ΜΑΘΗΜΑΤΙΚΑ.— **A note on the integrability of characteristic classes,** *by Stavros Papastavridis**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φ. Βασιλείου.

INTRODUCTION

Let M be an m-dimensional, closed, connected, C^∞ manifold with a complex structure on its stable normal bundle. We call such a manifold weakly almost complex manifold and in abbreviation w.a.c. manifold. For every w.a.c. manifold we denote $\nu_M : M \rightarrow BU$ the map classifying the stable normal bundle of M. By BU we denote BU(N), the classifying space of the universal N-dimensional complex bundle, when N is taken big enough in comparison with m.

Definition 0.1. We define $I_m^{2k} = \{x \varepsilon H^{2k}(BU;Q) : \nu_M^*(x) \varepsilon H^{2k}(M;Q)$ is an integral class for all m-dimensional w.a.c. manifolds$\}$.

It is an important question to compute I_m^{2k}. The group I_{2k}^{2k} has been computed and the computation is commonly referred as the Hattori-Stong Theorem, (see [**4**], [**1**]). The next Theorem provides a description of I_{2k+1}^{2k}.

Theorem 0.2. (a) If $m < m'$ then $I_{m'}^{2k}$ is contained in I_m^{2k}.

(b) $I_{2k+1}^{2k} = I_{2k}^{2k}$.

Let CP^∞ be the infinite dimensional complex projective space and $t \varepsilon H^2(CP^\infty; Z)$ the generator of its cohomology. By MU we denote MU(N), the Thom space of the universal N-dimensional complex bundle, provided N is very big in comparison with m. Let $U \varepsilon H^{2N}(MU; Z)$ be the Thom class of the universal N-complex bundle.

Our next Theorem gives a computation of I_{2k+2}^{2k}.

Theorem 0.3. We have $x \varepsilon I_{2k+2}^{2k}$ if and only if there exists $a \varepsilon \overline{KU}(MU \wedge CP_+^\infty)$ such that $ch_{N+k+1}(a) = (xU) \cdot t$.

Since the KU-cohomology of the space $MU \wedge CP_+^\infty$ can be computed and its Chern character too, then the above Theorem provides in principle a complete description of I_{2k+2}^{2k}. It should be noted though that if

* ΣΤΑΥΡΟΥ ΠΑΠΑΣΤΑΥΡΙΔΗ, Σχέσεις διαιρετότητος μεταξὺ στοιχείων πολλαπλοτήτων.

we have to decide for a specific class of $H^*(BU;Q)$ if it belongs to any one of the groups I^{2k}_{2k}, I^{2k}_{2k+1}, I^{2k}_{2k+2}, that may be a very hard technical question. Our next result illuminates a small aspect of this matter.

Theorem 0.4. If $x \varepsilon I^{2k}_{2k}$ then $(k+1)!(x) \varepsilon I^{2k}_{2k+2}$.

The paper is arranged as follows: In section one we reduce the problem of computing I^{2k}_{m} to a homotopy question by applying some ideas of Brown-Peterson (see [**3**]), and we prove Theorems 0.2 and 0.3. In section two we prove Theorem 0.4 by calculating certain Chern characters.

For an application of those ideas see [**9**].

REDUCTION TO A HOMOTOPY QUESTION

Let $K(Z, m-2k)$ be the well known Eilenberg-McLane space and i its fundamental class. If $m = 2k$ we put $K(Z, m-2k) = \text{point}$.

Next we consider the m-dimensional bordism groups $\Omega_m(K(Z, m-2k))$, whose elements consists of equivalenceclasses of pairs (M, f) where M is an m-dimensional w.a.c. manifold and $f: M \rightarrow K(Z, m-2k)$ is a map. For details on bordism e.t.c. a good reference is Stong's book [**8**].

Following Brown-Peterson, (see [**3**]) we define a map

$$l : H^{2k}(BU; Q) \rightarrow \mathrm{Hom}(\Omega_m(K(Z, m-2k)), Q)$$

by the formula

$$l(x)(M, f) = (\nu^*_M(x) \cdot f^*(i))(M) \varepsilon Q.$$

Note that $(M) \varepsilon H_m(M; Z)$ is the orientation class of the w.a.c. manifold M. It can be checked that l is well defined.

We will adopt the following notational convention: If we have an object which can be defined for both integral and rational coefficients, (i.e. the homology and cohomology of a space or the group $\mathrm{Hom}(A, *)$ where A is an abelian group e.t.c.) then we denote by s the map induced by the obvious inclusion $Z \rightarrow Q$, provided that such a map exists. So in this terminilogy a cohomology class of a space is called integral if it belongs to the image of s.

The next Theorem provides a reduction of the computation of I^{2k}_{m} to a homotopy question.

T h e o r e m 1. 1. The group I^{2k}_m consists of those elements $x \varepsilon H^{2k}(BU ; Q)$ such that $l(x)$ belongs to the image of s.

For the proof of this Theorem we need the following Lemma:

L e m m a 1. 2. Let M be an m-dimensional, closed, connected, oriented manifold. Let $x \varepsilon H^q(M ; Q)$ a rational cohomology class. Then x is integral if and only if for every $y \varepsilon H^{m-q}(M ; Z)$ we have $(x \cdot s(y)(M) \varepsilon Z \subseteq Q$.

P r o o f. Consider the following commutative diagram:

$$\begin{array}{ccc} H^q(M;Q) & \xrightarrow{f_1} & \mathrm{Hom}(H^{m-q}(M;Q), Q) \\ s\uparrow & & \uparrow s \\ H^q(M;Z) & \xrightarrow{f_2} & \mathrm{Hom}(H^{n-q}(M;Z), Z) \end{array}$$

where f_1 is defined by the formula $f_1(x)(y) = (x \cdot y)(M) \varepsilon Q$ for x in $H^q(M;Q)$ and y in $H^{m-q}(M;Q)$. The map f_2 is defined similarly.

By Poincare duality we have the isomorphisms $H^{m-q}(M; *) \simeq H_q(M; *)$ and by the universal coefficient Theorem we have epimorphisms $H^q(M; *) \rightarrow \mathrm{Hom}(H_q(M; *), *)$ where $*$ is Z or Q. Further more in the case where $*$ is Q the epimorphism is an isomorphism. So f_1 is isomorphism and f_2 is epi.

Since homomorphism into Q kills torsion we have an isomorphism $s : \mathrm{Hom}(H^{m-q}(M ; Z), Q) \rightarrow \mathrm{Hom}(H^{m-q}(M ; Q), Q)$.

Because of the previous remarks, the commutative diagram above is transformed to the following one:

$$\begin{array}{ccc} H^q(M;Q) & \xrightarrow{g_1} & \mathrm{Hom}(H^{m-q}(M;Z), Q) \\ s\uparrow & & \uparrow s \\ H^q(M;Z) & \xrightarrow{g_2} & \mathrm{Hom}(H^{m-q}(M;Z), Z) \end{array}$$

where both g_1 and g_2 are induced by the cup product pairings $H^q(M ; Q) x H^{m-q}(M ; Z) \rightarrow Q$ and $H^q(M ; Z) x H^{m-q}(M ; Z) \rightarrow Z$ respectively. But since g_1 is isomorphism and g_2 is onto, our Lemma follows free of charge.

P r o o f o f T h e o r e m 1. 1. Let us assume that $l(x)$ belongs to the image of s. By the definition of l this means that $(v^*_M(x) \cdot f^*(i))(M) \varepsilon Z \subseteq Q$ for all m-dimensional w.a.c. manifolds M, and all maps $f : M \rightarrow K(Z, m-2k)$.

By the fundamental property of Eilenberg-McLane spaces $f^*(i)$ could be any cohomology class of $H^{m-2k}(M;Z)$, and according to the previous Lemma the previous remark means that $\nu^*(x)$ is integral hence by definition x belongs to I_m^{2k}.

Now we are ready to prove Theorem 0.2.

Proof of Theorem 0.2. (a) Let us assume that $x \varepsilon I_{m'}^{2k}$ and let M be any w.a.c. m-dimensional manifold. Consider the manifold $MxS^{m'-m}$. The sphere $S^{m'-m}$ can be given an w.a.c. structure because its normal bundle is trivial, and so we can make the manifold $MxS^{m'-m}$ w.a.c. Because the stable normal bundle of $S^{m'-m}$ trivial we have $\nu^*_{MxS^{m'-m}}(x) = \nu^*_M(x) \otimes 1 \varepsilon H^{2k}(MxS^{m'-m};Q)$. But $\nu^*_{MxS^{m'-m}}(x)$ is integral by assumption so $\nu^*(x)$ is integral which means that x is an element of I_m^{2k}.

(b) It is well known that

$$\Omega_m(K(Z, m-2k)) \simeq \pi_{2N+m}(MU \wedge K(Z, m-2k)_+)$$

and under the identification of the two groups the map l is interpreted as follows: Let $a: S^{2N+m} \rightarrow MU \wedge K(Z, m-2k)_+$ be a map and (a) the corresponding homotopy element, then $l(x)(a) = a^*(xU \cdot i)(S^{2N+m}) \varepsilon Q$. For all those things a good reference is Stong's book. On the other hand $K(Z,1) = S^1$ and $K(Z,0) =$ point, and the groups Ω_m(point) and $\Omega_m(S^1)$ are isomorphic under the suspension isomorphism: $\pi_{2N+m}(MU) \simeq \pi_{2N+m+1}(MU \wedge S^1_+)$. Furthermore if for a map $a: S^{2N+m} \rightarrow MU$ and $x \varepsilon H^*(BU;Q)$ we have that $a^*(xU)(S^{2N+m}) \varepsilon Q$ is an integer then it will be the same for the suspension of a, b, namely $b^*(xU \cdot i)(S^{2N+m+1})$ will be an integer. And that ends the proof.

Proof of Theorem 0.3. For abbreviation let us call $X = MU \wedge K(Z,2)_+$, it is known that $K(Z,2) = CP^\infty$.
Following A. Hattori we consider the following commutative diagram:

$$\begin{array}{ccc} \widetilde{KU}(X) & \xrightarrow{f} & \mathrm{Hom}(\pi_{2N+2k+2}(X), \widetilde{KU}(S^{2N+2k+2})) \\ \downarrow ch_{N+k+1} & & \downarrow ch_{N+k+1} \\ H^{2N+2k+2}(X;Q) & \xrightarrow[g]{} & \mathrm{Hom}(\pi_{2N+2k+2}(X), H^{2N+2k+2}(S^{2N+2k+2};Q)). \end{array}$$

The vertical maps are defined via the Chern character. The map g defined by the formula $g(x)(a) = a^*(x)$ for all $x \varepsilon H^{2N+2k+2}(X; Q)$ and $a \varepsilon \pi_{2N+2k+2}(X)$, and the map f is defined by the formula $f(x)(a) = a^*(x)$ whenever x is an element of KU(X) and a is an element of $\pi_{2N+2k+2}(X)$.

Since the rational Hurewich homomorphism is an isomorphism: $\pi_*(X) \otimes Q \simeq H_*(X; Q)$, then the map g is an isomorphism.

Since the Chern character of an element of $KU(S^{2N+2k+2})$ is integral of follows that the image of the second collumn ch_{N+k+1} is included in the image of s.

Furthermore A. Hattori proved that the map f is an epimorphism, (actually Hattori proved it for the case $X = MU$ but the proof is the same for our case the main property being the fact that the space $K(Z, 2)$ is free of torsion).

With the above remarks it follows that the element

$$xU \cdot t \varepsilon H^{2N+2k+2}(X; Q)$$

is mapped by g into the image of s, if and only if it belongs to the image of KU(X) under the map ch_{N+k+1}. Those remarks with the interpretation of 1 given in the proof of Theorem 0.2 (b), namely that $x \varepsilon I^{2k}_{2k+2}$ means that $g(xU \cdot t)$ is integral, complete the proof.

The Theorem just proved provides a computation of the group I^{2k}_{2k+2} because the KU-group of the space X and its Chern character can be computed, (see [**1**], [**2**], [**5**]).

CHERN CHARACTER CALCULATIONS

In this section we are going to prove Theorem 0.4. The proof follows from Theorem 0.3 after some calculations. First we need a Lemma.

Let $z \varepsilon KU(CP^\infty)$ be the canonical line bundle over the infinite dimensional complex projective space. It is well known that $ch(z) = e^t$. Besides let ε be the element of $KU(CP^\infty)$ determined by the trivial bundle of complex dimension one.

L e m m a 2.1. Let d be an integer which is a multiple of $(k+1)!$ Then we can find integers $a_0, a_1, a_2, \ldots, a_{k+1}$ such that if

$$b = a_0 \varepsilon + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_{k+1} z^{k+1}$$

then we have $ch_i(b) = 0$ for $i = 0, 2, 3, \ldots, (k+1)$ and $ch_1(b) = d \cdot t$.

Proof. Since the Chern character is multiplicative we have $ch(z^i) = e^{it}$ so $ch_i(b) = (1/i!)\,(a_1 + a_2 2^i + a_3 3^i + \ldots + a_{k+1}(k+1)^i)$. So we have to prove that the following linear system has an integral solution:

$$\begin{aligned} a_0 + a_1 + a_2 + \ldots + a_{k+1} &= 0 \\ a_1 + 2a_2 + 3a_3 + \ldots + (k+1)\,a_{k+1} &= d \\ a_1 + 2^2 a_2 + 3^2 a_3 + \ldots + (k+1)^2 a_{k+1} &= 0 \\ &\ldots \\ a_1 + 2^{k+1} a_2 + 3^{k+1} a_3 + \ldots + (k+1)^{k+1} a_{k+1} &= 0. \end{aligned}$$

The matrix of the coefficients of our system of the $(k+1)$ last equations, is

$$\begin{bmatrix} 1 & 2 & 3 & \ldots & (k+1) \\ 1 & 2^2 & 3^2 & \ldots & (k+1)^2 \\ & & \ldots & & \\ 1 & 2^{k+1} & 3^{k+1} & \ldots & (k+1)^{k+1} \end{bmatrix}$$

Let us call A this matrix and $|A|$ its determinant. Let B be the following matrix:

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 2 & \ldots & (k+1) \\ & & \ldots & \\ 1 & 2^k & \ldots & (k+1)^k \end{bmatrix}$$

Then $|A| = (k+1)!\,|B|$. But $|B|$ is the Vandermonde determinant, (see **[6]** p. 15) and so it is non zero. Hence our system of equations has a unique solution. We must prove that the solution is integral, if d is a multiple of $(k+1)!$.

Let A_i be the matrix that we get from A by deleting the first row and the i-th collumn.

Let C be the matrix

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 2 & \ldots & k+1 \\ & & \ldots & \\ 1 & 2^{k-1} & \ldots & (k+1)^{k-1} \end{bmatrix}$$

and let C_i be the matrix which we get from C by deleting the i-th collumn. Clearly C_i is an kxk matrix, and $A_i = ((k+1)!/i) \cdot C_i$.

If we apply the rule of Cramer for the solution of the linear systems we get $a_i = ((-1)^i d \mid A_i \mid)/\mid A \mid = ((-1)^i d \mid C_i \mid)/i \mid B \mid$. But both C_i and B are Vandermonde matrices and their determinants are known (see [6] p. 15), and from the computations we have $a_i = \pm d/i!(k+1-i)!$. If the a_i's are integers for $i = 1, 2, \ldots, (k+1)$ then d must be a multiple of $(k+1)!$.

ΠΕΡΙΛΗΨΙΣ

Εἶναι γνωστὸν ὅτι ὡρισμένα ρητὰ στοιχεῖα (rational characteristic classes), γραμμικῶν δεσμῶν, εἶναι ἀκέραια χαρακτηριστικὰ στοιχεῖα διὰ τὰς ἐφαπτομένας δέσμας διαφορικῶν πολλαπλοτήτων τῆς αὐτῆς διαστάσεως (βλέπε εἰς βιβλ. ἐργασίας τῶν Hattori καὶ Stong.). Εἰς τὴν ἐργασίαν αὐτὴν ἐξετάζομεν τὸ φαινόμενον αὐτὸ διὰ πολλαπλοτήτας διαστάσεως κατὰ 1 ἢ 2 μεγαλυτέρας τῆς διαστάσεως τοῦ χαρακτηριστικοῦ στοιχείου, καὶ εὑρίσκομεν ὅλα τὰ χαρακτηριστικὰ στοιχεῖα ποὺ ἔχουν αὐτὴν τὴν ἰδιότητα, εἰς τὰς διαστάσεις αὐτάς.

Γεωμετρικὰς ἐφαρμογὰς τῶν συμπερασμάτων αὐτῶν, ἐκθέτομεν εἰς [9].

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Ὁ Ἀκαδημαϊκὸς κ. **Φ. Βασιλείου**, παρουσιάζων τὴν ἀνωτέρω ἐργασίαν εἶπε τὰ ἑξῆς:

Εἰς τὴν ἐργασίαν τοῦ κ. Σταύρου Παπασταυρίδη, μὲ τὸν ἀγγλικὸν τίτλον «A note on the integrability of characteristic classes», ἐργασίαν τὴν ὁποίαν ἔχω ἐπίσης τὴν τιμὴν νὰ ἀνακοινώσω εἰς τὴν Ἀκαδημίαν Ἀθηνῶν, ὁ συγγραφεὺς εἰδικὸς ἐπιστήμων τοῦ Πανεπιστημίου Ἀθηνῶν ἀναχωρεῖ ἀπὸ τὸν ὁρισμὸν μιᾶς «σχεδὸν μιγαδικῆς πολλαπλότητος». Εἰς κάθε σχεδὸν μιγαδικὴν πολλαπλότητα ὁρίζονται ὡρισμένα στοιχεῖα, λεγόμενα στοιχεῖα τοῦ Chern, εἰσαχθέντα τὸ 1946 ἀπὸ τὸν ἀμερικανὸν μαθηματικὸν Chern μὲ σκοπὸν τὴν μελέτην τῶν μιγαδικῶν πολλαπλοτήτων, ὅπως π. χ. τῆς μετρικῆς Hermite κ. ἄ.

Μὲ τὴν ἀνάπτυξιν τῆς Ἀλγεβρικῆς Τοπολογίας, τὰ στοιχεῖα Chern ἀποτελοῦν τὰς πλέον σημαντικὰς ἀλγεβρικὰς ἀναλλοιώτους διὰ τὴν μελέτην τῶν διαφορικῶν πολλαπλοτήτων, ὅπως λεπτομερῶς ἀναφέρει ὁ συγγραφεὺς εἰς τὴν παροῦσαν ἐργασίαν του. Ἐνδεικτικῶς ἀναφέρονται ἐδῶ, ἀφ' ἑνὸς ἡ ὑπὸ τοῦ Hirzebruch γενόμενη ἀπόδειξις τοῦ λεγομένου «θεωρήματος τοῦ δείκτου» — θεωρήματος ἐκφράζοντος τὸν δείκτην μιᾶς διαφορικῆς πολλαπλότητος ὡς πολυωνύμου στοιχείων τοῦ Chern, καὶ ἀφ' ἑτέρου αἱ ἐργασίαι τῶν C. T. C. Wall καὶ J. Milnor διὰ τῶν ὁποίων ἀποδεικνύεται, ὅτι τὰ στοιχεῖα Chern καθορίζουν πότε μία διαφορικὴ πολλαπλότης εἶναι τὸ σύνορον μιᾶς ἄλλης πολλαπλότητος. Πρέπει νὰ σημειωθῇ, ὅτι τὰ ὡς ἄνω ἐξαγόμενα ἐγένοντο μὲ τὸν συνδυασμὸν ἀλγεβρικῶν καὶ γεωμετρικῶν μεθόδων, συνδυασμὸν τόσον χαρακτηριστικὸν τῶν ἐπιτυχιῶν εἰς τὴν Ἀλγεβρικὴν Τοπολογίαν.

Εἰς τὴν παροῦσαν ἀνακοίνωσιν, ὁ συγγραφεὺς ὁρίζει ὡς χαρακτηριστικὸν στοιχεῖον σχεδὸν μιγαδικῆς πολλαπλότητος ὡρισμένον ὁμογενὲς πολυώνυμον στοιχείων Chern καὶ ὡς χαρακτηριστικὸν ἀριθμὸν σχεδὸν μιγαδικῆς πολλαπλότητος ἕνα χαρακτηριστικὸν στοιχεῖον βαθμοῦ ὅπως εἶναι ἡ διάστασις τῆς πολλαπλότητος.

Ἀπὸ τὸν Milnor καὶ ἄλλους εἶχεν ἤδη ἀποδειχθῇ ὅτι ὡρισμένοι χαρακτηριστικοὶ ἀριθμοὶ ἱκανοποιοῦν σχέσεις τινας διαιρετότητος. Διετυπώθη μάλιστα τότε, καὶ ἡ εἰκασία, ὅτι ὅλαι αἱ τοιαῦται σχέσεις διαιρετότητος πρέπει νὰ προέρχωνται ἀπὸ τὸ λεγόμενον θεώρημα τῶν Riemann - Roch, εἰκασία ἡ ὁποία ἀργότερον ἀπεδείχθη ὀρθὴ ἀπὸ τοὺς μαθηματικοὺς Hattori καὶ Stong.

Παρέμενεν ὅμως ἀνοικτὸν τὸ πρόβλημα τῆς εὑρέσεως τῶν σχέσεων διαιρετότητος δι' ὅλα τὰ χαρακτηριστικὰ στοιχεῖα, οἱουδήποτε βαθμοῦ, ὄχι δὲ μόνον τῶν χαρακτηριστικῶν ἀριθμῶν.

Ὁ συγγραφεὺς τῆς παρούσης ἀνακοινώσεως λύει τὸ πρόβλημα αὐτὸ εἰς τὴν περίπτωσιν βαθμῶν κατὰ 1 ἢ 2 μικροτέρων τῆς διαστάσεως τῆς πολλαπλότητος.

Λεπτομερείας θέλει εὕρῃ ὁ ἐνδιαφερόμενος εἰς τὰ Πρακτικὰ τῆς Ἀκαδημίας.