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ΟΥΡΑΝΙΟΣ ΜΗΧΑΝΙΚΗ.—**The motion of a projectile around the earth under the influence of the earth's gravitational atraction and a thrust***, *by Dem. G. Magiros* **. Ἀνεκοινώθη ὑπὸ τοῦ κ. Ἰωάνν. Ξανθάκη.

Abstract.

In this paper the motion of a projectile around the earth under the influence of the gravitational attraction of the earth and a thrust is discussed. The orbit of the projectile and its velocity along the orbit are found in two cases, namely when the thrust is suddenly applied to the projectile,

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** ΔΗΜ. ΜΑΓΕΙΡΟΥ, Ἡ κίνησις βλήματος πέριξ τῆς γῆς ὑπὸ τὴν ἐπίδρασιν τῆς ἑλκτικῆς δυνάμεως τῆς γῆς καὶ μιᾶς ὠστικῆς δυνάμεως.

and when it is gradually applied. The types of the Keplerian orbit, when the thrust is suddenly removed, are also discussed.

Introduction.

The following problem is discussed:

A projectile is moving around the earth in a Keplerian orbit, when a thrust is applied. Find the motion of the projectile during the action of the thrust.

This problem is solved for a general thrust vector, either suddenly or gradually applied to the projectile and acting continuously for non-infinitesimal time. The problem is specialized in the case of constant thrust vector, case which is identical with the problem of Quantum Mechanics in connection with «Stark Effect». Conditions are given in connection of the types of the Keplerian orbit, when the thrust is suddenly removed from the projectile.

I. *Mathematical formulation of the problem.*

If $\underline{T}$ is the thrust per unit mass, acting for time τ, the motion of the projectile during the time τ is governed by the differential equation:

$$\ddot{\underline{r}} = -\frac{G}{r^3}\underline{r} + \underline{T} \tag{1}$$

where r is the magnitude of $\underline{r}$, G a constant equal to $\frac{k(M+m)}{m}$, M and m are the masses of the earth and the projectile, respectively; k the constant of gravitation.

The time τ is split according to:

$$\tau = t_0 + t_1, \tag{2}$$

where t_0 is infinitesimal, and we take as initial conditions to the differential Eq. (1) the following conditions:

$$\underline{r}(t)_{t=t_0} = \underline{r}_0 \;,\; \dot{\underline{r}}(t)_{t=t_0} = \dot{\underline{r}}_0 + \underline{I}_0, \tag{3}$$

where $\underline{r}_0$ and $\dot{\underline{r}}_0$ are the position vector and velocity vector, respectively, at the point of the original orbit where the thrust is applied to the projectile; and $\underline{I}_0$ is the impulse per unit mass of the thrust $\underline{T}$ acting on the projectile in the infinitesimal time t_0. Such a selection of initial conditions leads to a solution which is very helpful to practical problems.

II. *Solution of the problem.*

By changing the time t according to:

$$\bar{t} = t - t_o \tag{2.1}$$

the dif. Eq. (1) keeps its form, and the initial conditions (3) in the new scale time are:

$$\underline{r}(\bar{t})_{\bar{t}=o} = \underline{r}_o, \qquad \dot{\underline{r}}(\bar{t})_{\bar{t}=o} = \dot{\underline{r}}_o + \underline{I}_o \tag{3.1}$$

A solution of the system (1) and (3.1) of the form:

$$\underline{r}(\bar{t}) = a(\bar{t})\,\underline{r}_o + b(\bar{t})\,(\dot{\underline{r}}_o + \underline{I}_o) + c(\bar{t})\,\underline{T}_o \tag{4}$$

is going to be established by calculating the functions $a(\bar{t})$, $b(\bar{t})$, $c(\bar{t})$ appropriately. $\underline{T}_o$ is the thrust at $\bar{t}=o$.

For $\bar{t}=o$ the function of Eq. (4) gives

$$a(o) = 1, \quad b(o) = o, \quad c(o) = o \tag{4.1}$$

and its derivative:

$$\dot{a}(o) = o, \quad \dot{b}(o) = 1, \quad \dot{c}(o) = o \tag{4.2}$$

The function of Eq. (4) must satisfy identically the differential Eq. (1); then:

$$\ddot{a}(\bar{t})\,\underline{r}_o + \ddot{b}(\bar{t})\,(\dot{\underline{r}}_o + \underline{I}_o) + \ddot{c}(\bar{t})\,\underline{T}_o \equiv$$

$$-\frac{G}{r^3}\,a(\bar{t})\,\underline{r}_o - \frac{G}{r^3}\,b(\bar{t})\,(\dot{\underline{r}}_o + \underline{I}_o) - \frac{G}{r^3}\,c(\bar{t})\,\underline{T}_o + \underline{T}(\bar{t}). \tag{5}$$

If the projections of $\underline{T}(\bar{t})$ on the constant vectors $\underline{r}_o$, $\dot{\underline{r}}_o + \underline{I}_o$ and $\underline{T}_o$ are, by omitting constant factors, $T_1(\bar{t})$, $T_2(\bar{t})$, and $T_3(\bar{t})$, respectively from the identity (5) we can get:

$$\ddot{a}(\bar{t}) + \frac{G}{r^3}\,a(\bar{t}) = T_1(\bar{t}), \tag{6.1}$$

$$\ddot{b}(\bar{t}) + \frac{G}{r^3}\,b(\bar{t}) = T_2(\bar{t}) \tag{6.2}$$

$$\ddot{c}(\bar{t}) + \frac{G}{r^3}\,c(\bar{t}) = T_3(\bar{t}) \tag{6.3}$$

The determination of $a(\bar{t})$, $b(\bar{t})$ and $c(\bar{t})$ from Eqs. (6), subject to the initial conditions (4.1) and (4.2), requires r, T_1, and T_2, and T_3 to be known functions of time $\bar{t}$. For a prescribed thrust $\underline{T}$, the functions $T_1(\bar{t})$, $T_2(\bar{t})$ and $T_3(\bar{t})$ are known, but r is unknown, then we can not solve the Eqs. (6) for $a(\bar{t})$, $b(\bar{t})$, and $c(\bar{t})$.

In spite of that, approximations of these functions can be found in the following way.

We restrict ourselves without loss of generality to the case of any thrust constant in magnitude and direction. For such a thrust the Eqs. (6) may be replaced by:

$$\ddot{a}(\bar{t}) + \frac{G}{r^3}\, a(\bar{t}) = 0 \tag{7.1}$$

$$\ddot{b}(\bar{t}) + \frac{G}{r^3}\, b(\bar{t}) = 0 \tag{7.2}$$

$$\ddot{c}(\bar{t}) + \frac{G}{r^3}\, c(\bar{t}) = 1. \tag{7.3}$$

Let us assume for $a(\bar{t})$, $b(\bar{t})$, and $c(\bar{t})$ the Meclaurin's expensions:

$$a(\bar{t}) = a(0) + \dot{a}(0)\bar{t} + \ddot{a}(0)\frac{\bar{t}^2}{2} + \cdots a^{(n)}(0)\frac{\bar{t}^n}{n!} + \cdots \tag{8.1}$$

$$b(\bar{t}) = b(0) + \dot{b}(0)\bar{t} + \ddot{b}(0)\frac{\bar{t}^2}{2} + \ldots b^{(n)}(0)\frac{\bar{t}^n}{n!} + \ldots \tag{8.2}$$

$$c(\bar{t}) = c(0) + \dot{c}(0)\bar{t} + \ddot{c}(0)\frac{\bar{t}^2}{2} + \ldots c^{(n)}(0)\frac{\bar{t}^n}{n!} + \ldots \tag{8.3}$$

We can calculate as many coefficients of these series as we want, by using Eqs. (7), (4.1), (4.2) and (1). The first two coefficients are known from the conditions (4.1) and (4.2). The third coefficients, found from Eqs. (7), are:

$$\ddot{a}(0) = -\frac{G}{r_0^3},\quad \ddot{b}(0) = 0,\quad \ddot{c}(0) = 1 \tag{4.3}$$

The fourth coefficients can be obtained from Eqs. (7) by differentiation, and then using conditions (4.1), (4.2) and (3.1). We can get:

$$\dddot{a}(\bar{t}) = -G(r\dot{a} - 3a\dot{r})/r^4, \tag{4.4}$$

and similar formulae for $\dddot{b}(\bar{t})$ and $\dddot{c}(\bar{t})$, then:

$$\dddot{a}(0) = 3G(\dot{r}_0 + I_0)/r_0^4,\quad \dddot{b}(0) = -G/r_0^3,\quad \dddot{c}(0) = 0 \tag{4.5}$$

For the fifth coefficients we use the formula (1). From the Eq. (4.4) we get:

$$\ddddot{a}(\bar{t}) = -G\left\{\frac{\ddot{a}}{r^3} - 3\,\frac{2\dot{a}\dot{r} + a\ddot{r}}{r^4} + 12a\,\frac{\dot{r}^2}{r^5}\right\} \tag{4.6}$$

and similar formulae for $\ddddot{b}(\bar{t})$ and $\ddddot{c}(\bar{t})$, which for $\bar{t} = 0$ give conditions where all the quantities of their right-hand members are known, the value

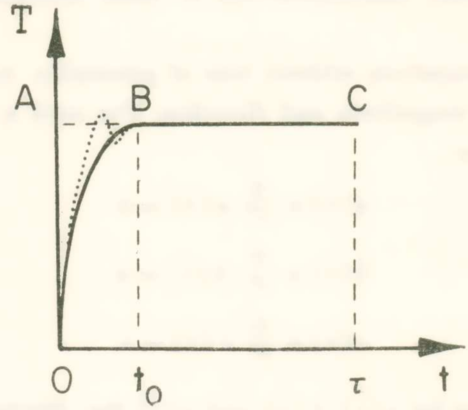

Fig. 1. The solid line is in accordance with the relation (13), the dotted line with practical problems where $t_0 = 0.02$ seconds, approximately.

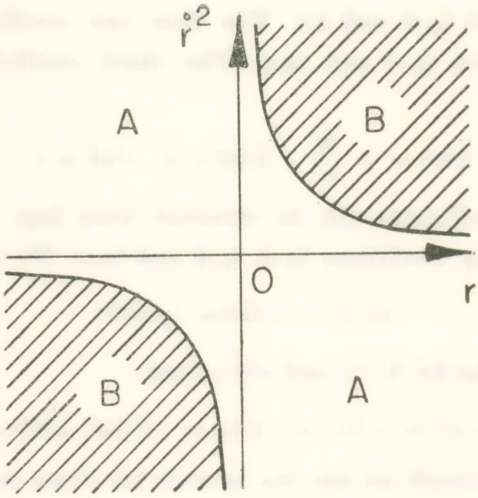

Fig. 2. The points of the rectangular hyperbola $r \cdot \dot{r}^2 = 2\mu$ in the $r \cdot \dot{r}^2$—plane give a parabolic orbit; the points of the regions A give an elliptic orbit, and that of B a hyberbolic.

$\ddot{r}_o$ being given by the Eq. (1). For the other coefficients we follow the same procedure.

If we restrict ourselves to the approximations up to the order $\bar{t}^2$, we have:

$$a(\bar{t}) = 1 - \frac{G}{2r_o^3}\bar{t}^2,\ b(\bar{t}) = \bar{t},\ c(\bar{t}) = \frac{1}{2}\bar{t}^2 \tag{9}$$

then an approximation of the solution (4), if we come back to the original scale time by using (2,1), is:

$$\underline{r}(t) = \left\{1 - \frac{G}{2r_o^3}(t - t_o)^2\right\}\underline{r}_o + (t - t_o)(\dot{\underline{r}}_o + \underline{I}_o) + \frac{1}{2}(t - t_o)^2\underline{T}_o, \tag{10}$$

from which we get:

$$\dot{\underline{r}}(t) = -\frac{G}{r_o^3}(t - t_o)\underline{r}_o + (\dot{\underline{r}}_o + \underline{I}_o) + (t - t_o)\underline{T}_o. \tag{11}$$

III. *The solution when the thrust is either suddenly or gradually applied.*

The formulae (10) and (11) are velid in $t_o \leq t \leq \tau$, and the thrust $\underline{T}_o$ and its impulse $\underline{I}_o$ are the values of the thrust and the impulse at $t = t_o$. The impulse $\underline{I}_o$ during the time t_o is, by definition, given by:

$$\underline{I}_o = \int_0^{t_o} \underline{T}(t)\,dt. \tag{12.1}$$

If the thrust $\underline{T}(t)$ is, during the time t_o, a constant, $\underline{T}_o = T_o\underline{n}$ say, then:

$$\underline{I}_o = t_o \underline{T}_o, \tag{12.2}$$

and we speak about «*sudden application*» of the thrust. If the thrust is changed in $0 \leq t \leq t_o$, following any law, being zero at $t = 0$, we speak about «*gradual application*» of the thrust.

If the thrust is, say, constant in its direction but its magnitude varies accarding to parabolic law during the infinitesimal time t_o, being constant at the remainder time, i.e.

$$\underline{T}(t) = T(t)\underline{n},\quad \underline{n} = \text{constant},\quad T(t) = \begin{cases} at^{1/2}, & 0 \leq t \leq t_o \\ at_o^{1/2}, & t_o \leq t \leq \tau \end{cases} \tag{13}$$

where $\underline{n}$ is the unit vector, and the number a, which characterizes the parameter of the parabola, is a large positive constant, the impulse is:

$$\underline{I}_o = \frac{2}{3} t_o \underline{T}_o,\quad \underline{T}_o = at_o^{1/2}\underline{n}. \tag{12.3}$$

The above is an example of «gradual application», which can approximate the practical problems. In Fig. 1, the solid line is the graph of the thrust (13), while the dotted line is the thrust according to practical problems, t_o being approximately equal to: 0.02 seconds.

The cases (12.2) and (12.3) can be written as:

$$\underline{I}_o = m t_o \underline{T}_o, \quad \underline{T}_o = a t_o^{1/2} \underline{n} \tag{12.4}$$

where $m=1$ corresponds to sudden application, and $m={}^2/_3$ to gradual application, then the formulae (10) and (11) can be written as:

$$\underline{r}(t) = \left\{1 - \frac{G}{2r_o^3}(t-t_o)^2\right\}\underline{r}_o + (t-t_o)\dot{\underline{r}}_o + \left\{m t_o (t-t_o) + \frac{1}{2}(t-t_o)^2\right\}\underline{T}_o, \tag{10.1}$$

$$\dot{\underline{r}}(t) = -\frac{G}{r_o^3}(t-t_o)\underline{r}_o + \dot{r}_o + \left\{m t_o + (t-t_o)\right\}\underline{T}_o. \tag{11.1}$$

These formulae describe the motion of the projectile at time $t_o \leq t \leq \tau$; $m=1$ corresponds to a sudden application of the constant thrust $\underline{T} = a t_o^{1/2} \underline{n}$, $m=2/3$ corresponds to a gradual application of a thrust according to (13).

IV. *The type of the Keplerian orbit if the thrust is suddenly removed.*

The position vector $\underline{r}(t)$ and the velocity vector $\dot{\underline{r}}(t)$, given above, determine the motion of the projectile on the non-Keplerian arc of its orbit during the action of the thrust. When the thrust is suddenly removed at time τ, the values $\underline{r}(\tau)$ and $\dot{\underline{r}}(\tau)$ determine completely the Keplerian orbit of the projectile after time τ.

The type of this Keplerian orbit depends upon the sign of the right-hand member of the relation:

$$\frac{1}{A_1} = \frac{2}{r} - \frac{\dot{r}^2}{\mu} \tag{14}$$

where A_1 is the semi-major axis of the orbit, $\mu = k(M+m)$, and if this member is bigger, equal to or less than zero, we have an ellipse, parabola or hyperbola, respectively, and this leads to the restrictions:

$$r \cdot \dot{r}^2 \lesseqgtr 2\mu \tag{15}$$

for an ellipse, parabola or hyperbola, respectively. In fig. 2 the graph of (15) in the r, $\dot{r}^2$ — plane is given. The regious A correspond to an ellipse; the regious B to a hyperbola, and their boundary, the curve $r\dot{r}^2 = 2\mu$, to a parabola.

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ΠΕΡΙΛΗΨΙΣ

Ἐνταῦθα σπουδάζεται ἡ κίνησις ὀχήματος πέριξ τῆς γῆς ὑπὸ τὴν ἐπίδρασιν τῆς ἑλκτικῆς δυνάμεως τῆς γῆς καὶ μιᾶς ὠστικῆς δυνάμεως. Ἡ τροχιὰ καὶ ἡ ταχύτης τοῦ ὀχήματος εὑρίσκονται εἰς δύο περιπτώσεις, ὅταν δηλαδὴ ἡ ὠστικὴ δύναμις ἐφαρμόζεται ἐπὶ τοῦ ὀχήματος εἴτε ἀκαριαίως εἴτε βραδέως. Δίδονται ἐπίσης καὶ αἱ συνθῆκαι ὑπὸ τὰς ὁποίας διακρίνομεν τὰ εἴδη τῆς Κεπλερείου τροχιᾶς, ὅταν ἡ ὠστικὴ δύναμις παύσῃ ἀκαριαίως νὰ δρᾷ.

ΒΙΟΛΟΓΙΚΗ ΧΗΜΕΙΑ.— **Περὶ τῆς περιεκτικότητος τοῦ ἥπατος καὶ τοῦ σπληνὸς ἰχθύων τῆς θαλάσσης εἰς αἱμοσιδερίνην καὶ σίδηρον,** ὑπὸ **Γ. Γίτσα, Ἀ. Δημητριάδου καὶ Ἀ. Χρηστομάνου.** Ἀνεκοινώθη ὑπὸ τοῦ κ. Γεωργ. Ἰωακείμογλου.
