreover, $\Pi^c(\sigma,\rho,\alpha)$ is the complementary energy density and $C_{ijkl}\,(x,\rho(x),\alpha(x))$ is the fourth order elastic compliance tensor.

In principle, more general **cost functions** can be assumed in the optimal design problems (1) and (4). Nevertheless, the marginal function form used here, where the value of the minimization subproblem is either maximized in (1), or minimized in (4), has certain advantages from both the theoretical and the numerical point of view.

Without entering into details here we mention that continuity and, in some cases, differentiability or convexity/concavity information for the marginal function (inner subproblem in (1) or in (4)) as a function of the design variables $(\rho(x),\alpha(x))$ can be extracted as established in ref. [5].

3. THE MULTILEVEL DECOMPOSITION

For the needs of the applications treated in this paper it is sufficient to assume that the density of the structure at each point $x \in \Omega$ (cell or finite element, for the numerical application) depends on the local design variables $\alpha(x)$, i.e. $\rho(x) = f(\alpha(x))$, $\forall\ x \in \Omega$. In order to exploit the pointwise nature of the above relation we decompose the optimal design problem as follows. For problem (3):

$$\max_{\rho(x)\in A_1}\ \max_{\alpha(x)\in A_2(\rho)}\ \min_{u\in U_{ad}}\left\{\frac{1}{2}\int_\Omega E_{ijkl}(x,\rho(x),\alpha(x))\,\varepsilon_{ij}(u)\,\varepsilon_{kl}(u)\,d\Omega - l(u)\right\} \tag{6}$$

For problem (5), respectively,

$$\min_{\rho(x)\in A_1}\ \min_{\alpha(x)\in A_2(\rho)}\ \min_{\sigma\in\Sigma_{ad}}\left\{\frac{1}{2}\int_\Omega C_{ijkl}(x,\rho(x),\alpha(x))\,\sigma_{ij}\,\sigma_{kl}\,d\Omega\right\} \tag{7}$$

Here $A_1 = \{\rho(x) : 0 \le \rho(x) \le 1, \int_\Omega \rho(x)\, d\Omega \le V,\ \ x \in \Omega\}$ and $A_2 = \{\alpha(x) : \rho(x) = \text{given} = f(\alpha(x)), x\in\Omega\}$. Obviously, the following relation holds:

$$A_{ad} = \{\rho(x),\alpha(x) : \rho(x) \in A_1,\ \alpha(x) \in A_2(\rho)\}.$$

Following [1], [6] we change the order of the second and third operators in problems (6) and (7) and, by observing that the restrictions of set $A_2(\rho)$ above can be assumed to hold elementwise, we get the following problems:

$$\max_{\rho(x)\in A_1}\ \min_{u\in U_{ad}}\left\{\int_\Omega\left\{\max_{\alpha(x)\,S.t.\,\rho(x)=f(\alpha(x))}\frac{1}{2}E_{ijkl}(x,\rho(x),\alpha(x))\varepsilon_{ij}(u)\varepsilon_{kl}(u)d\Omega\right\} - l(u)\right\} \tag{8}$$

and:

$$\min_{\rho(x)\in A_1}\ \min_{\sigma\in\Sigma_{ad}}\left\{\int_\Omega\left\{\min_{\alpha(x)\,S.t.\,\rho(x)=f(\alpha(x))}\frac{1}{2}C_{ijkl}(x,\rho(x),\alpha(x))\,\sigma_{ij}\,\sigma_{kl}\,d\Omega\right\}\right\} \tag{9}$$

From the above form of the optimal design problem a hierarchical iterative solution strategy is straightforward [1], [6]. The inner maximization (respectively minimization) problem is solved in a decoupled, elementwise form. The second level minimization problem is solved at the structural level by a general purpose finite element code. Both above-mentioned subproblems are solved for a given density distribution $\rho(x)$, which in turn is updated by solving the first level optimization problem in (8) (respectively in (9)) with respect to the variable $\rho(x)$.

A few remarks are in order here. If the relation $E_{ijkl}\,(\alpha(x))$ (respectively, $C_{ijkl}(\alpha(x))$) is given analytically, then the lower level local optimization problem in (8) (respectively in (9)) can be solved also analytically. This is the case e.g. of optimal topology design problems through **homogenization**, which is studied in detail in [1] and [6], among others. In this case the second level problem can be formulated as a fictitious non-linear elastic structural analysis problem, with a nonlinear and, in general, nondifferentiable strain-energy density, which is given by the previously mentioned local optimization problem. For the solution of the latter problem the methods developed within the theory of nonsmooth mechanics can be employed following ref. [6].

In this paper, since the above-mentioned relation is not given explicitly, but it is produced from a complete analysis of a **unit representative cell**, or it is interpolated from such solutions summarized in the form of look-up tables, problems (8) and (9) have to be solved iteratively. Thus, the second level problems i.e. the nonlinear and possibly nondifferentiable structural analysis problem are approximated by linear problems, which are defined by the current values of E_{ijkl} (respectively C_{ijkl}), as they are given from the solution of the lower level (local) subproblems.

It is worthwhile noting that, in some applications, like the design of fiber - reinforced composites, the mass distribution is approximately constant for the whole structure, irrespectively of the values of the design variables. In this case the previously formulated problems along with the corresponding solution algorithms are appropriately simplified. Alternatively the **mass variable** may be kept in the formulation to measure the cost, associated with the material changes performed in the course of the algorithm. In this case the **mass constraint** should be interpreted as a maximum allowable cost constaint.

4. THE MULTILEVEL SOLUTION ALGORITHM

The solution algorithm for problem (8) is summarized as follows:

Step 0: Initialization

$l = 0$, Choose $\rho^l(x)$, $\alpha^l(x)$.

Step 1: Iteration

set iteration counter $l = l + 1$.

Step 2: Local subproblem

for each finite element and for a given e^l, solve the local problem:

$$\max_{\alpha(x)\in A_2(\rho^l)} \pi(e^l,\rho^l,\alpha) = \max_{\alpha(x)\in A_2(\rho^l)} \frac{1}{2} E_{ijkl}(x,\rho^l(x),\alpha(x))\varepsilon^l_{ij}\varepsilon^l_{kl} \tag{10}$$

The solution $a^{l}{}_{+}{}^{1}$ defines the current design (e.g. microstructure) of the finite element.

Step 3: Structural analysis subproblem

solve the potential energy minimization problem:

$$\min_{u\in U_{ad}} \left\{ \int_\Omega \pi(\varepsilon.\rho^l,\alpha^{l+1})d\Omega - l\,(u) \right\} = \min_{u\in U_{ad}} \left\{ \frac{1}{2}\int_\Omega E_{ijkl}(x,\rho^l(x),\alpha^{l+1}(x))\varepsilon_{ij}\varepsilon_{kl}d\Omega - l(u) \right\} \tag{11}$$

with solution u^{l+1}.

Step 4 : Density updating by the iterative formula:

$$\rho^{l+1} = \max\left\{ \rho_{\min}, \min\left\{ \left[\frac{1}{\Lambda_e} \frac{\partial\pi}{\partial\rho} (\varepsilon^l,\rho,\alpha^{l+1}) \right]^\eta \rho^l, 1 \right\} \right\} \tag{12}$$

where η is a nonnegative constant used for numerical efficiency and the Lagrange multiplier Λ_e, which acts as a scaling factor here, is determined by the maximum mass constraint $\int_\Omega \rho^{l+1}d\Omega = V$.

Step 5: Convergence check

if no convergence, then continue with Step 1.

For the stress based formulation given by (9), an analogous algorithm can be constructed.

The numerical realization of the above algorithm is done within the environment of an open, home-made finite-element programme. In the two-dimensional applications, which are presented in this paper, we encountered checkerboard, parasitic, patterns in the results. This phenomenon is well known in the specialized literature [1], [8]. In this paper we do not study topology optimization problems, where the optimal design of a microstructure within the material is sought. Accordingly, we do not use a very fine finite element discretization of the analyzed examples. The checherboard problem has been suppressed by the simple element-to-nodal-density technique, described in ref. [8].

5. APPLICATION TO CELLULAR MATERIALS

Let us consider a cross section of a transversely isotropic material. A plane-strain elasticity problem is considered, where the in-plane constitutive law reads:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} E(\alpha) & -\dfrac{E(\alpha)}{\nu(\alpha)} & 0 \\ -\dfrac{E(\alpha)}{\nu(\alpha)} & E(\alpha) & 0 \\ 0 & 0 & 2(1-\nu(\alpha)) \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{13}$$

where $E(\alpha)$ and $\nu(\alpha)$ denote the transverse elastic modulus and Poisson's ratio of the material, which both depend on the design variables α of the material.

Let us assume that this material is a cellular microstructure. For instance, in classical cellular material theory, [9], [10], the following relation can be used for open-walled cellular solids:

$$E(a) = E_s k_l \left[\frac{\rho}{\rho_s} \right]^2 \tag{14}$$

where $\alpha = \rho/\rho_s$ is the comparative density of the cellular material, ρ is the real density, ρ_s is the density and E_s the elastic modulus of the matrix material and k_1 is a proportionality factor. Poisson's ratio is usually taken equal to zero. This model has been used for the mechanical behaviour of tissues in plants [10]. A more complicated lattice continuum model can be extracted from the Koiter general anisotropic model of [11], which has been used for bone modelling and remodelling studies [12].

The method can be extended to apply for the above outlined model by considering cellular structures with reentrant corner cells, which lead to overall elastic properties with an effective negative Poisson's ratio [4]. Since no analytical formulae exist for this type of microstructures the results of a parametric numerical investigation, by using numerical homogenization techniques are used here (see [4] for more details). However, for comparison, a power law type adaptive material is considered of the form introduced in ref. [14]:

$$E = \alpha^4 E_0,\ 0 \le \alpha \le 1,\ \nu = \text{const.} \tag{15}$$

6. A NUMERICAL EXAMPLE

An optimal (stiff or flexible) structure will be designed by the numerical algorithm presented in this paper. We consider two models for the modification of the internal (composite) material: the composite material with star-shaped inclusions, which has been studied by the numerical homogenization method [4], and the power law attributed to Mlejnek [14], where the elastic constants of the initial material are simply modified by multiplication with the scalar material variable in the power of four (15).

In general, the initial and the final deformed shape of the discretized structure will be given for the graphical evaluation of the procedure. The displacements are scaled with respect to initial plot. The history of the material density, along with the history of the achieved modification of the stiffness, document the iterative procedure in each case. It is assumed in this example that a predefined upper (or lower) bound on the total density is determined, as a function of the material modifications, which is chosen to be the stopping criterion.

Moreover, the stiffness is measured by the square root of the sum of the squares of all diagonal elements of the stiffness matrix of the discretized matrix, which changes in each iteration of the algorithm, depending on distribution of the current design variables. This quantity is scaled by the initial stiffness value. The optimal design is finally demonstrated for the problem. Here the material (design) variable α ranges between zero and unity. For the composite material considered here [4] and for the reference material (15), the variation of the elastic modulus and the Poisson ratio with respect to the variable α are shown in Fig. 1. It is to be understood that, for the composite material, the variable α can be physically explained as the internal angle of the nonconvex, star-shaped microstructure of the composite material [4], while for the material of equation (15) it can be explained as a material density measure. Fig. 2a presents a series of star-shaped void inclusions with re-entrant corners representing unit cells creating negative Poisson's ratios in the cellular material, while Fig. 2b gives the variation of Poisson's ratio of these unit cells versus the angles at the corners at the star-shaped inclusions for the cellular material introduced and studied in ref. [4].

Let us consider now a plate structure, discretized by triangular finite elements, fixed at its left hand-side boundary and loaded by a compressive stress

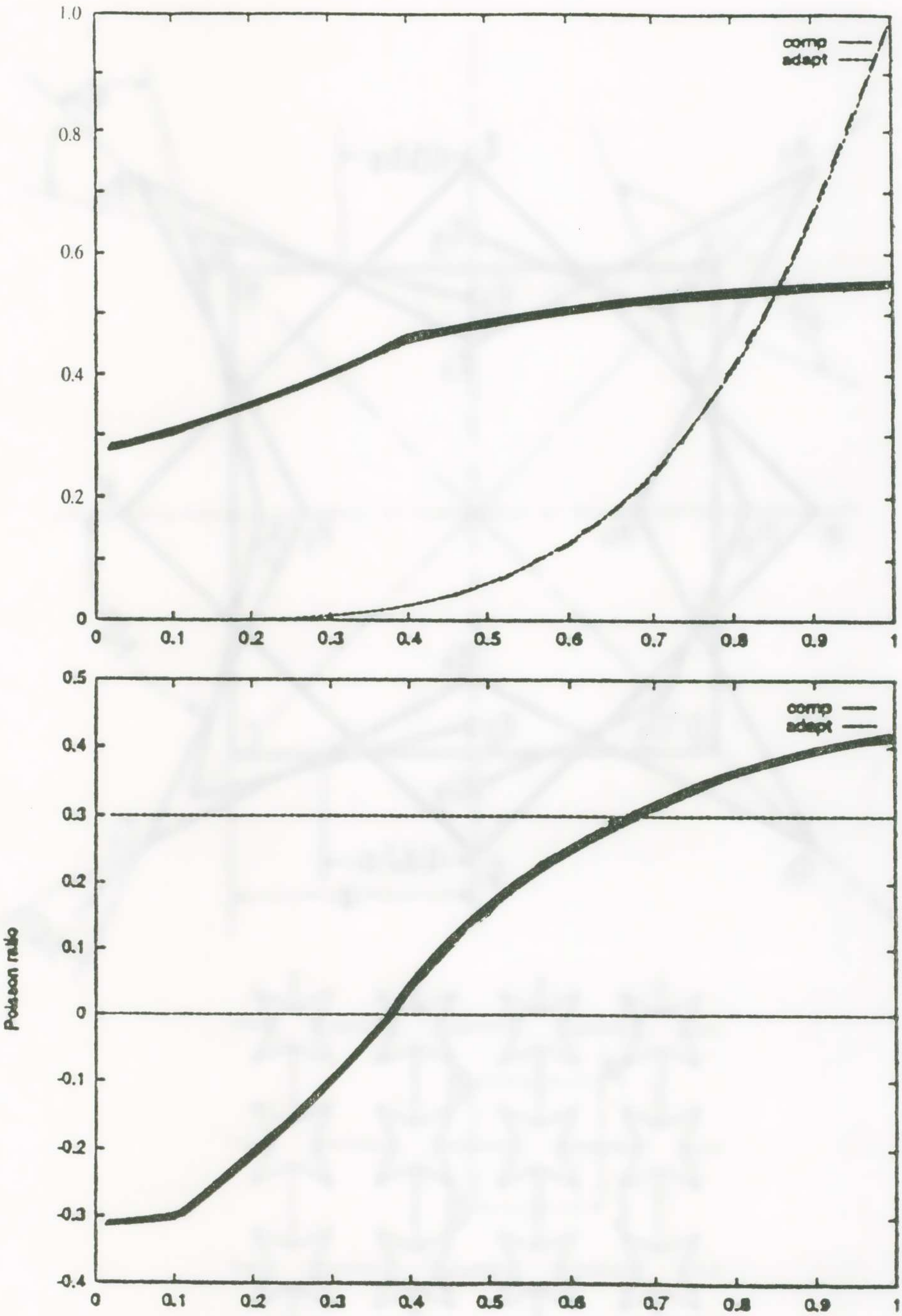
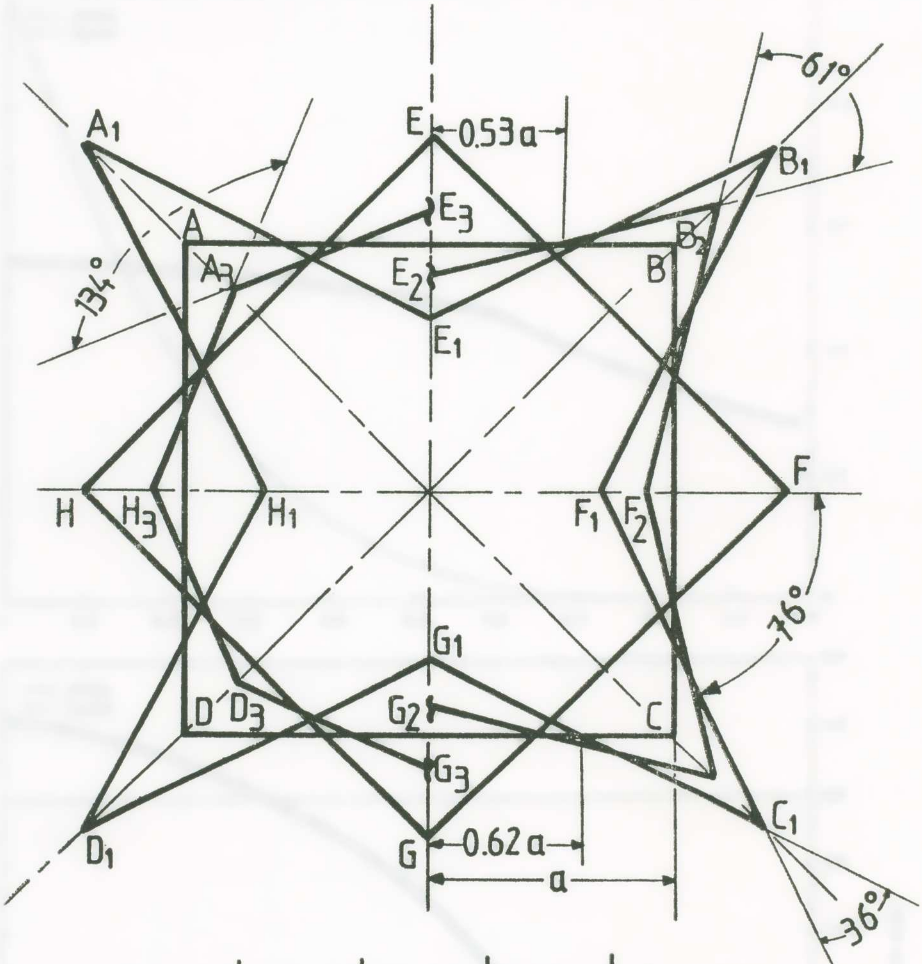


Fig. 1. Variation of the elastic modulus E, and the Poisson ratio, ν, for the two materials used in the investigations (*comp* is the composite material and *adapt* is the adaptive material).



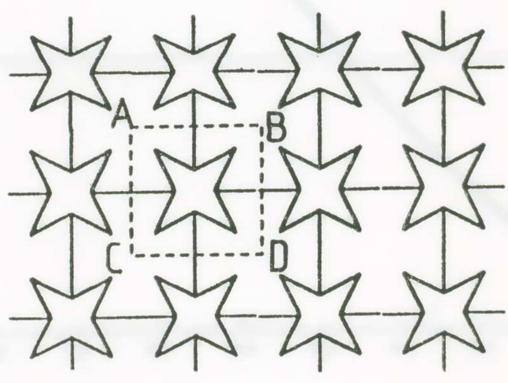


Fig. 2(a). Periodic fiber-reinforced composite with star-shaped inclusions of five different angles of the re-entrant corners.

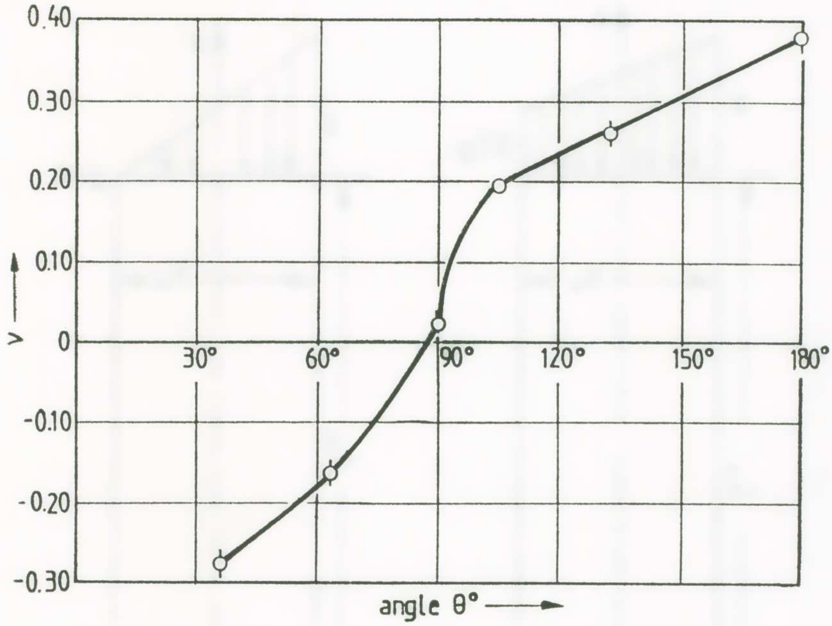


Fig. 2 (b). The variation of Poisson's ratio of the composite versus the angles of the corners of the inclusions.

distribution at the opposite boundary. The whole plate is thus subjected to compression (or tension) and bending, which results from the total boundary loading. This example could be seen as a first attempt towards the modelling of a bone adaption process justifying the bone density variation existing in trabecular and cordical bones. The real problem is, of course, much more complicated than the simple models presented here. Figures 3a, b present the forms and loading situations of the cantilever. This may be considered as an attempt to study the mode of loading and the density arrangement of thin plates sectioned from a human diaphysis sectioned at different orientations lengthwise of a femur bone, as indicated in Fig. 4a. Figure 4b indicates a scanning electron micrograph showing the cellular structure of a section of the femoral head of the condyle of a human femur, indicating clearly the form of the cancellous bone presenting a low-density open-cell structure.

First we consider a compressive loading distribution, linear along the short boundary, with values between p and 0,5 p (Fig. 3a). Let us seek the more flexible structure, by using the composite material distribution. For a

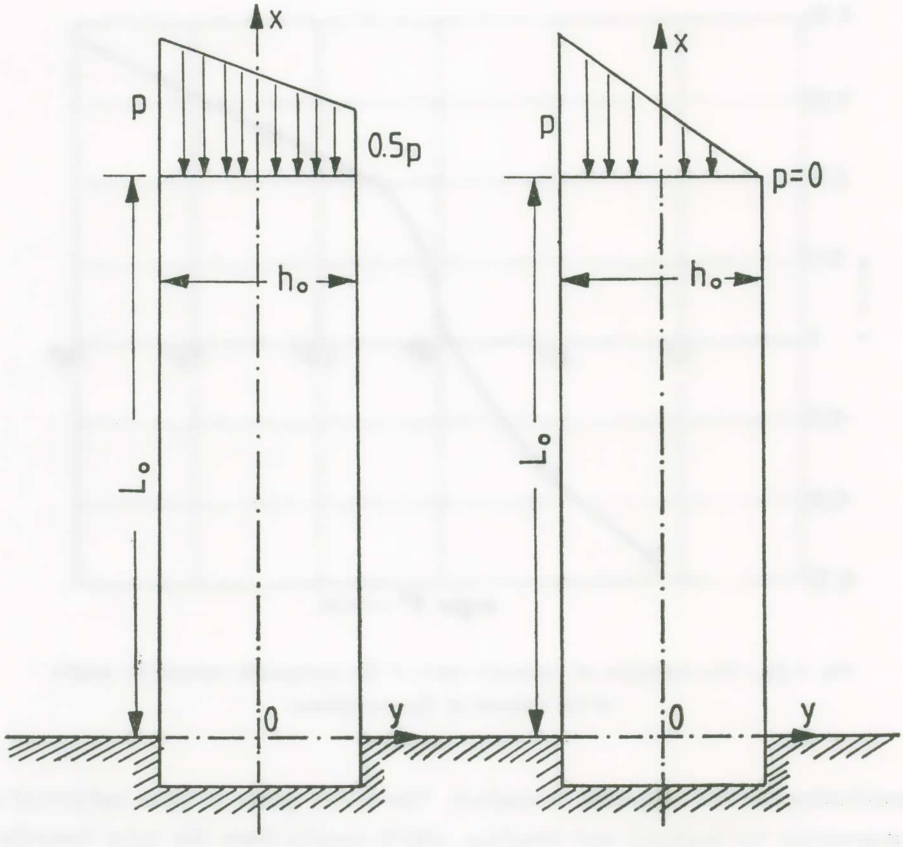
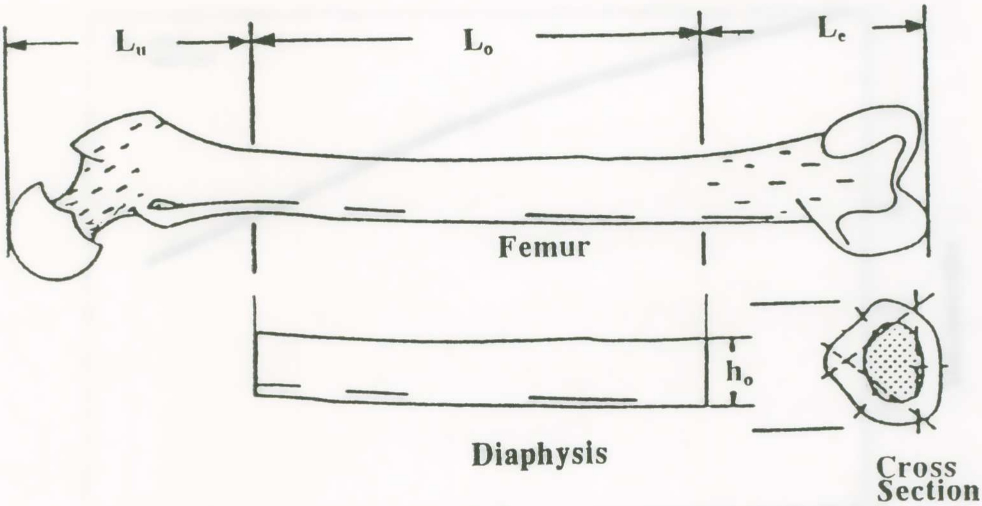


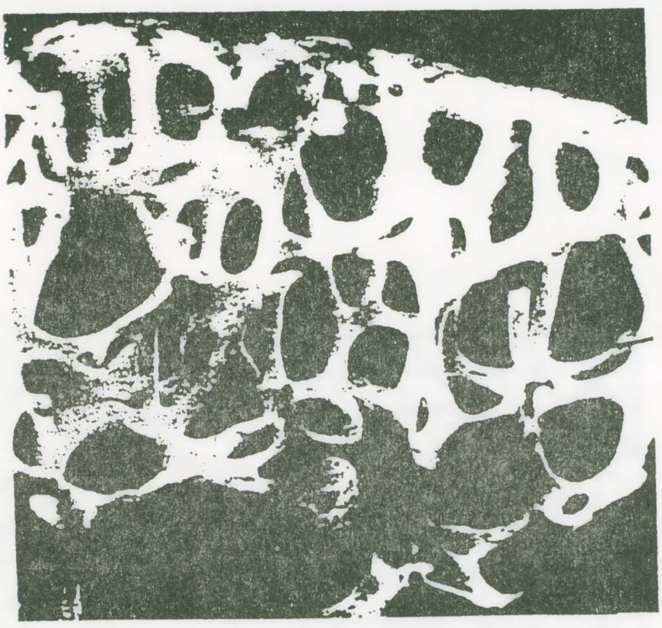
Fig. 3. Two modes of loading of cantilever beams representing sections of the diaphysis of a human felur.

uniform (homogeneous) starting material with $\alpha = 1.0$ and for stopping percentage of total material density equal to 0.50 the numerical procedure and the results are given in Figs. 5(a-d). For the same support and loading configuration let us seek the more stiff structure. Now the starting material design paramater is $\alpha = 0.1$. For stopping percentage of total material density equal to 0.50, the numerical procedure and the results are given in Figs. 6(a-d).

As a next example we consider now the same plate, but loaded along the whole of its transverse boundary (Fig. 3b). This loading case is now considered with a larger applied bending moment. To this end a compressive loading dis-

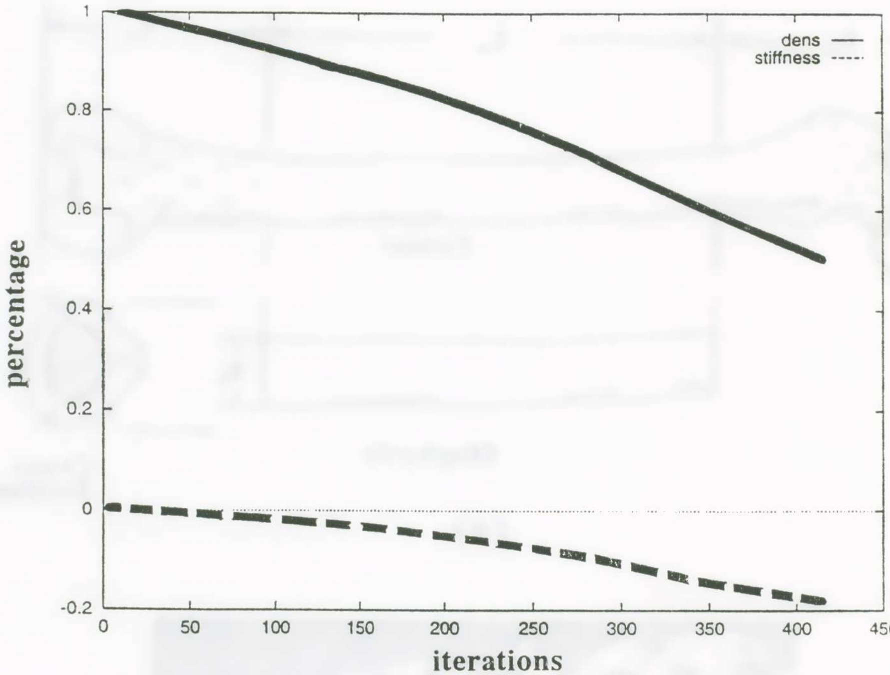


(a)



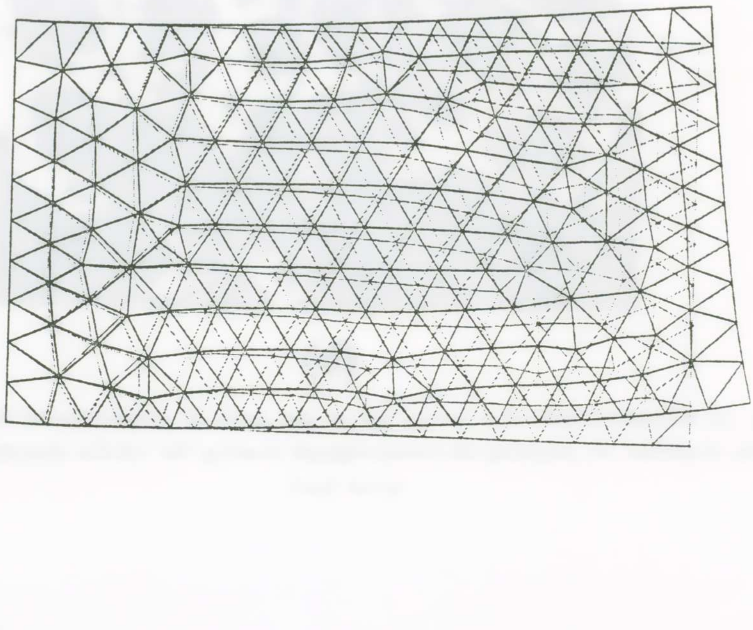
(b)

Fig. 4. (a) Schematic view of a human femur and mode of preparation of the specimens from its diaphysis, (b) scanning electromicrograph showing the cellular structure of the femoral head.

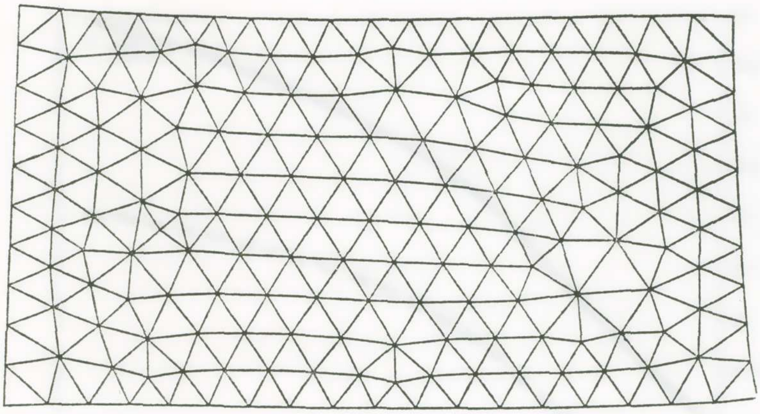


dens
stiffness
1
0.8
0.6
0.4
0.2
0
-0.2
0
50
100
150
200
250
300
350
400
450
percentage
iterations

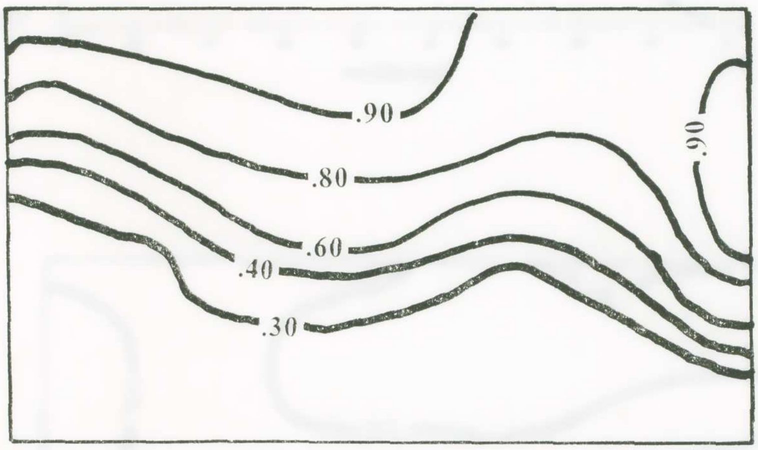
(a)



(b)



(c)

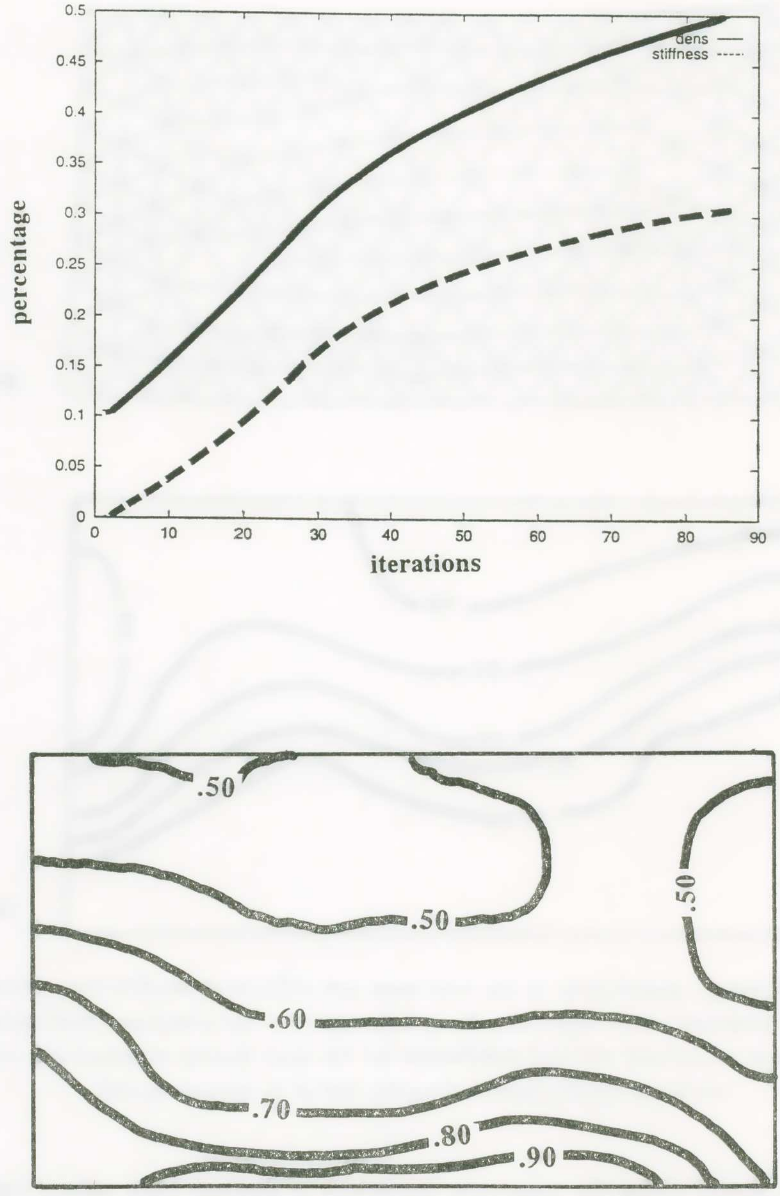


(d)

Fig. 5. Percentage modification of the total mass and stiffness of the structure versus the number of iterations of the algorithm, (b, c) deformation of the initial and final optimized structure and (d) optimal material distribution for the most flexible structure of a cantilever beam linearly loaded along the half of its transverse end.

tribution, linear along the tranverse boundary is applied with values varying between p and 0. Let us seek first the more flexible structure, by using the composite material distribution. For a uniform (homogeneous) starting material with $\alpha = 1.0$ and for stopping percentage of total material density equal to 0.50 the numerical procedure and the results are given in Figs. 7(a-d).

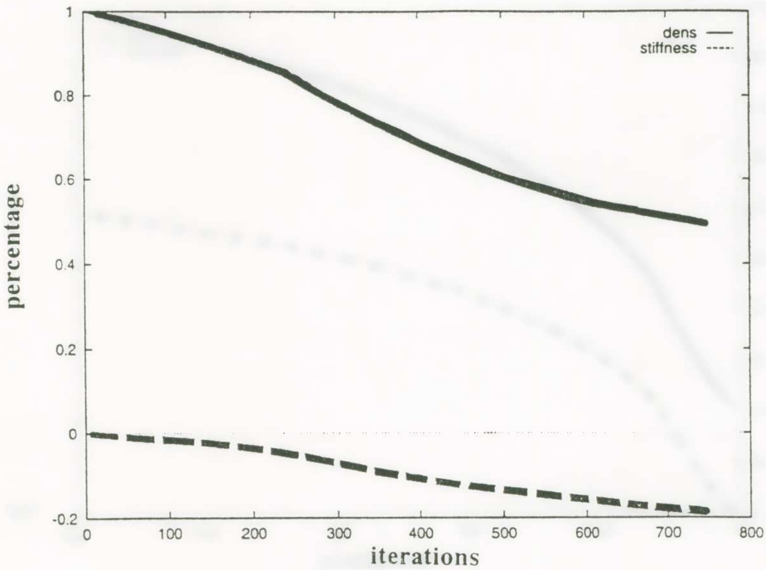
For the same support and loading configuration let us seek the more stiff



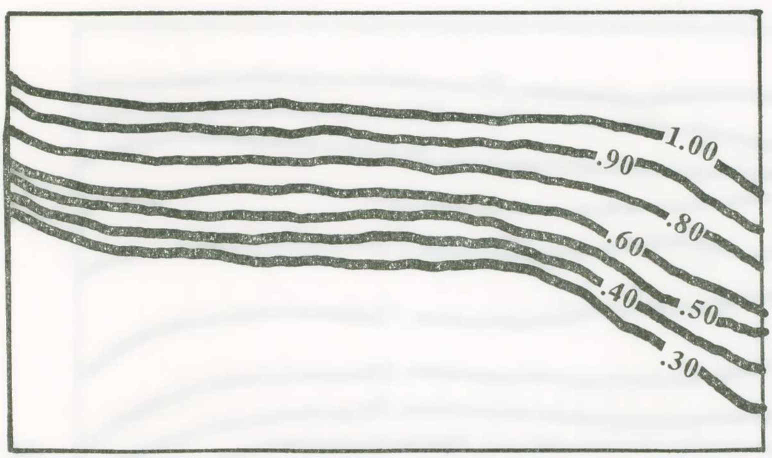
(a)

(b)

Fig. 6. The same data for the same type of loading of the cantilever beam, for the case of most stiff-structure.

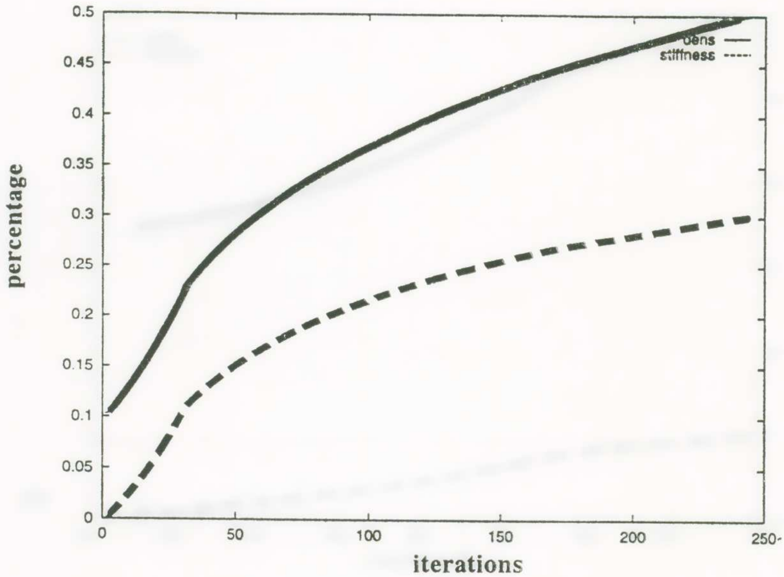


(a)

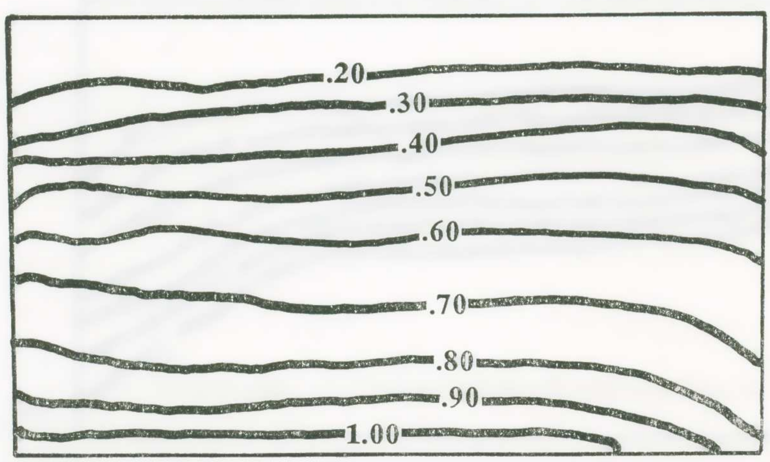


(b)

Fig. 7. The case of a cantilever beam as in Fig. 3b subjected to a linearly varying load along its transverse boundary for the case of the most flexible structure.



(a)



(b)

Fig. 8. The case of a cantilever beam as in Fig. 3b subjected to a linearly varying load along its transverse boundary for the case of the most stiff structure.

structure. Now the starting material design parameter is $\alpha = 0.1$. For stopping percentage of total material density equal to 0.50, the numerical procedure and the results are given in Figs. 8(a-d).

7. CONCLUSIONS

By using results of multilevel distributed optimization and optimal design of structures and results of our previous work on homogenized elastic constants of composite materials exhibiting extreme material values, as it is for instance the negative Poisson's ratio, we have proposed and tested in this paper an algorithm for the optimal structural design by using composite materials.

The use of composites with variable material constants, and especially the ones studied in our previous contribution [4], has been shown to be a feasible alternative for the optimal structural design. It is technologically more possible that materials with variable microstructure of the type used here will be constructed by automated construction machines, than the microstructures proposed by analogous approaches based on numerical homogenization techniques and multilayer materials [1]. The relevance of our approach to problems arising in biomechanics has not been investigated in detail here. However, the link has been mentioned within this study of composite and the foamy materials, which exhibit extremal properties.

Note, that the same materials are used in the investigation of the present paper and that stress- and strain- driven microstructural modification mechanisms (either growth and densification, or microdamage) are believed to be the basic components of bone adaptation strategies. This fact is only pointed out here, without more discussion, since the problems considered in biomechanics are certainly much more complicated than the academic, laboratory-tailored models, which have been adopted here. For instance, it is conjectured that the internal structure of the animal (and human) bones should lead to some kind of optimal structural response, but, for instance, not a stiffer or more flexible structure for one given loading case is an appropriate model. A case where, for example, two loading cases are consideted may be more realistic. Then the stiffer response, which leads to more loading carrying capacity, is sought for the one loading case and for the second loading case the more flexible mechanical response, with enhanced energy absorption characteristics, and thus greater fracture toughness, is considered. Extension of our investigation into the previously mentioned directions are still an open area for further investigations.

ACKNOWLEDGMENTS

The research work contained in this paper has been supported by a research programme No 200/246 allocated by the research Fund of the Academy. The author expresses his thanks for this support. He expresses also his gratitude to Mrs Anna Zografaki for helping him in typing the manuscript and plotting the figures of the paper.

REFERENCES

[1] M. P. Bendsoe, *Optimization of structural topology, shape and material*, Springer 1995.

[2] G. I. N. Rozvany, *Structural design via optimality criteria. The Prager approach to structural optimization*, Kluwer Academic, Dordrecht, 1989.

[3] L. J. Gibson, M. F. Ashby, F. R. S., «The mechanics of two-dimensional cellular materials», *Proc. Royal Soc. London*, A382, (1982), 43-59.

[4] P. S. Theocaris, G. E. Stavroudakis, P. D. Panagiotopoulos, «Negative Poisson's ratio in materials with a star-shaped microstructure. A numerical homogenization approach», *Archive of Applied Mechanics*, (1997) 67(4), 274-286.

[5] G. E. Stavroulakis, H. Günzel, «Optimal structural design in nonsmooth mechanics». *Multilevel Optimization: Algorithms and Applications*, Eds A. Migdalas, P. Pardalos, P. Varbrand, Kluwer Academic, (1997), in press.

[6] C. S. Jog, R. B. Haber, M. P. Bendsoe, «Topology design with optimized, self-adaptive materials», *Computer Meth. in Appl. Mech. and Engng.* 37, (1994) 1323-1350.

[7] P. D. Panagiotopoulos, *Inequality problems in mechanics and applications. Convex and nonconvex energy functions*, Birkhaüser Verlag, Basel-Boston Stuttgart (1985), see also: P. D. Panagiotopoulos, *Hemivariational inequalities. Applications in mechanics and Engineering*, Springer Verlag, Berlin-Heidelberger-New York, (1993).

[8] Sung-Kie Youn, Sang-Houn Park, «A study on the shape extraction process in the structural topology optimization using homogenized material», *Computers and Structures*, 62(3), (1997), 527-538.

[9] M. F. Ashby, «The mechanical properties of cellular solids», *Amer. Inst. Mech. Eng. Metal. Trans.*, A 14 (1983), 1755-1769.

[10] K. J. Niklas, *Plant Biomechanics. An engineering approach to plant form and function*, The University of Chicago Press, Chicago and London, 1992.

[11] W. T. Koiter, «Couple-stresses in the theory of elasticity», *Proc. K. Ned. Akad., Wet.*, Vol. B-67, (1964), 1-17.

[12] P. S. Theocaris, *The mesophase concept in composites*, Springer Verlag, Berlin, Heidelberg, 1987, see also:
P. S. Theocaris, A. G. Varias, «The influence of the mesophase on the transverse and longitudinal moduli and the major Poisson ratio in fibrous composites», *Colloid and Polymer Science*, 264, (1986), 1-9.

[13] M. P. Bendsoe, Kikuchi, N., «Generating optimal topologies in structural design using a homogenization method», *Computer Methods in Applied Mechanics and Engineering*, 71, (1988), 197-224.

[14] H. P. Mlejnek, «Some aspects of the genesis of structures», *Structural Optimization*, 5, (1992), 64-69.

ΠΕΡΙΛΗΨΙΣ

Πολυβάθμιος ὑπολογισμὸς βελτιστοποιήσεως κυψελωτῶν κατασκευῶν προσομοιαζουσῶν πρὸς ἐμβίους δομάς, διὰ τῆς μεθόδου τῆς ἀριθμητικῆς ὁμογενοποιήσεως

Ἡ ἀνακοίνωσις αὐτὴ ἀφορᾶ εἰς τὸ πρόβλημα βελτίστης κατανομῆς τοῦ ὑλικοῦ κατασκευῆς, εἴτε ἐκ συνθέτων ὑλικῶν, εἴτε ἐκ κυψελωτῶν τοιούτων, διὰ χρησιμοποιήσεως τῶν κατανεμομένων πολυβαθμίων τεχνικῶν βελτιστοποιήσεως. Ἡ μέθοδος ἐπιλύσεως αὐτῶν τῶν προβλημάτων ἔχει τύχει τελευταίως μεγάλου ἐνδιαφέροντος, λόγῳ τῆς μεγάλης προόδου τῶν μεθόδων τοπολογικῆς βελτιστοποιήσεως βασιζομένων εἰς τεχνικὰς ἀριθμητικῆς ὁμογενοποιήσεως. Θεωρώντας τὴν διαδικασίαν αὐτὴν λύσεως ὡς αὐτόματον, καθοδηγουμένην ὑπὸ τῆς κατασκευῆς, ὥστε αὐτὴ νὰ ἀποτελῆ τὴν βελτίστην διαμόρφωσίν της, ἐπιτυγχάνεται ἡ καλυτέρα προσαρμογὴ τῶν στοιχείων τῆς κατασκευῆς διὰ τὴν βελτίστην συμπεριφοράν της εἰς ἐξωτερικὰς καταπονήσεις, καὶ δι' αὐτοῦ τοῦ τρόπου εἶναι δυνατὸν νὰ θεωρηθῆ ὅτι ἀνθρώπιναι κατασκευαὶ ἀρχίζουν νὰ προσομοιάζουν πρὸς ἀντιστοίχους κατασκευὰς τῆς φύσεως, ἰδίως πρὸς αὐτὰς τῶν ἐμβίων ὄντων.

Τὸ πρόβλημα τῆς βελτίστης κατανομῆς τοῦ ὑλικοῦ εἰς τὴν κατασκευὴν διατυποῦται εἰς τὴν γενικήν του μορφήν, βασιζομένην ἐπὶ τῶν τελευταίων ἐξελίξεων βελτιστοποιήσεως τοῦ σχήματος καὶ τῆς τοπολογίας τῆς ὑπ' ὄψιν κατασκευῆς, τῆ βοηθείᾳ ἀριθμητικῆς ὁμογενοποιήσεως. Οὕτω ἡ ἀνθεκτικωτέρα κατασκευὴ ἐπιτυγχάνεται διὰ μεγιστοποιήσεως τῆς δυναμικῆς ἐνεργείας τῆς κατασκευῆς ἐν ἰσορροπίᾳ, ἢ ἰσοδυνάμεως, δι' ἐλαχιστοποιήσεως τῆς συμπληρωματικῆς ἐνεργείας της. Ἡ κατάλληλος ἐπιλογὴ τοῦ ὑλικοῦ εἰς ἕκαστον σημεῖον τῆς κατασκευῆς καθορίζεται ἀπὸ τὰς μεταβλητὰς ὑπολογισμοῦ ὡς αὐταὶ προκύπτουν ἐκ τῆς βελτίστης λύσεως τοῦ προβλήματος.

Ἐκ τῆς διαμορφώσεως τοῦ προβλήματος καὶ δεδομένου ὅτι ἡ μορφὴ τῆς κατασκευῆς ἐν ἰσορροπίᾳ ἐπιβάλλει τὴν δυναμικὴν ἐνέργειαν νὰ γίνεται ἐλαχίστη, συνάγεται ὅτι ἡ μέθοδος τῶν πολυβαθμίων τεχνικῶν εἶναι ἡ καταλληλοτέρα μέθοδος διὰ τὴν ἐπίλυσιν τοῦ προβλήματος. Τοιουτοτρόπως, ὁ πολυβάθμιος διαχωρισμὸς τοῦ προβλήματος εἰς δύο ὑπο-προβλήματα εἶναι δυνατός, ὅπου τὸ πρῶτον ὑπο-πρόβλημα εἶναι τὸ πρόβλημα τῆς δομικῆς ἀναλύσεως τῆς κατασκευῆς, καὶ τὸ δεύτερον ὑποπρόβλημα ἀσχολεῖται μὲ τὴν εὕρεσιν καὶ τὸν καθορισμὸν τῆς προσαρμογῆς τοῦ καταλλήλου ὑλικοῦ διὰ τὴν κατασκευήν. Τὸ δεύτερον τοῦτο πρόβλημα ἐπιλύεται τοπικῶς ἀπὸ σημείου εἰς σημεῖον τῆς κατασκευῆς διὰ διακεκριμένων προβλημάτων μικρᾶς κλίμακος προοδευτικῶς μεταβαλλομένων.

Ἡ μέθοδος ἐφηρμόσθη εἰς διάφορα ὑλικά, πορώδη, κυψελωτὰ καὶ σύνθετα ὑλικὰ εἰς τυπικὰς κατασκευὰς παραδειγμάτων. Θεωροῦμεν πρὸς τοῦτο, πρόβλημα μεταβαλλομένης παραμέτρου, παρουσιάζοντος ἐλάχιστον ὅριον **τανυστοῦ ἐνδόσεως**, περιγραφομένου ἀπὸ χαρακτηριστικὸν ἄνυσμα μεταβαλλομένης ἐντάσεως ἀπὸ θέσεως εἰς θέσιν καὶ ἐξαρτωμένου ἀπὸ τὰς χαρακτηριστικὰς μεταβλητὰς τῆς κατασκευῆς, αἱ ὁποῖαι καθορίζουν τὴν μικροδομὴν τοῦ ὑλικοῦ της.

Τὸ πρόβλημα ἐλαχιστοποιήσεως τῆς ἐνδόσεως τοῦ ὑλικοῦ, συναρτήσει ἐκφράσεων βασιζομένων εἰς τὰς παραμορφώσεις, ἐκφράζεται συνοπτικῶς ὡς:

$$\max_{[\rho(x),\alpha(x)]\in A_{ad},x\in\Omega}\ \min_{u\in U_{ad}} \left\{ \overline{\Pi}(u,\rho,\alpha) = \Pi\,(e(u),\rho,\alpha)-l(u) \right\} \tag{1}$$

ὅπου A_{ad} δίδει τὰς ἐπιτρεπομένας τιμὰς τῶν μεταβλητῶν ὑπολογισμοῦ, αἱ ὁποῖαι, ὑπὸ τὴν προϋπόθεσιν ὅτι ζητεῖται κατασκευὴ μὲ τὴν μεγίστην μᾶζαν V, ἐκφράζονται ὡς ἡ σχέσις:

$$A_{ad} = \left\{ \rho(x),\ \alpha(x),\ x\in\Omega \text{ τοιοῦτον ὥστε } \int_{\Omega}\rho d\Omega \le V,\ 0\le\rho\le 1 \right\} \tag{2}$$

Ἡ συνολικὴ δυναμικὴ ἐνέργεια $\Pi(u,\rho,\alpha)$ συντίθεται ἀπὸ τὴν πυκνότητα ἐνεργείας $\Pi(e(u),\rho,\alpha)$, καὶ τὸν ὅρον $l(u)$, τὴν δυναμικὴν ἐνέργειαν ἐκ τῆς φορτίσεως, ἐνῶ ἡ U_{ad} δίδεται ἀπὸ τὴν φύσιν τοῦ ἐξεταζομένου προβλήματος.

Διὰ γραμμικὰς ἐλαστικὰς κατασκευὰς καὶ μικρὰς μετατοπίσεις ἡ σχέσις (1) γράφεται:

$$\max_{[\rho(x),\alpha(x)]\in A_{ad}} \min_{u\in U_{ad}} \left\{\frac{1}{2}\int_\Omega E_{ijkl}(x,\rho(x),\alpha(x))\,\varepsilon_{ij}(u)\,\varepsilon_{kl}(u)\,d\Omega - l(u)\right\} \quad (3)$$

Ἀντιστοίχως τὸ πρόβλημα τῶν τάσεων λαμβάνει τὴν μορφήν:

$$\min_{[\rho(x),\alpha(x)]\in A_{ad}} \min_{\sigma\in\Sigma_{ad}} \left\{\frac{1}{2}\int_\Omega C_{ijkl}(x,\rho(x),\alpha(x))\,\sigma_{ij}\,\sigma_{kl}\,d\Omega\right\} \quad (4)$$

ὅπου Σ_{ad} εἶναι τὸ σύνολον τῶν ἐπιτρεπομένων τάσεων καὶ ἰσχύει, ἐπὶ πλέον, ἡ συνθήκη ἰσορροπίας:

$$\mathrm{div}\boldsymbol{\sigma} + p = 0 \quad (5)$$

Διὰ τὴν ἐπίλυσιν τοῦ προβλήματος ὑποθέτομεν ὅτι ἡ πυκνότης τῆς κατασκευῆς ἐξαρτᾶται ἀπὸ τὰς τοπικὰς τιμὰς τῶν συνθηκῶν τῆς κατασκευῆς, ἤτοι $\rho(x) = f[\alpha(x)]$.

Διὰ τὴν ἐφαρμογὴν τοῦ προβλήματος, τῇ βοηθείᾳ τῶν τοπικῶν τιμῶν τῶν συναρτήσεων αὐτῶν ἀποσυνδέομεν τὸ πρόβλημα, λαμβάνοντες διὰ τὴν σχέσιν (3) τὴν ἀκόλουθον σχέσιν:

$$\max_{\rho(x)\in A_1} \max_{\alpha(x)\in A_2(\rho)} \min_{u\in U_{ad}} \left\{\frac{1}{2}\int_\Omega E_{ijkl}(x,\rho(x),\alpha(x))\,\varepsilon_{ij}(u)\,\varepsilon_{kl}(u)\,d\Omega - l(u)\right\} \quad (6)$$

καὶ διὰ τὴν σχέσιν (5) τὴν ἀντίστοιχον:

$$\min_{\rho(x)\in A_1} \min_{\alpha(x)\in A_2(\rho)} \min_{\sigma\in\Sigma_{ad}} \left\{\frac{1}{2}\int_\Omega C_{ijkl}(x,\rho(x),\alpha(x))\,\sigma_{ij}\,\sigma_{kl}\,d\Omega\right\} \quad (7)$$

μὲ τὰς καταλλήλους τιμὰς διὰ τὰς μεταβλητὰς A_1 καὶ A_2 ἀντιστοίχως.

Ἀλλάζοντας τὴν διαδοχὴν τοῦ δευτέρου καὶ τρίτου τελεστοῦ εἰς τὰς σχέσεις (6) καὶ (7) ἔχομεν τὰς ἀκολούθους ὁριστικὰς σχέσεις:

$$\max_{\rho(x)\in A_1} \min_{u\in U_{ad}} \left\{\int_\Omega \left\{ \max_{\alpha(x)\,\mathrm{S.t.}\,\rho(x)=f(\alpha(x))} \frac{1}{2} E_{ijkl}(x,\rho(x),\alpha(x))\,\varepsilon_{ij}(u)\,\varepsilon_{kl}(u)\,d\Omega\right\} - l(u)\right\} \quad (8)$$

$$\min_{\rho(x)\in A_1} \min_{\sigma\in\Sigma_{ad}} \left\{\int_\Omega \left\{ \min_{\alpha(x)\,\mathrm{S.t.}\,\rho(x)=f(\alpha(x))} \frac{1}{2} C_{ijkl}(x,\rho(x),\alpha(x))\,\sigma_{ij}\,\sigma_{kl}\,d\Omega\right\}\right\} \quad (9)$$

Ἐκ τῆς τελευταίας διαμορφώσεως τοῦ προβλήματος βελτίστης λύσεως συνάγεται ἡ ἄμεσος ἱεραρχικὴ ἐπαναληπτικὴ στρατηγικὴ λύσεώς του. Ἡ ἐσωτερικὴ μεγιστοποίησις ἢ ἀντιστοίχως ἐλαχιστοποίησις τῶν συναρτήσεων εἶναι δυνατὸν νὰ λυθοῦν ὑπὸ τὴν μορφὴν τῶν ἀποσυνδεδεμένων στοιχειωδῶν περιοχῶν. Ἡ δευτέρα βαθμὶς τοῦ προβλήματος ἐλαχιστοποιήσεως ἐπιλύεται εἰς τὴν στάθμην τῆς κατασκευῆς, τῇ βοηθείᾳ στοιχειωδῶν πεπερασμένων κωδικῶν γενικῆς χρήσεως. Ἀμφότερα τὰ ἀνωτέρω ὑποπροβλήματα ἐπιλύονται διὰ δοθεῖσαν διανομὴν πυκνότητος $\rho(x)$, ἡ ὁποία προσαρμόζεται καταλλήλως διὰ τῆς ἐπιλύσεως τοῦ ἑκάστοτε ὑποπροβλήματος πρώτης βαθμίδος.

Διὰ τὴν περίπτωσιν τῶν ἐπιλυομένων προβλημάτων εἰς τὴν ἐργασίαν αὐτὴν γίνεται δεκτὸν ὅτι ἡ μεταβολὴ τοῦ τανυστοῦ $E_{ijkl}\,(a/x)$ δὲν εἶναι ἀναλυτικῶς γνωστή. Διὰ τὴν πλήρη ἐπίλυσιν τοῦ προβλήματος καθορίζεται μοναδιαία ἀντιπροσωπευτικὴ κυψελὶς τῆς ὁποίας εἶναι γνωσταὶ αἱ μηχανικαὶ ἰδιότητες, καὶ τὸ πρόβλημα ἐπιλύεται διὰ παρεμβολῆς ἐνδιαμέσων τιμῶν μεταβολῶν τῶν ἀντιστοίχων παραμέτρων τῇ βοηθείᾳ πινακοποιημένων μεταβολῶν. Ἐν συνεχείᾳ, τὰ δευτεροβάθμια προκύπτοντα ὑποπροβλήματα ἐκφράζονται ὡς μὴ γραμμικαὶ καὶ πιθανὸν μὴ διαφορίσημοι τιμαί, αἱ ὁποῖαι προκύπτουν ἐκ τῆς ἀντιστοίχου γραμμικοποιήσεως τῶν σχετικῶν περιπτώσεων, εἰς τρόπον ὥστε νὰ προσδιορίζωνται αἱ ἑκάστοτε τιμαὶ τοῦ τανυστοῦ E_{ijkl} ἢ τοῦ C_{ijkl}, ὡς αὐταὶ προκύπτουν ἀπὸ τὴν ἑκάστοτε ἐπίλυσιν τοῦ πρωτοβαθμίου τοπικοῦ προβλήματος.

Ἡ ἔκφρασις τοῦ ἀλγορίθμου τῆς πολυβαθμιδωτῆς λύσεως ἀναπτύσσεται λεπτομερῶς εἰς τὴν ἐργασίαν. Δι' ἑκάστην βαθμίδα ἐπαναλήψεως ὁρίζεται στοιχειώδης μικρὰ ποσότης τῆς συναρτήσεως ἐλέγχου, ἡ ὁποία ὁρίζει τὸ πέρας τῶν ἐπαναλήψεων καὶ τὴν σύγκλισιν τῆς λύσεως πρὸς τὴν ἀληθῆ τιμήν της.

Ἡ ἐφαρμογὴ τῆς μεθόδου γίνεται εἰς κυψελιδωτὰς κατασκευάς, αἱ ὁποῖαι προσομοιάζουν μὲ τὰς δομὰς τῶν ἐμβίων ὄντων εἰς τὴν φύσιν. Δι' ἐγκαρσίως ἰσότροπον ὑλικόν, ὑπὸ συνθήκας ἐπιπέδου παραμορφώσεως, ἡ καταστατικὴ σχέσις μεταξὺ τῶν τάσεων καὶ παραμορφώσεων δίδεται ὡς ἀκολούθως:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} E(\alpha) & -\dfrac{E(\alpha)}{\nu(\alpha)} & 0 \\ -\dfrac{E(\alpha)}{(\nu(\alpha)} & E(\alpha) & 0 \\ 0 & 0 & 2(1-\nu(\alpha)) \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{10}$$

Περαιτέρω δεχόμεθα ὅτι τὸ ὑλικὸν ἔχει κυψελιδωτὴν μικροκατασκευήν, ἐκφραζομένην ὑπὸ τῆς σχέσεως:

$$E(a) = E_s k_1 \left\{ \frac{\rho}{\rho_s} \right\}^2 \quad (11)$$

ὅπου $a = \rho / \rho_s$ ἐκφράζει τὴν σχετικὴν πυκνότητα τοῦ ὑλικοῦ.

Διὰ κυψελιδωτὰ ὑλικὰ μὲ φυσαλίδας μορφῆς εἰσδυουσῶν γωνιῶν, τὰ ὁποῖα δίδουν σύνθετα ὑλικὰ μὲ ἀρνητικοὺς λόγους Poisson, ἡ προηγουμένη σχέσις (11) δίδεται ὑπὸ τὴν προσεγγιστικὴν μορφήν:

$$E = a^4 E_0 \quad 0 \leq a \leq 1.0 \quad \nu = \sigma\tau\alpha\theta. \quad (12)$$

Τὰ ὑλικὰ αὐτὰ εἶναι λίαν ἐνδιαφέροντα διότι ἀντιστοιχοῦν εἰς κατασκευὰς μὲ ἀρνητικὸν λόγον τοῦ Poisson, ὁ ὁποῖος ἀπαντᾶται γενικῶς εἰς τὴν ὕλην τῶν ἐμβίων ὄντων.

Ὡς παράδειγμα ἐφαρμογῆς τῆς μεθόδου θὰ ἐπιλυθῇ στοιχειῶδες πρόβλημα βελτίστης δυσκάμπτου ἢ εὐκάμπτου κατασκευῆς, τῇ βοηθείᾳ τοῦ ἀλγορίθμου τοῦ εἰσαχθέντος εἰς τὸ ἄρθρον. Δύο χαρακτηριστικαὶ περιπτώσεις προτύπων ὑλικῶν θεωροῦνται διὰ τὴν διαμόρφωσιν τοῦ καταλλήλου ὑλικοῦ, ἤτοι, εἴτε σύνθετα ὑλικὰ μὲ ἀστεροειδεῖς μορφὰς κενῶν μὲ εἰσεχούσας γωνίας, τὰ ὁποῖα εἶναι εἰδικὰ ὑλικὰ παρουσιάζοντα ἀρνητικοὺς λόγους Poisson, εἴτε ὑλικὰ ὑπακούοντα εἰς τὴν σχέσιν (12). Εἰς τὰς περιπτώσεις αὐτὰς τὰ ἀρχικὰ καὶ τελικὰ παραμορφωμένα σχήματα τῶν προβλημάτων τῶν διακεκριμένων κατασκευῶν δίδονται τῇ βοηθείᾳ γραφικῶν διαδικασιῶν.

Αἱ μετατοπίσεις κλιμακοῦνται σχετικῶς πρὸς τὸ ἀρχικὸν σχῆμα, ἐνῶ τὸ ἱστορικὸν τῆς μεταβολῆς τῆς πυκνότητος τῆς κατασκευῆς μετὰ τοῦ ἱστορικοῦ τῆς μεταβολῆς τῆς δυσκαμψίας καθορίζει τὴν ἐπαναληπτικὴν διαδικασίαν δι' ἑκάστην περίπτωσιν. Εἰς τὰ παραδείγματα αὐτὰ ἁπλαῖ δοκοὶ εἰς ἐπίπεδον τάσιν φορτίζονται ἐπί τινος τῶν συνόρων των καὶ καθορίζονται ἐκ τῶν προτέρων τὰ ἄνω ἢ κάτω ὅρια μεταβολῆς τῶν χαρακτηριστικῶν των δι' ἑκάστην περίπτωσιν, ὡς συναρτήσεις τῶν ὁρίων μεταβολῶν τῶν ἰδιοτήτων τῶν ὑλικῶν.