ΜΑΘΗΜΑΤΙΚΑ.— **More on automorphisms of a finite order,** *by D. A. Anapolitanos**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φίλωνος Βασιλείου.

INTRODUCTION

In [**1**], [**2**] and [**4**] two quite different axiomatisations of the notion of an automorphism of order n, for given $n \in \omega - \{0, 1\}$, have been presented. In [**3**] it has been shown that the notion of an automorphism of order n, for given $n \in \omega - \{0, 1\}$ is not Σ_m-axiomatisable for any $m \in \omega$, for a very large class of first order languages. In the present paper we use the axiomatisation that we found in [**1**] and [**2**] and we characterise completely the existence of a model, for a given first order theory, admitting an automorphism of order n for every finite n. We also find a Σ_1-axiomatisation of this notion for a language with only unary predicate symbols, constant symbols and no fuction sumbols at all.

In order to denote structures for a given first order language L we shall use tha capital letters A, B, ... For their universes we shall use the notation $|A|$, $|B|$, ... respectively. For the rest of our model theoretic terminology see [**6**].

Given a structure A, an automorphism of order n on A is an automorphism f on A such that $f^n = I$ and $f^m \neq I$ for all $1 \leqslant m < n$ where I is the identity function on A.

Given a first order language we say that the notion of an automorphism of order n, for given $n \in \omega - \{0, 1\}$ is Σ_m-axiomatisable if there exists a set X of Σ_m sentences of L such that $A \equiv B$ for some B admitting an automorphism of order m iff $A \models X$, where A, B are structures for L.

D e f i n i t i o n 1: Let L de a first-odrer language and $\overline{\overline{L}} = k$. (By $\overline{\overline{L}}$ we denote the cardinality of L and by k the corresponding cardinal. The axion of Choice is presupposed). In case the language L has less than k variable symbols we extend the language L to a new one adding

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k new variable symbols. We identify this new language with L, because there is no fundamental difference between them.

Now given an ordinal $\alpha < k$ we can define the notion of an α-sequence of variable symbols of L as a function $\vec{x}_\alpha : \alpha \to U$ where U is the set of variable symbols of L. Given $\vec{x}_\alpha$ let $\vec{x}_\alpha x$ be $\vec{x}_\alpha \cup \{<\alpha, x>\}$ where $x \in U$.

For given $n \geqslant 2$ we define:

$$J_{n,1}^{\vec{x}^1_\alpha, \ldots, \vec{x}^n_\alpha} = \{\exists x\, \varphi(\vec{x}^1_\alpha, \ldots, \vec{x}^n_\alpha, x) \to \exists x_1 \exists x_2 \ldots \exists x_n \big(\varphi(\vec{x}^1_\alpha, \vec{x}^2_\alpha, \ldots, \vec{x}^n_\alpha, x_1) \wedge \varphi(\vec{x}^2_\alpha, \vec{x}^3_\alpha, \ldots, \vec{x}^n_\alpha, \vec{x}^1_\alpha, x_2) \wedge \ldots \wedge \varphi(\vec{x}^n_\alpha, \vec{x}^1_\alpha, \vec{x}^2_\alpha, \ldots, \vec{x}^{n-1}_\alpha, x_n)\big) \mid \varphi(\vec{x}^1_\alpha, \ldots, \vec{x}^n_\alpha, x) \text{ a formula of L with free variables from } \vec{x}^1_\alpha, \vec{x}^2_\alpha, \ldots, \vec{x}^n_\alpha, x\}.$$

$$J_{n,m+1}^{\vec{x}^1_\alpha, \ldots, \vec{x}^n_\alpha} = \{\exists x\, \varphi(\vec{x}^1_\alpha, \ldots, \vec{x}^n_\alpha, x) \to \exists x_1 \exists x_2 \ldots \exists x_n \big(\varphi(\vec{x}^1_\alpha, \vec{x}^2_\alpha, \ldots, \vec{x}^n_\alpha, x_1) \wedge \varphi(\vec{x}^2_\alpha, \vec{x}^3_\alpha, \ldots, \vec{x}^n_\alpha, \vec{x}^1_\alpha, x_2) \wedge \ldots \wedge \varphi(\vec{x}^n_\alpha, \vec{x}^1_\alpha, \vec{x}^2_\alpha, \ldots, \vec{x}^{n-1}_\alpha, x_n) \wedge \Delta\big) \mid \text{where } \Delta \text{ is a finite subset of } \bigcup_{1 \leqslant k \leqslant m} J_{n,k}^{\vec{x}^1_\alpha, \vec{x}^2_\alpha, \ldots, \vec{x}^n_\alpha} \text{ and } \varphi(\vec{x}^1_\alpha, \ldots, \vec{x}^n_\alpha, x) \text{ a formula of L with free variables from } \vec{x}^1_\alpha, \vec{x}^2_\alpha, \ldots, \vec{x}^n_\alpha, x\}.$$

$$J_n^{\vec{x}^1_\alpha, \ldots, \vec{x}^n_\alpha} = \bigcup_{k \in \omega - \{0\}} J_{n,k}^{\vec{x}^1_\alpha, \ldots, \vec{x}^n_\alpha}.$$

Let $J_n = \{\exists x_1 \ldots \exists x_n \big(\Sigma(x_1, \ldots, x_n) \wedge \bigwedge (x_i \neq x_j)\big) \mid \Sigma(x_1, \ldots, x_n)$ the conjuction of any finite subset of $J_n^{x_1, \ldots, x_n}\}$.

T h e o r e m 2: Let T be a theory in a first-order language L. T has a model A admitting an automorphism $f : A \to A$ of order n iff $T \cup J_n$ is consistent.

P r o f : In can be found in **[1]** and [**2**].

Note 3: Given a theory T in L and models $A \models T$, $B \models T$ such that A admits an automorphism f_1, of order m and B admits an automorphism f_2 of order n such that $m < n$ and m does not divide n, it is not certain that there is o model $C \models T$ admitting an automorphism g of order n and an automorphism f of order m, as the following example shows:

Let T be a theory in a first-order language with only non-logical symbols a binary relation symbol $<$ and a ternary one B_3, and let T describe models of exactly one of two sorts.

(a) $A \models T$, contains an infinite linear ordering and exactly two elements which are not included in the range of the order relation, all the other elements being linearly ordered under $<$. Also $B_3^A = \varnothing$.

(b) $B \models T$, contains an infinite linear ordering as well, and exactly 3 elements a, b, c which are not included in the range of the order relation, all the other elements being linearly ordered under $<$. Also $B_3^B = \{<a, b, c>, <b, c, a>, <c, a, b,>\}$.

Now it is obvious that T has models admitting an automorphism of order 2 (case a) and models admitting an automorphism of order 3 (case b), but not models admitting both. Hence $T \cup J_2$ is consistent, $T \cup J_3$, is consistent but $T \cup J_2 \cup J_3$ is inconsistent.

The problem now is: given a set $X \subseteq \omega$ can we have a set of sentences S such that given a theory T we can say:

«T has a model A which for each $n \in X$ admits an automorphism of order n iff $T \cup S$ is consistent»?

In the sequel we answer the above question positively.

Lemma 4: Let L be a first-order language and A a structure for it. Then:

(a) A is k-saturated for every k iff A is finite.

(b) Suppose $\overline{\overline{L}} \leqslant k$ and $\omega \leqslant \overline{\overline{|A|}} \leqslant 2^k$. Then there is an k^+-saturated elementary extension of A having cardinality 2^k.

Proof: See [5].

Lemma 5: Suppose T is a complete theory in a first-order language L, with a model $B \models T$ admitting an automorphism f of order 2. If $A \models T$ is saturated then it admits an automorphism g of order 2 as well.

Proof: In case $\overline{\overline{L}} < \omega$ the result is immediate because the notion of an automorphism of order 2 can be expressed as a Σ_1^1 sentence and it is well-known that if A is saturated then A satisfies every Σ_1^1 - sentence which is cinstistent with Th (A) (see [**3**]). Here by $\overline{\overline{L}} < \omega$ we mean the cardinality of the set of non-logical symbols of the language.

We will try to prove the theorem without putting any restrictions on the cardinality of L.

Let, $A \models T$ be saturated and $A \equiv B$ where A has an automorphism of order 2, f say.

We add a unary function symbol to the language of T which we intend to interpet as f in B, denoting it by f as well.

We now use Lemma 4 and we get a model $(B^*, f^*) \equiv (B, f)$ which is $|\overline{\overline{A}}|$-saturated, and $\overline{\overline{|B^*|}} > \overline{\overline{|A|}}$.

We now well-order $|A|$ and $|B^*|$ in such a way that the first element b_0 in the well-ordering of $|B^*|$ is sent to an element $f^*(b_0)$ by f^* such that $f^*(b_0) \neq b_0$ which is possible because as we have said $(B^*, f^*) \equiv (B, f)$.

We now use a back and forth argument as follows:

Let $\Gamma_{b_0}(x)$ be the type of b_0 (in the language L) and consider an element a_{b_0} in $|A|$ such that $A \models \Gamma_{b_0}(\underline{a}_{b_0})$ (possible as A is saturated). So $(A, a_{b_0}) \equiv (B^*, b_0)$.

We now consider the element $f^*(b_0) \in |B^*|$ and its type in (B^*, b_0), $\Gamma_{f^*(b_0)}$ say. Then there is an element $a_{f^*(b_0)} \in |A|$ such that $(A, a_{b_0}) \models \Gamma_{f^*(b_0)}(\underline{a}_{f^*(b_0)})$. (It is clear that $a_{b_0} \neq a_{f^*(b_0)}$). Then we get $(A, a_{b_0}, a_{f^*(b_0)}) \equiv (B^*, b_0, f^*(b_0))$.

We now get the element of $|A| - \{a_{b_0}, a_{f^*(b_0)}\}$ with the smallest index in the well-ordering of A, a_0 say, and its type $\Gamma a_0(x)$ in $(A, a_{b_0}, a_{f^*(b_0)})$. We find, by a similar argument as above, elements b_{a_0}, $f^*(b_{a_0}) \in |B^*|$, $a_{f^*(b_{a_0})} \in |A|$ such that

$$(A, a_{b_0}, a_0, a_{f^*(b_0)}, a_{f^*(b_{a_0})}) \equiv (B^*, b_0, b_{a_0}, f^*(b_0), f^*(b_{a_0})).$$

We again pick up the element of $|A| - \{a_{b_0}, a_0, a_{f^*(b_0)}, a_{f^*(b_{a_0})}\}$ with the smallest index in the well-ordering of $|A|$, a_1 say, and we continue this way until we exhaust all elements of $|A|$ (possible as $|A| < \overline{\overline{|B^*|}}$).

We now define $g : |A| \to |A|$ by $g(a_{b_0}) = a_{f^*(b_0)}$ and

$$g(a_{f^*(b_0)}) = a_{b_0} \text{ and for } c \in |A| - \{a_{b_0}, a_{f^*(b_0)}\}$$

$$g(c) = a_{f^*(b_c)} \text{ if } c \neq a_{f^*(b_a)} \text{ for any } a \in |A| - \{a_{b_0}, a_{f^*(b_0)}\}$$

$$g(c) = a \text{ if } c = a_{f^*(b_a)} \text{ some } a \in |A| - \{a_{b_0}, a_{f^*(b_0)}\}.$$

It can be clearly and easily shown now that g is an automorphism of A of order 2. (of order 2 because $a_{b_0} \neq a_{f^*(b_0)}$).

The above proof obviously works for any Σ_1^1 sentence as well.

Lemma 6: Suppose T is a complete theory in a first-order language L, with a model $B \models T$ admitting an automorphism f of order $n \geqslant 2$. Let $A \models T$ be saturated. Then A has an automorphism of order n.

Proof: Similar to that of lemma 5.

Theorem 7: Given a set $A \subseteq \omega$ and a theory T in a first-order language L, the following two statements are equivalent.

(1) $T + \bigcup_{n \in A} J_n$ is consistent.

(2) T has a model A which for each $n \in A$ admits an automorphism g of order n.

Proof: (1) $\Rightarrow$ (2). There is a model $B \models T + \bigcup_{n \in A} J_n$.

We then take Th(B) and let $A \models \mathrm{Th}(B)$ be saturated. Then as Th(B) is complete by Lemma 6 we get that A is the required model.

(2) $\Rightarrow$ (1). The proof comes immediately from Theorem 2.

Theorem 8: Let L be a first order language with only unary predicate symbols, constant symbols and no function symbols et all. Then the notion of an automorphism of order n, for given $n \in \omega - \{0, 1\}$, is Σ_1-axiomatisable.

Proof: Let $\{P_i \mid i \in I\}$ and $\{c_j \mid j \in J\}$ be the sets of unary predicate symbols and constant symbols respectively, of the language L. We consider the following set of sentenses:

$$X = \Big\{ \exists x_1 \ldots \exists x_n \Big(\bigwedge_{1 \leqslant i \neq j \leqslant n} x_i \neq x_j \wedge \bigwedge_{1 \leqslant j \leqslant n} \big(\bigwedge_{k \in K} x_j \neq c_k \big)$$

$$\wedge \bigwedge_{1 \leqslant j \leqslant n-1} \Big(\bigwedge_{r \in R} \big(P_r(x_j) \longleftrightarrow P_r(x_{j+1}) \big) \Big) \Big) \,\Big|\, K \subseteq J \text{ and } R \subseteq I \text{ and } K, R$$

finite$\Big\}$.

We claim that the set X, Σ_1-axiomatises the notion of an automorphism of order n. That the set X is a set of Σ_1 sentences is obvious.

We now suppose that A and B are structures for L such that $A \equiv B$ and B admits an automorphism of order n. Let $f : B \to B$ be such an automorphism of order n and $b \in |B|$ such that $f^m(b) \neq b$ for $1 \leqslant m < n$. Then for any sentence φ of X we have $B \models \varphi$ because:

$$B \models \bigwedge_{1 \leqslant i \neq j \leqslant n} \big(\underline{f^i(b)} \neq \underline{f^j(b)}\big) \wedge \bigwedge_{1 \leqslant j \leqslant n} \big(\bigwedge_{k \in K} \underline{f^j(b)} \neq c_k \big)$$

$$\wedge \bigwedge_{1 \leqslant j \leqslant n-1} \big(\bigwedge_{r \in R} \big(P_r\big(\underline{f^j(b)}\big) \longleftrightarrow P_r\big(\underline{f^{j+1}(b)}\big)\big)\big).$$

Hence $B \models X$ and so $A \models X$ because $A \equiv B$.

We will prove now the opposite direction.

Let A be a structure for the language L such $A \models X$. We extend the language L to a language L′ adding the new constant symbols $d_1, d_2, \ldots, d_n$. Given a sentence $\varphi \in X$ of the form

$$\exists x_1 \ldots \exists x_n \Big(\bigwedge_{1 \leqslant i \neq j \leqslant n} x_i \neq x_j \wedge \bigwedge_{1 \leqslant j \leqslant n} \big(\bigwedge_{k \in K} x_j \neq c_k \big)$$

$$\wedge \bigwedge_{1 \leqslant j \leqslant n-1} \Big(\bigwedge_{r \in R} \big(P_r(x_j) \longleftrightarrow P_r(x_{j+1}) \big) \Big) \Big)$$

we shall use the notation $\Psi_{<K, R>}$ for the furmula

$$\bigwedge_{1 \leqslant i \neq j \leqslant n} d_i \neq d_j \wedge \bigwedge_{1 \leqslant j \leqslant n} \big(\bigwedge_{k \in K} d_j \neq c_k \big) \wedge \bigwedge_{1 \leqslant j \leqslant n-1} \Big(\bigwedge_{r \in R} \big(P_r(d_j) \longleftrightarrow P_r(d_{j+1}) \big) \Big).$$

It is now abvious that for each $\Psi_{<K, R>}$ there sxists a structure $A_{<K, R>}$ for L′ such that $A_{<K, R>} \models Th(A)$ and $A_{<K, R>} \models \Psi_{<K, R>}$. So to each $\Psi_{<K, R>}$ we assing exactly one structure $A_{<K, R>}$ and we consider the family of all such structures

$$\{ A_{<K, R>} \mid <K, R> \in P_\omega(J) \times P_\omega(I) \}$$

where $P_\omega(J)$ and $P_\omega(I)$ denote the sets of all finite subsets of J and I respectively.

We now consider the set $\Lambda \subseteq P(P_\omega(J) \times P_\omega(I))$ defined as follows, (where $P(D)$ denotes the power set of D): $\lambda \in \Lambda \Longleftrightarrow$ there exists an ordered pair $<k, r>$ such that

$$\lambda = \{<K, R> \mid A_{<K,R>} \models \bigwedge_{1 \leqslant i \neq j \leqslant n} d_i \neq d_j \wedge \bigwedge_{1 \leqslant j \leqslant n} d_j \neq c_k \wedge \bigwedge_{1 \leqslant j \leqslant n-1} (P_r(d_j) \longleftrightarrow P_r(d_{j+1}))\}.$$

We can now see that Λ has the finite intersection property.

Indeed if by $<k_i, r_i>$ we denote an ordered pair that generates $\lambda_i \in \Lambda$ according to the above definition, then considering a finite set $\{\lambda_1, \lambda_2, \ldots, \lambda_m\} \subseteq \Lambda$ we can see that

$$A_{<K,R>} \in \lambda_1 \cap \lambda_2 \cap \ldots \cap \lambda_m \quad \text{where} \quad <K, R> = <\bigcup_{1 \leqslant i \leqslant m}\{k_i\}, \bigcup_{1 \leqslant i \leqslant m}\{r_i\}>$$

Hence we can extend J to a non-principal ultrafilter F and denote the corresponding ultraproduct by B'. We can now use Łós Theorem (see (5) and (6)) and get that

$$B' \models Th(A) \quad \text{and}$$

$$B' \models \{\Psi_{<K,R>} \mid <K, R> \in P_\omega(J) \times P_\omega(I)\}.$$

We now define a function $f : B' \to B'$ by

$$f(d_i^{B'}) = d_{i+1}^{B'} \quad \text{for all} \quad 1 \leqslant i \leqslant n-1$$

$$f(d_n^{B'}) = d_1^{B} \quad \text{and}$$

$$f(b) = b \quad \text{for all} \quad b \in |B'| - \{d_1^{B'}, \ldots, d_n^{B'}\}$$

where $d_1^{B'}, \ldots, d_n^{B'}$ denote the interpretations of $d_1, \ldots, d_n$ in the structure B'.

Now it is obvious that the function f is an automorphism of order n. If we consider the structure B' restricted to the language L we get a new structure B such that $B \equiv A$ and B admits an automorphism f of order n.

ΠΕΡΙΛΗΨΙΣ

Εἰς τὰ [1], [2] καὶ [4] ἔχουν παρουσιασθῆ δύο τελείως διαφορετικαὶ ἀξιωματικοποιήσεις τῆς ἐννοίας ἑνὸς αὐτομορφισμοῦ πεπερασμένης τάξεως. Εἰς τὴν ἐργασίαν [3] ἐπίσης ἔχει ἀποδειχθῆ ὅτι ἡ ἔννοια ἑνὸς αὐτομορφισμοῦ τάξεως n, ὅπου n φυσικὸς ἀριθμὸς διάφορος τοῦ 1 καὶ τοῦ 0, δὲν εἶναι $\sum_m$ - ἀξιωματικοποιήσιμη, εἰς τὰς πλείστας τῶν περιπτώσεων, ὅπου m εἶναι τυχὼν φυσικὸς ἀριθμός.

Εἰς τὴν παροῦσαν ἐργασίαν χρησιμοποιοῦμεν τὸ σύνολον τῶν προτάσεων μὲ τὸ ὁποῖον ἀξιωματικοποιήσαμε τὴν ἔννοιαν τοῦ αὐτομορφισμοῦ τάξεως n, ὅπου n φυσικὸς ἀριθμὸς διάφορος τοῦ 1 καὶ 0, καὶ ἐπιτυγχάνομεν νὰ χαρακτηρίσωμεν πλήρως, δεδομένης μιᾶς πρωτοβαθμίου θεωρίας T, τὴν ὕπαρξιν ἑνὸς μοντέλου της τὸ ὁποῖον νὰ δέχεται ἕνα αὐτομορφισμὸν τάξεως n διὰ κάθε $n > 0, 1$.

Ἐπίσης ἀποδεικνύομεν τὴν ὕπαρξιν μιᾶς $\sum_1$ - ἀξιωματικοποιήσεως τῆς ἀνωτέρω ἀναφερθείσης ἐννοίας διὰ πρωτοβαθμίους γλώσσας περιεχούσας μόνον μονομελῆ κατηγορηματικὰ σύμβολα ὡς καὶ σταθερὰ σύμβολα.

REFERENCES

1. D. A. Anapolitanos, «Cyclic Indiscernibles and automorphisms of finite order». Ph. D. Thesis, Manchester (1977).
2. D. A. Anapolitanos, «Automorphisms of finite order» to appear in «Zeitschrift für Mathematische Logik und Grundlagen der Mathematik».
3. D. A. Anapolitanos - J. Väänänen, «On the axiomatisability of the notion of an automorphism of a finite order» to appear in «Zeitschrift für Mathematische Logik und Grundlagen der Mathematik».
4. J. Barwise, «Some applications of Henkin Quantifiers», Israel. Journal of Mathematics, Vol. 25 (1976), pp. 47 - 67.
5. J. L. Bell - A. B. Slomson, «Models and Ultraproducts», North - Holland, Amsterdam (1971).
6. C. C. Chang - H. J. Keisler, «Model Theory», North-Holland, Amsterdam (1973).
7. G. E. Sacks, «Saturated Model Theory», Mathematics Lecture Note Series, W. A. Benjamin, Inc, (1972).