ΔΙΑΦΟΡΙΚΗ ΓΕΩΜΕΤΡΙΑ.— **On the geodesic components of a C^k positive metric,** *by P. A. Bozonis**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ὀ. Πυλαρινοῦ.

1. Introduction. A *positive* (definite) *metric* or *Riemannian metric* on a C^k manifold M is a covariant tensor field g of degree 2 which satisfies

(1) $g(X, X) \geqq 0$ for all vector fields $X \in TM$ where TM is the tangent bundle of M and $g(X, X) = 0$ if and only if $X = 0$ and

(2) $$g(X, Y) = g(Y, X) \quad \text{for all} \quad X, Y \in TM.$$

In other words, g assigns an *inner product* g_x in each tangent space $T_x(M)$, $x \in M$ [2]. This inner product (hence the metric) is required to be of class C^k in the sense that if X and Y are C^k vector fields on M, then $g(X, Y)$ is a real-valued function of class C^k on M.

In terms of a local coordinate system $x^1, x^2, \ldots, x^n$ the components of g are given by

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

Because of (1) and (2), the matrix

$$A_g = \begin{Vmatrix} g_{11}, & \cdot & \cdot & \cdot, g_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ g_{n1}, & \cdot & \cdot & \cdot, g_{nn} \end{Vmatrix}$$

is symmetric and positive definite in the usual sense of linear algebra.

Because of the bilinearity of the C^k positive metric g, the matrix A_g completely determines g.

Actually, let $r: U \to M$ be a parametrization of a surface $M \subset E^3$ and U an open subset of E^2. The well-known functions

$$g_{11} = r_{u^1} \cdot r_{u^1}, \quad g_{12} = r_{u^1} \cdot r_{u^2}, \quad g_{22} = r_{u^2} \cdot r_{u^2},$$

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completely determine the dot product of tangent vectors at points of M, for if

$$v = v^1 r_{u^1} + v^2 r_{u^2}, \qquad w = w^1 r_{u^1} + w^2 r_{u^2}$$

then

$$v \cdot w = g_{11} v^1 w^1 + g_{12}(v^1 w^2 + v^2 w^1) + g_{22} v^2 w^2.$$

Let now g_{ij} and $\bar{g}_{ij}$ be two C^k symmetric positive definite covariant tensors, we say that they are equivalent if there is a C^k diffeomorphism $h : U \to \bar{U}$ which transforms the metric g_{ij} to $\bar{g}_{ij}$.

So we can define the C^k positive metric as an equivalence class of pairs (U, g_{ij}) under the equivalence defined above.

In the sequel we shall identify a metric g with its components g_{ij} in some local coordinate system.

It is an important question to find an appropriate diffeomorphism simplifying the original metric g_{ij}. Perhaps, the most interesting case is the form which the metric of a geodesic parametrization of a surface in E^3 has (which we call a *geodesic metric*).

Since the determining of a realization in E^3 of a given C^k positive metric (which is not completely solved) is presupposed for the assigning of the above suitable diffeomorphism, this problem is yet interesting.

The purpose of this paper is to find a C^k diffeomorphism $h : U \to \bar{U}$ which simplifies the given C^k positive metric g_{ij} to the geodesic metric $\bar{g}_{ij}$:

$$\bar{g}_{11} = 1, \qquad \bar{g}_{12} = 0, \qquad \bar{g}_{22} = \bar{g}_{22}(\bar{u}^1, \bar{u}^2).$$

2. Definitions. Here the notions of the *chart*, C^r *atlas*, C^r *structure*, C^r - n - *manifold*, C^k *Riemannian* C^r - n - *manifold*, *covariant derivative* (∇), *angle function* and C^r *realization* have the usual meaning of the corresponding terms of the Differential Geometry [**2**], [**4**].

Definition 2.1. Let M be a C^r - m - manifold and N a C^r - n - manifold and Δ and Δ' their C^r structures, respectively. Suppose for each $x \in M$ there is a chart $(U, \varphi) \in \Delta'$ with $x \in U$ such that $(U \cap M, \varphi) \in \Delta$, then we say that M is a *submanifold* of N of class C^r.

Definition 2.2. A *patch* $r : U \to E^n$ is an injective map of an open set $U \subset E^m$ into E^n $(m < n)$ whose derivative map f_* is also injective.

If f is not injective, then it is called a *parametrization* of $r(U)$.

Definition 2.3. A *frame-field* on an arbitrary C^k Riemannian C^r - 2 - manifold M consists of 2 mutually orthogonal unit vector fields E_1, E_2 defined on some open set of M.

The *associated frame field* E_1, E_2 of an orthogonal patch $r: U \rightarrow M$ consists of the orthogonal unit vector fields E_1 and E_2 whose values at each point $r(u^1, u^2)$ of $r(U)$ are

$$\frac{r_{u^1}(u^1, u^2)}{\sqrt{g_{11}(u^1, u^2)}}, \qquad \frac{r_{u^2}(u^1, u^2)}{\sqrt{g_{22}(u^1, u^2)}}.$$

3. The geodesic metric. Let M be an arbitrary 2 - dimensional differentiable manifold furnished with the given C^k positive metric g_{ij} $(3 \leqq k \leqq \infty)$, then by J. Nash there exists a C^k isometric imbedding $f: M \rightarrow E^{51}$ of M into 51 - dimensional Euclidean space **[3]**.

Since M is compact and f is injective and has rank 2, it follows that f(M) is a submanifold of E^{51} **[6]**. Consequently, for each chart (U, φ) of the C^k Riemannian C^r - 2 - manifold M, the map $f \circ \varphi^{-1} = r: U \rightarrow E^{51}$ is a patch which realizes the given metric (we chose the chart such thatf $\varphi(U) = Z$ is the domain of the given metric g_{ij}).

Let us consider an arbitrary curve α passing through the point $p(u^1, u^2)$ on the range of $f|U$ and the geodesics intersecting this curve orthogonally. It follows from the theory of differential equations (**[5]**, p. 157) that the above geodesic consist of a family of curves which in a neighborhood of the point $p(u^1, u^2)$ are given by means of the equation

$$\varphi(u^1, u^2) = c, \qquad [(\varphi_{u^1})^2 + (\varphi_{u^2})^2 \neq 0].$$

It is also known that a second family of curves exists which is orthogonal to the first. As a consequence of these propositions we conclude that a neighborhood W of each $p \in f(U)$ can be always parametrized in such a way that the parameter curves $u^2 = c$ are the geodesics and the orthogonal trajectories of these geodesics are the curves $u^1 = c'$.

Our goal now is to prove that the geodesic curvature k_g of parameter curves $u^2 = c$ in the neighborhood W depends on the first fundamental form of $f(V) = W$.

Let β be a *unit-speed* curve whose range is contained in the open set W oriented by a frame field E_1, E_2 and φ is an angle function from E_1 to β' along β, then

$$k_g = \frac{d\varphi}{ds} + \omega_{12}(\beta'), \tag{1}$$

where k_g is the geodesic curvature of β ([4], p. 329) and ω_{12} is the 1-form

$$\omega_{12}(X) = \nabla_X E_1 . E_2 . \tag{2}$$

For associated frame field and along the parameter curve $\gamma : u^2 = c$ we obtain

$$\omega_{12}(r_{u^1}) = \left(\frac{r_{u^1}}{\sqrt{g_{11}}}\right)_{u^1} \cdot \frac{r_{u^2}}{\sqrt{g_{22}}} \tag{3}$$

since covariant derivatives along the parameter curves reduce to partial derivatives.

Assuming that $\gamma : u^2 = c$ has an arc-length parametrization, we have

$$\omega_{12}(r_s) = r_{ss} \cdot \frac{r_{u^2}}{\sqrt{g_{22}}} \tag{4}$$

hence

$$\begin{aligned}(kg)_{u^2=c} &= \frac{1}{\sqrt{g_{22}}}(r_{u^1u^1}\dot{u}^1\dot{u}^1 + r_{u^1}\ddot{u}^1) . r_{u^2} \\ &= \frac{1}{\sqrt{g_{22}}}(\Gamma^{\varepsilon}_{11} r_{u^\varepsilon} + b^{\gamma}_{11} n_\gamma)\dot{u}^1\dot{u}^1 r_{u^2} \quad \begin{pmatrix}\varepsilon = 1, 2 \\ \gamma = 1,2\ldots, 49\end{pmatrix} \\ &= \frac{1}{\sqrt{g_{22}}} \Gamma^2_{11}\dot{u}^1\dot{u}^1 r_{u^2} . r_{u^2} \qquad (5) \\ &= \frac{\sqrt{g_{22}}}{g_{11}} \Gamma^2_{11}\end{aligned}$$

since $\dot{u}^1 = \frac{1}{\sqrt{g_{11}}}$, $\varphi = 0$ and the formulae of Gauss are also valid as in the case of E^3.

In the particular case of an orthogonal coordinate system, we have

$$\Gamma^2_{11} = -\frac{1}{2g_{22}}(g_{11})_{u^2}. \tag{6}$$

Relation (5) in view of (6) can be written

$$(kg)_{u^2=c} = -\frac{1}{2g_{11}\sqrt{g_{22}}}(g_{11})_{u^2} . \tag{7}$$

But it is known that a *regular* curve (that is, a curve whose all velocity vectors are different from zero) in a 2-dimensional Riemannian manifold is a geodesic if and only if it has constant speed and geodesic curvature $k_g = 0$. Since γ is a unit-speed curve and geodesic, it follows that

$$(g_{11})_{u^2} = 0$$

that is, g_{11} depends only on u^1

$$g_{11} = g_{11}(u^1).$$

And if we introduce a new parameter

$$\bar{u}^1 = \int_0^{u^1} \sqrt{g_{11}}\, du^1$$

the ds^2 asumes the form

$$ds^2 = (du^1)^2 + g_{22}(u^1, u^2)(du^2)^2$$

where we have replaced $\bar{u}^1$ again by u^1.

We sum up the preceding into the following

Proposition 3.1. *If* g_{ij} *is a* C^k $(3 \leqq k \leqq \infty)$ *positive metric defined on a neighborhood* $U \subset E^2$, *then there exists an appropriate* C^k *diffeomorphism* $h: V \to \overline{V}$, *on aneighborhood* V *of every point* $p \in U$, *which transforms the given metric into a* C^k *geodesic metric.*

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Ὁ Ἀκαδημαϊκὸς κ. **Ὄ. Πυλαρινὸς** κατὰ τὴν ἀνακοίνωσιν τῆς ἀνωτέρω ἐργασίας εἶπε τὰ κάτωθι :

Ἡ εἰς μίαν ἀνοικτὴν περιοχὴν διδιαστάτου χώρου, τῆς ὁποίας τὰ σημεῖα ἀντιστοιχοῦν εἰς συστήματα τιμῶν δύο ἀνεξαρτήτων ἀπ᾽ ἀλλήλων παραμέτρων u^1, u^2, ἀνηκουσῶν ἀντιστοίχως εἰς ὡρισμένα διαστήματα, καθοριζομένη μετρικὴ ὑπὸ διαφορικῆς τετραγωνικῆς μορφῆς τῶν u^1, u^2, λέγεται *γεωδαισιακή*, ὅταν ἐκ τῶν συντελεστῶν τῆς τετραγωνικῆς ταύτης μορφῆς ὁ πρῶτος ἢ ὁ τρίτος εἶναι σταθερῶς ἴσος πρὸς τὴν μονάδα, ὁ τρίτος ἢ ὁ πρῶτος ἀντιστοίχως εἶναι συνάρτησις τῶν u^1, u^2 ὡρισμένη ὡς θετικὴ δι᾽ ὅλα τὰ εἰς τὰ σημεῖα τῆς περιοχῆς ἀντιστοιχοῦντα συστήματα τιμῶν τῶν u^1, u^2, ὁ δὲ δεύτερος εἶναι σταθερῶς ἴσος πρὸς μηδέν.

Τούτου τεθέντος, ὅταν ἡ εἰς τὴν ἐν λόγῳ περιοχὴν μετρικὴ καθορίζηται ὑπὸ διαφορικῆς τετραγωνικῆς μορφῆς τῶν παραμέτρων u^1, u^2, τῆς ὁποίας οἱ συντελεσταὶ εἶναι συναρτήσεις τῶν u^1, u^2 ὡρισμέναι, ἔχουσαι παραγώγους ὡς πρὸς u^1 u^2 μέχρι καὶ τῶν τῆς τρίτης τοὐλάχιστον τάξεως ὡρισμένας καὶ πεπερασμένας δι᾽ ὅλα τὰ εἰς τὰ σημεῖα τῆς περιοχῆς ἀντιστοιχοῦντα συστήματα τιμῶν τῶν παραμέτρων τούτων, οὔσης δ᾽ ἐπὶ πλέον θετικῶς ὡρισμένης ἐφ᾽ ὁλοκλήρου τῆς περιοχῆς, τίθεται τὸ πρόβλημα τῆς ἐξακριβώσεως τῆς δυνατότητος ἀλλαγῆς παραμέτρων εἰς τὴν περιοχὴν τοιαύτης ὥστε δι᾽ αὐτῆς ἡ εἰς τὴν περιοχὴν ἀναφερομένη εἰς τὰς νέας παραμέτρους μετρικὴ καθίσταται γεωδαισιακὴ καὶ τοῦ εἰς τὴν περίπτωσιν ταύτην καθορισμοῦ τῶν σχέσεων μεταξὺ τῶν ἀρχικῶν καὶ τῶν νέων παραμέτρων, τῇ βοηθείᾳ τῶν ὁποίων ἡ εἰς τὴν περιοχὴν μετρικὴ ἀνάγεται εἰς γεωδαισιακήν.

Μία τοιαύτη ἀλλαγὴ παραμέτρων εἰς τὴν θεωρουμένην περιοχὴν ἔχει ἤδη ἀποδειχθῆ ὅτι εἶναι δυνατὴ ὑπὸ τὴν προϋπόθεσιν ὅτι ὑπάρχει τμῆμα ἐπιφανείας ἑνὸς τριδιαστάτου Εὐκλειδείου χώρου, τοῦ ὁποίου τὰ σημεῖα ἀντιστοιχοῦν κατὰ τρόπον ἀμφιμονότιμον εἰς τὰ σημεῖα τῆς περιοχῆς, ἡ δὲ εἰς τὸ τμῆμα τοῦτο μετρικὴ καθορίζεται ὑπὸ τῆς δοθείσης διαφορικῆς τετραγωνικῆς μορφῆς, ἐπὶ τοῦ ὁποίου (τμήματος) δηλονότι ἡ θεωρουμένη περιοχὴ ἀπεικονίζεται ἰσομετρικῶς.

Ὁ κ. Μποζώνης εἰς τὴν ἐργασίαν ταύτην ἀποδεικνύει ὅτι μία ἀλλαγὴ παραμέτρων εἰς τὴν θεωρουμένην περιοχήν, διὰ τῆς ὁποίας ἡ εἰς αὐτὴν μετρικὴ ἀνάγεται εἰς γεωδαισιακήν, εἶναι δυνατὴ καὶ εἰς τὴν γενικωτέραν περίπτωσιν καθ᾽ ἣν δὲν ὑποτίθεται γνωστὸν ὅτι ὑπάρχει τμῆμα ἐπιφανείας ἑνὸς τριδιαστάτου Εὐκλειδείου χώρου, ἐπὶ τοῦ ὁποίου ἡ περιοχὴ αὕτη ἀπεικονίζεται ἰσομετρικῶς.