ΜΑΘΗΜΑΤΙΚΑ.— **The Helson constant of the Union of independent Countable Helson sets,** *by E. Galanis* *. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φίλ. Βασιλείου.

Introduction and notation. Let G be a locally compact abelian group and let $\hat{G}$ be its dual group, i. e. the multiplicative group of all continuous homomorphisms

$$x : G \to T \text{ of } G \text{ into } T = R \pmod{2\pi}.$$

We shall denote by M(G) the set of all bounded complex valued Radons measures on G.

Let E be a compact subset of G; we shall write Gp(E) for the group generated by E in G.

Definition. A compact subset E of G is called a Helson a-set (Ha-set), if there exists a constant $a > 0$ such that

$$\|\hat{\mu}\|_{\infty} = \sup_{x \in \hat{G}} |\hat{\mu}(x)| \geqslant a \|\mu\|$$

for every $\mu \in M(E)$, (observe that then $0 < a \leqslant 1$).

In this paper we shall prove the following theorems.

Theorem 1. Let $E_1, E_2 \subset G$ countable compact Ha_1, Ha_2 subsets of G such that

$$Gp(E_1 + x) \cap Gp(E_2 + x) = \{0\}$$

except for a countable set of x, N say.

Then $E_1 \cup E_2$ is an $H_{\min(a_1, a_2)}$ — set.

Theorem 2. There exist two finite disjoint Helson — 1 sets $E_1, E_2 \in R^2$ with $(0, 0) \notin E_1, E_2$ and $Gp(E_1) \cap Gp(E_2) = \{0\}$ yet with $E_1 \cup E_2$ not Helson — 1.

Remarks: The result of Theorem 1 should be contrasted with the case of two disjoint independent perfect Helson sets of constants a_1, a_2 where it is known [1] that the union may has constant at most $\dfrac{a_1 a_2}{a_1 + a_2}$.

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* Ε. ΓΑΛΑΝΗ, **Ἡ σταθερὰ Helson τοῦ συνενώματος ἀνεξαρτήτων ἀπαριθμητῶν συνόλων Helson.**

Proof of Theorem 2.

Take $E_1 = \{(x, 0), (-x, 0)\}$, $E_2 = \{(0, y), (0, -y)\}$, $(x, y \neq 0)$.

Since $E_1 + (0, \sqrt{2\pi})$ is independent, E_1 is independent so Kronecker so E_1 is Helson — 1. Similarly E_2 is Helson — 1. Clearly $0 \notin E_1, E_2$ and $Gp(E_1) \cap Gp(E_2) = \{0, 0\}$. On the other hand setting

$$\mu = (\delta_{(x,0)} - \delta_{(-x,0)} + \delta_{(0,y)} + \delta_{(0,-y)})/4$$

we have $\operatorname{supp}\mu = E_1 \cup E_2$, $\|\mu\|_M = 1$ and

$$|\hat{\mu}(\chi)| = |\chi_{(x,0)} - \overline{\chi_{(x,0)}} + \chi_{(0,y)} + \overline{\chi_{(0,y)}}|/4 =$$

$$= \sqrt{(\chi_{(x,0)} - \overline{\chi_{(x,0)}})^2 + (\chi_{(0,y)} + \overline{\chi_{(0,y)}})^2} \leqslant \sqrt{2^2 + 2^2}/4 = \frac{1}{\sqrt{2}} < 1.$$

(Notting that $\chi_{(x,0)} - \overline{\chi_{(x,0)}} = 2\,\mathrm{Im}\,\chi_{(x,0)}$, $\chi_{(0,y)} + \overline{\chi_{(0,y)}} = 2\,\mathrm{Re}\,\chi_{(0,y)}$).

So that $E_1 \cup E_2$ has Helson constant at most $\frac{1}{\sqrt{2}}$.

Proof of Theorem 1. We shall prove first

L e m m a : Let E_1, E_2 finite subsets of G, such that

$$Gp(E_1) \cap Gp(E_2) = \{0\}.$$

Then given $\chi_1, \chi_2 \in \hat{G}$ and $\varepsilon > 0$, there exists $\chi_3 \in \hat{G}$ such that

$$\|\chi_1 - \chi_3\|_{C(E_1)} \leqslant \varepsilon$$
$$\|\chi_2 - \chi_3\|_{C(E_2)} \leqslant \varepsilon$$

P r o o f : Let χ_1, χ_2 two characters. Then

$$\chi_1|_{Gp(E_1)} : Gp(E_1) \to T$$
$$\chi_2|_{Gp(E_2)} : Gp(E_2) \to T$$

Let $H = Gp(E_1) \cap Gp(E_2) \subset G$ and

$$\psi : H \to T \text{ such that } \psi(h_1 h_2) = \chi_1(h_1)\,\chi_2(h_2)$$

Then $\exists\, \tilde{\psi}$ continuous character, $G \to T$ such that

$$\tilde{\psi}|_{E_1} = \chi_1|_{E_1}$$
$$\tilde{\psi}_2|_{E_2} = \chi_2|_{E_2}$$

Case 1. E_1, E_2 are finite.

E_1 finite, so there exist finite sets of measures $\mu_1, \mu_2, \ldots, \mu_n$ such that

(i) $\|\mu_i\| = 1$

(ii) If $\mu \in M(E)$, $\|\mu\| = 1$ then $\|\mu - \mu_i\| \leqslant \varepsilon$.

For each $1 \leqslant i \leqslant n$, there exists a sequence $n_{i,k} \to \infty$ as $k \to \infty$ such that

$$|\hat{\mu}_i(n_{i,k})| \to a_1 .$$

Similarly for σ_i for E_2.

Claim we can find a single point y such that

(1) $y \notin N$

(2) $e^{i(\arg \hat{\mu}_i(n_{i,k}) + n_{i,k} y)}$ is dense in T

$$\forall\ t \in T \quad \lim\ |\hat{\mu}_i(n_{i,k})\, e^{in_{i,k} y} - a_1 e^{it}| = 0.$$

Choose $\lambda_1, \lambda_2, \ldots$ $|\lambda_i| = 1$ λ_i dense in T. Set $N = \{x_1, \ldots\}$.

Now suppose we have constructed $y_{i,k}$ and $\varepsilon_{i,k} > 0$. Two cases arise either $i < n$ in which case set $i' = i + 1$, $k' = k$ or

$i = n$ in which case set $i' = 1$, $k' = 1$.

Now since $n_{i',r} \to \infty$ as $r \to \infty$, we can find an $n_{i',r'}$ such that

$$n_{i',r'}\ \varepsilon_{i,k} \geqslant \frac{16}{\pi}.$$

Now we can find a $\delta_{i',k'}$, $|\delta_{i',k'}| \leqslant \dfrac{2\pi}{n_{i',r'}}$, $e^{n_{i',k'}\,\delta_{i',k'}} = \lambda_{k'} e^{-i \arg \hat{\mu}_i(n_{i',k'})}$.

Set $y_{i',k'} = y_{i,k} + \delta_{i',k'} + w_{i',k'}$ with

$$|w_{i',k'}| \leqslant \frac{\delta_{i',k'}}{2}, \qquad |y_{i',k'} - x_{i'}| \geqslant \frac{\delta_{i',k'}}{2} \tag{1}$$

observe
$$|y_{i',k'} - y_{i,k}| \leqslant |\delta_{i',k'}| \leqslant \frac{\varepsilon_{i,k}}{8} \tag{2}$$

choose
$$0 < \varepsilon_{i',k'} \leqslant \frac{\delta_{i',k'}}{2\pi} \cdot 2^{-k'-16} \tag{3}$$

Then provided $|y - y_{i',k'}| \leqslant \varepsilon_{i',k'}$ we have by (1)

(a) $|y - x_{i'}| \geqslant \dfrac{\delta_{i',k'}}{4}$

(b) $|e^{i(\arg \hat{\mu}_i (n_{i',k'}) + n_{i',k'} y)} - \lambda_{k'}|$

$\leqslant |e^{i(\arg \hat{\mu}_i (n_{i',k'}) + n_{i',k'}(y_{i',k'}))} - \lambda_{k'}| + |e^{in_{i',k'}(y - y_{i',k'})} - 1|$

$\leqslant 0 + 2^{-k'} = 2^{-k'}$.

Now $y_{i,k} \to y$ with $|y_{i,k} - y| \leqslant \varepsilon_{i,k}$.

So by (b) $\lim\limits_{i \to \infty} |e^{i(\arg \hat{\mu}_i (n_{i,k}) + n_{i,k} y)} - \lambda_i| = 0$, by (a) follows

$|y - x'_i| \geqslant \dfrac{\delta_{i',k'}}{4} > 0$. So $y \notin N$.

Now since $y \notin N$ we can apply lemma. Take $k \in M^+(E_1 \cup E_2 + y)$, $\|K\| = 1$.

Then $\exists\ \mu_i, \sigma_i, \lambda_1, \lambda_2, \lambda_1 + \lambda_2 = 1$ such that

$$\|K - \lambda_1 \mu_i * \delta_y - \lambda_2 \sigma_i * \delta_y| \leqslant \varepsilon$$

Now $\exists\ r_1, r_2$ such that

$$|\hat{\mu}_i (r_1)\, e^{ir_1 y} - a_1| \leqslant \varepsilon$$

$$|\hat{\sigma}_i (r_2)\, e^{ir_2 y} - a_2| \leqslant \varepsilon$$

by lemma $\exists\ r_3$ such that

$$\|x_{r_3} - x_{r_1}\|_{C(E_1)} \leqslant \varepsilon$$

$$\|x_{r_3} - x_{r_2}\|_{C(E_2)} \leqslant \varepsilon.$$

Then
$$|\hat{\mu}_i (r_3)\, e^{ir_3 y} - \hat{\mu}_i (r_1)\, e^{ir_1 y}| \leqslant \varepsilon$$

$$|\hat{\sigma}_i (r_3)\, e^{ir_3 y} - \hat{\sigma}_i (r_2)\, e^{ir_2 y}| \leqslant \varepsilon$$

So $|\hat{\mu}_i (r_3)\, e^{ir_3 y} - a_1| \leqslant |\hat{\mu}_i (r_3)\, e^{ir_3 y} - \hat{\mu}_i (r_1) e^{ir_1 y}| + |\hat{\mu}_i (r_1)\, e^{ir_1 y} - a_1| \leqslant 2\varepsilon$

and similarly

$$|\hat{\sigma}_i (r_3)\, e^{ir_3 y} - a_2| \leqslant 2\varepsilon$$

Then
$$|\hat{K} (r_3) - \lambda_1\, \hat{\mu}_i (r_3)\, e^{ir_3 y} - \lambda_2\, \hat{\sigma}_i (r_3)\, e^{ir_3 y}| \leqslant \varepsilon$$

So $\lambda_1 a_1 + \lambda_2 a_2 - \hat{K}(r_3) \leqslant |\lambda_1 a_1 + \lambda_2 a_2 - \lambda_1 \hat{\mu}_i(r_3)\, e^{ir_3y} - \lambda_2 \hat{\sigma}_i(r_3) e^{ir_3y}| +$

$$+ |\lambda_1 \hat{\mu}_i(r_3)\, e^{ir_3y} + \lambda_2 \hat{\sigma}_i(r_3)\, e^{ir_3y} - \hat{K}(r_3)| \leqslant$$

$$\leqslant \lambda_1 |a_1 - \hat{\mu}_i(r_3)\, e^{ir_3y}| + \lambda_2 |a_2 - \hat{\sigma}_i(r_3)\, e^{ir_3y}| +$$

$$+ |\lambda_1 \hat{\mu}_i(r_3)\, e^{ir_3y} + \lambda_2 \hat{\sigma}_i(r_3)\, e^{ir_3y} - \hat{K}(r_3)| \leqslant 3\varepsilon$$

and finally

$$\hat{K}(r_3) \geqslant \lambda_1 a_1 + \lambda_2 a_2 - 3\varepsilon \geqslant \min(a_1, a_2) - 3\varepsilon.$$

So $E_1 \cup E_2$ is Helson, with constant $\min(a_1, a_2) - 3\varepsilon$. As ε is arbitrary $E_1 \cup E_2$ is Helson constant $\min(a_1, a_2)$.

Case 2. E_1, E_2 countable.

Let $\mu \in M(E_1 \cup E_2)$. We write

$$\mu = \mu_1 + \mu_2 + \mu_3 + \mu_4, \quad \mu_1, \mu_2 \in M(E_1), \quad \mu_3, \mu_4 \in M(E_2).$$

Supp μ_1 is finite subset of E_1, $\|\mu_2\| \leqslant \varepsilon$

Supp μ_3 is finite subset of E_2, $\|\mu_4\| \leqslant \varepsilon$.

Then $$2\varepsilon + \|\hat{\mu}\|_\infty \geqslant \|\widehat{\mu_1 + \mu_3}\|_\infty \geqslant \min(a_1, a_2)\, \|\mu_1 + \mu_3\| \geqslant$$

$$\geqslant \min(a_1, a_2)\, (\|\mu\| - \varepsilon) \geqslant \min(a_1, a_2) - 2\varepsilon$$

or $$\|\hat{\mu}\|_\infty \geqslant \min(a_1, a_2)\, \|\mu\| - 4\varepsilon$$

and since ε is arbitrary, this completes the proof of the Theorem.

It is my pleasure to express my gratitude to Dr. T. W. Körner who posed the problem considered in this paper.

REFERENCES

1. T. W. Körner: Some results on Kronecker, Dirichlet and Helson sets.— Ann. Ins. Fourier (Grenoble) 1970, pp. 219-324.

ΠΕΡΙΛΗΨΙΣ

Ἔστωσαν E_1, E_2 δύο σύνολα τοῦ Helson οὕτως ὥστε

$$Gp(E_1) \cap Gp(E_2) = \{0\},\ 0 \notin E_1 \cup E_2$$

μὲ σταθερὰς τοῦ Helson a_1, a_2 ἀντιστοίχως.

Ὁ Βαρόπουλος ἔχει ἀποδείξει ὅτι ἡ ἕνωσις $E_1 \cup E_2$ εἶναι ἓν σύνολον τοῦ Helson μὲ σταθερὰν τουλάχιστον $\frac{a_1^2 a_2^2}{a_1^2 + a_2^2}$ καὶ ὁ Körner ἔχει ἀποδείξει ὅτι ἡ ἕνωσις δύο συνόλων τοῦ Helson δύναται νὰ εἶναι ἀκριβῶς ἴση μὲ $\frac{a_1 a_2}{a_1 + a_2}$.

Ἐπίσης ὁ Βαρόπουλος ἔχει ἀποδείξει ὅτι καὶ εἰς τὴν γενικὴν περίπτωσιν ἡ ἕνωσις δύο συνόλων τοῦ Helson εἶναι ἓν σύνολον τοῦ Helson.

Ὑποθέτομεν ὅτι ἔχομεν δύο κλειστὰ ἀριθμήσιμα σύνολα τοῦ Helson E_1, E_2 μὲ σταθερὰς a_1, a_2 καὶ $Gp(E_1) \cap Gp(E_2) = \{0\}$. Τότε ἡ εἰκασία τοῦ Körner εἶναι ὅτι ἡ ἕνωσις $E_1 \cup E_2$ εἶναι ἓν σύνολον τοῦ Helson μὲ σταθερὰν $\min(a_1, a_2)$.

Εἰς τὴν παροῦσαν ἐργασίαν τὸ θ ε ώ ρ η μ α 2 ἀποδεικνύει ὅτι ἡ ἀνωτέρω εἰκασία τοῦ Körner δὲν εἶναι ἀληθής, ἐνῷ τὸ θ ε ώ ρ η μ α 1 ἀποδεικνύει ὅτι, ἐὰν ὑποθέσωμεν ὅτι

$$Gp(E_1 + x) \cap Gp(E_2 + x) = \{0\}$$

ἰσχύει δι' ὅλα τὰ x ἐκτὸς ἴσως ἑνὸς ἀριθμησίμου πλήθους, τότε ἡ ἕνωσις δύναται νὰ ἔχῃ σταθερὰν ἴση μὲ $\min(a_1, a_2)$.

★

Ὁ Ἀκαδημαϊκὸς κ. **Φίλ. Βασιλείου** παρουσιάζων τὴν ἀνωτέρω ἀνακοίνωσιν λέγει τὰ ἑξῆς:

Ἔχω ἐπίσης τὴν τιμὴν νὰ ἀνακοινώσω εἰς τὴν Ἀκαδημίαν Ἀθηνῶν ἐργασίαν τοῦ διδάκτορος τοῦ Πανεπιστημίου τοῦ Cambridge τῆς Ἀγγλίας κ. Ε. Γαλανῆ, ἐργασίαν συντεταγμένην ἀγγλιστὶ καὶ ἡ ὁποία, ἐν μεταφράσει, ἔχει τὸν συμπληρωμένον τίτλον «Ἡ σταθερὰ Helson τοῦ συνενώματος ἀνεξαρτήτων ἀπαριθμητῶν συνόλων Helson».

Ἡ ἐργασία αὐτὴ ἀναφέρεται εἰς δημοσιευθείσας μελέτας τῶν Ν. Βαροπούλου καὶ T. W. Körner. Τούτων, ὁ μὲν Βαρόπουλος ἀπέδειξεν ὅτι τὸ συνένωμα δύο συνόλων Helson, μὲ σταθερὰς Helson a_1, ἀντιστ. a_2, εἶναι ἐπίσης σύνολον Helson μὲ σταθερὰν τοὐλάχιστον ἴσην μὲ $a^2_1 a^2_2 (a^2_1 + a^2_2)$, ἐνῷ ὁ Körner διετύπωσε τὴν εἰκασίαν ὅτι τὸ ἐν λόγῳ συνένωμα ἔχει σταθερὰν Helson ἴσην μὲ $\min(a_1, a_2)$.

Εἰς τὴν παροῦσαν ἀνακοίνωσιν ὁ συγγραφεὺς ἀποδεικνύει ὅτι γενικῶς ἡ ἄνω εἰκασία τοῦ Körner δὲν ἀληθεύει. Εὐσταθεῖ μόνον ὑπὸ ὡρισμένην προϋπόθεσιν, τὴν ὁποίαν καὶ διατυπώνει ὁ κ. Γαλανῆς εἰς τὴν παροῦσαν ἀνακοίνωσίν του.