ΜΑΘΗΜΑΤΙΚΑ.— **Actual Mathematical Solutions of Problems Posed by Reality, II. (Applications)***, *by D. G. Magiros* **. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ἰω. Ξανθάκη.

INTRODUCTION

In the previous paper **3(a)**, we discussed a classical procedure for finding actual mathematical solutions of real systems in many physical or social fields. The main phases of the procedure were:

A. The creation of a theory of the system, which helps its modeling;
B. The selection of a «well-posed-model» of the system, which gives a well-posed mathematical problem, and
C. The construction of the solution of this problem, which is the «actual solution» of the system.

In the present paper we give some applications of the above classical method, by which we can see the difficulties of its application and its advantages in case this method can be applied. We select the applications from thermodynamics, astrodynamics, non-linear mechanics, biology, etc.

1st Application **Problem of Thermodynamics: (4)**

Forward and Backward Heat Flow Problem.

The Step: «$S_c \rightarrow M$» of the classical method characterizes the whole study of the problem. The problem is: *«To study the heat flow in a given medium»*. To make this physical problem correctly stated, one accepts for the medium to be homogenious and isotropic with respect to the heat flow, and that the heat flow is towards the decreasing temperature. Based on these hypotheses, the mathematical idealization, the model, is the partial differential equation:

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$$U_{xx} + U_{yy} + U_{zz} = U_t \qquad (1)$$

where : $u = u(x, y, z, t)$ is the temperature in the x, y, z - space and t-time. In the equation (1) there is a coefficient depending on density, specific heat, and thermal conductivity and this coefficient is here taken equal to unity.

In case of a «one-dimensional medium», if the «data-initial condition» is :

$$u(x, 0) = n \,.\, \sin nx, \qquad n = \text{integer} \qquad (2)$$

one can check that the solution of equation (1), satisfied by (2), is:

$$u(x, t) = n \,.\, e^{-n^2 t} \,.\, \sin nx \qquad (3)$$

and it is unique, when the first two Hadamard's restrictions are satisfied. We distinguish here two cases :

a. It $t > 0$, when one has the «forward heat problem», the solution (3) $\to 0$ and the condition (2) $\to \infty$, as $n \to \infty$, then the solution (3) satisfies also the third Hadamard's restriction, when the function (3) is accepted as an «actual solution» of the «forward heat problem», which is a «well-posed-problem».

b. If $t < 0$, when one has the «backward heat problem», the solution (3) and the condition (2) $\to \infty$, as $n \to \infty$, then the solution (3) violates the third Hadamard's restriction, when the function (3) is a «formal solution» of the «backward heat problem», which is a «non-well-posed-problem».

2nd Application **Problem of Orbital Mechanics: 3 (b)**

An artificial celestial body is moving under the influence of a central force obeying the inverse square Newton's law toward the attractive center. A general force is applied, acts for an interval of time, then it is removed. Find the motion of the body during the action of the general force.

A model of this problem is:

$$\left.\begin{aligned} &\ddot{\underline{r}} = -\frac{\mu}{r^3(\tau)}\,\underline{r}(\tau) + \underline{T}(\tau) \\ &\underline{r}(0) = \underline{r}_0\,, \quad \dot{\underline{r}}(0) = \dot{\underline{r}}_0 + \underline{I}_0 \\ &D_1 : |\underline{r}(\tau)| < M_1\,, \quad |\dot{\underline{r}}| < M_2 \\ &D : \ 0 \leqslant \tau \leqslant \tau' \end{aligned}\right\} \tag{4}$$

where T the general force, $\underline{r}$ the radial vector from the attractive center to the center of mass of the body, $\underline{I}_0$ the impulse, which is given by:

$$\underline{I}_0 = \int_0^{t_0} T(t)\,dt\,, \qquad \tau = t - t_0\,. \tag{4. 1}$$

If we take the function:

$$\underline{r}(\tau) = a_1(\tau)\,\underline{r}_0^* + a_2(\tau)\,\underline{s}_0^* + a_3(\tau)\,\underline{T}_0^* \tag{5}$$

as a «trial solution», where $\underline{r}_0^*$, $\underline{s}_0^*$, $\underline{T}_0^*$ are special unit vectors, the coefficients a_1, a_2, a_3 must satisfy the following conditions in order that the function (5) is a «formal solution» of (4):

$$\left.\begin{aligned} &\ddot{a}_1 + \frac{\mu}{r^3}\,a_1 = T_1\,; \quad a_1(0) = r_0\,, \quad \dot{a}_1(0) = 0 \\ &\ddot{a}_2 + \frac{\mu}{r^3}\,a_2 = T_2\,; \quad a_2(0) = 0\,, \quad \dot{a}_2(0) = s_0 \\ &\ddot{a}_3 + \frac{\mu}{r^3}\,a_3 = T_3\,; \quad a_3(0) = 0\,, \quad \dot{a}_3(0) = 0 \end{aligned}\right\} \tag{6}$$

If T_1, T_2, T_3 are differentiable, $\dot{T}_1$, $\dot{T}_2$, $\dot{T}_3$ continuous, $r \neq 0$; a_1, a_2, a_3 twice differentiable, and $\ddot{a}_1$, $\ddot{a}_2$, $\ddot{a}_3$ continuous, we see that equations (6) satisfy the Hadamard's restrictions, when the functions: $a_1(\tau)$, $a_2(\tau)$, $a_3(\tau)$ can be uniquely determined from equations (6), and are continuous functions of the initial conditions of (6). Therefore, the solution (5) of equation (4), after the above restrictions of the force T and its derivative, is unique and depends continuously on the initial conditions of (4), then it can be accepted as an actual solution of the equation (4).

3rd Application **Problem of Non-Linear Mechanics: 3 (c)**

The Problem of Principal Modes of Non-Linear Systems.

The concept of «principal modes» of linear systems plays a predominant role in the analysis of the oscillatory systems of many fields.

The principal modes in linear systems are, by definition, the fundamental set of solutions of which a linear combination gives the general solution of the linear system; or, physically speaking, they are the special modes of oscillations of the linear system in terms of which we can discuss any kind of oscillations of the system.

Since the «principle of superposition» does not hold in non-linear systems, the concept of principal modes, as given above, is meaningless in non-linear systems, and the following problem may arise: *«Has the problem of principal modes of non-linear systems a physical meaning?»*; or *«How one can make the problem of principal modes of non-linear systems a well-posed problem?»*

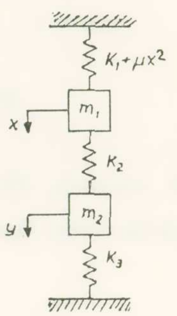

Figure 1.

The writer has publisched some papers in connection with this important problem, and transfers here some appropriate thoughts, techniques and results in order to give this problem as an example of the classical approach of the preceding paper.

We can find a new definition of the concept of principal modes for both the linear and non-linear systems, and such that the known definition in linear systems comes as a result from the new definition. The writer gave two new definitions which, under some conditions, are equivalent.

After that we try to make the physical problem correctly stated and the mathematical idealization well-posed.

We take a trial solution and make it formal, first, and then actual.

If we restrict ourselves to a «two-degrees-of-freedom» mechanical non-linear system, as shown in Figure 1, the equations of motion of the «two-masses-three springs» non-linear system are:

$$\left.\begin{aligned} \ddot{x} + \omega_1^2 x - \lambda_2 y + \lambda_1 x^3 &= 0 \\ \ddot{y} + \omega_2^2 y - \lambda_3 x &= 0 \end{aligned}\right\} \qquad (7)$$

where :

$$\omega_1^2 = \frac{K_1 + K_2}{m_1}, \quad \omega_2^2 = \frac{K_2 + K_3}{m_2}, \quad \lambda_1 = \frac{\mu}{m_1}, \quad \lambda_2 = \frac{K_2}{m_1}, \quad \lambda_3 = \frac{K_2}{m_2} \tag{7a}$$

and μ characterizes the non-linearity of one anchor spring.

By using the transformation :

$$x = x_1, \quad \dot{x} = x_2, \quad y = x_3, \quad \dot{y} = x_4 \tag{8}$$

the system (7) can be reduced to its normal form :

$$\left.\begin{array}{l} x_i = f_i\,(x_1, x_2, x_3, x_4), \quad i = 1, 2, 3, 4 \\ f_1 = x_2, \quad f_2 = -\omega_1^2 x_1 + \lambda_2 x_3 - \lambda_1 x_1^3, \quad f_3 = x_4, \quad f_4 = \lambda_3 x_1 - \omega_2^2 x_3 \end{array}\right\} \tag{9}$$

valid in a region R :

$$R : \; |x_i| < h, \quad i = 1, 2, 3, 4 \tag{9a}$$

The appropriate initial conditions for «principal modes» are in R :

$$x_1(0) = x_{10}, \quad x_2(0) = 0, \quad x_3(0) = x_{30}, \quad x_4(0) = 0 \tag{9b}$$

where x_{10} and x_{30} are appropriately related to each other.

Now we remark that the nature of the functions f_i of (9) are such that all Hadamard's restrictions are satisfied. These functions f_i are continuous in R, then bounded; they have continuous partial derivatives $\partial f_i / \partial x_k$ in R, when they satisfy Lipschitz conditions with respect to x_i in R for a Lipschitz constant $s = \mathrm{l.u.b}\, /\partial f_i / \partial x_k/$. The above properties assure the *unique* existence of the solution of (9) and (9b) in a region $R' \subset R$. As the initial point x_{i0}, $i = 1, 2, 3, 4$ varies in R', the solution satisfies the three Hadamard's restrictions, and the problem is «well-posed».

4th Application **A Problem of Underwater Warfare:**

The Problem of Domes. 1

The problem of domes arose in the winter of 1942 - 1943 in connection with «underwater warfare». As is known, underwater sound ranging depends on sending out a sound beam in water and, attached to a fast-moving ship, the water steaming around the plate causes serious disturbances. For elimination of these disturbances, the projector is closed in a so-called «dome», Figure 2, which is a convex shell of metal or other material filled with water. Such domes interfere only slightly with the formation of a concentrated sound beam. During 1942 - 1943, a large number of small submarines chases were built and equipped with sound gear similar to, but smaller than, the gear used before. While the manufacture of domes to fit this smaller gear was underway, it was discovered that these smaller domes led to an intolerable diffusion on the sound beam. At that time, a quick remedy was imperative, and a mathematical analysis of the problem was needed to support and speed-up experimented work.

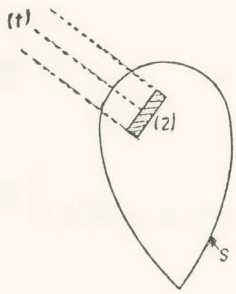

Figure 2.
(1) Axis of beam sound
(2) Projector
S Surface of Dome

The mathematical problem, related to the above physical problem, was to solve the differential equation:

$$\left.\begin{aligned} &\nabla^2 P + K^2 P = 0 \\ &\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \end{aligned}\right\} \qquad (10)$$

in which $K = \omega / c$, ω the frequency, c the sound velocity and K has for our problem, unfortunately, different values within the shell of the dome and outside.

This mathematical idealization was not a suitable one for the problem.

They found the suitable mathematical idealization by the following process. The actual dome of small finite thickness was replaced by an extremely thin surface, then the influence of the dome was simply replaced by conditions for jump discontinuities of the disturbance q of the beam across the surface.

These conditions are:

$$\left.\begin{aligned} &[q] = \frac{p_1}{p_0^{-1}} \cdot \frac{\partial p}{\partial n} \\ &\left[\frac{\partial q}{\partial n}\right] = \frac{p_0}{p}\left(K_0^2 - K_1^2\right) p - \left(1 - \frac{p_0}{p}\right)\left(\frac{\partial^2 p}{\partial n^2} + 2H \frac{\partial p}{\partial n}\right) \end{aligned}\right\} \qquad (11)$$

where the symbol $[\cdot]$ means jump of the quantity of the symbol across the surface, q is the disturbance of the acoustic pressure p caused by the dome and the normal derivatives $\frac{\partial}{\partial n}$ are to be evaluated on the surface S of the dome. The quantity H is the mean curvature of S, i. e. the average of the curvature of any two normal plane sections at right angles to each other. In addition to conditions (11) to be satisfied by q on S, q should be a solution of the equation :

$$\nabla^2 q + K_0^2 q = 0 \tag{12}$$

same behavior as P at ∞. This problem possess the unique solution :

$$q = -\frac{1}{4\pi}\iint\limits_{S}\left[\frac{\partial q}{\partial n}\right]\frac{e^{ik_0 r'}}{r'}ds + \frac{1}{4\pi}\iint\limits_{S}[q]\frac{\partial}{\partial n}\left(\frac{e^{ik_0 r'}}{r'}\right)dS \tag{13}$$

The quantities in the brackets are given by conditions (11), r′ is the distance from a fixed point (x, y, z) at which q (x, y, z) is to be determined to the point of integration on S. This formula yields the disturbance as the effect due to a layer of point sources and a layer of dipoles disturbed on S with intensities which are known as soon as the original pressure p is known, since the quantitites in brackets are fixed in value due to conditions (11). The relative directional disturbance :

$$\left|\frac{p_1}{p}\right| R.c.h\left(\frac{q_1}{p_1} - \frac{q_0}{p_0}\right)$$

would, finally, be obtained from (13). The solution (13) is valid for a shell of constant thickness, but it could be extended without essential error to cases in which the dome shell is made up of a not too large number of pieces, each of which is of constant thickness. All that would be necessary would be to insert a numerical factor d in the integrands on the right-hand side of expression (13), which would be precise constant on S. By this formula, one can analyze the contribution to the distortion of various factors, such as the curvature of the dome and the density and sould velocity within it.

The above kind of mathematical idealization, even without detailed numerical computation, proved helpful to the designing engineer.

5th Application **Biology, Ecology, Economics: 2, 5**

The Problem of Mixed Populations: Two Species Competing for a Common Food Supply.

For the study of the growth of two mixed populations of species in mutual interdependence of any kind, e.g. in competing for a common food supply, several models have been proposed. One of these models, of which the formulation is based on determinizing the time-rate of change of quantities as a function of the quantities and some parameters, is :

$$\left.\begin{aligned} \dot{x} &= a\,[b - x - f_1(y)]\,x \\ \dot{y} &= c\,[d - y - f_2(x)]\,y \end{aligned}\right\} \qquad (14)$$

x and y are the numbers (or masses) of individuals of the species present at any time, and a, b, c, d parameters of which the domain of possible change define the environment of the model.

The model (14) is either «well-posed» or «non-well-posed», depending on properties of the functions $f_1(y)$ and $f_2(x)$.

Physically, the quantities x and y are non-negative, when the region D of the validity of the model (14) is the first quadrant of the x, y - plane. The initial conditions x_0, y_0 of (14) is the starting point of the solution, if this solution exists, and this point lies in the region D.

If the functions $f_1(y)$ and $f_2(x)$ are defined, single-valued and continuous in the region D, then the right-hand numbers of (14) are continuous functions of all their arguments, when a solution of (14) necessarily exists through the point (x_0, y_0), and the first Hadamard's restriction is satisfied.

If, in addition, the functions $f_1(y)$ and $f_2(x)$ have continuous derivatives in y and x, respectively, then the right-hand numbers of (14) have continuous partial derivatives with respect to all their arguments, and the solution through (x_0, y_0) is unique and depends continuously on the x_0, y_0, when the second and third Hadamard's restrictions are satisfied, and the model (14) is a «well-posed» one.

We remark that : the solution of the model (14), which starts from the inital point (x_0, y_0), tends, as t increases, to a point $\bar{x}$, $\bar{y}$, and we may have three cases. First, the point $(\bar{x}, \bar{y})$ may be a point inside the region D, both $\bar{x}$ and $\bar{y}$ positive, when one can speak about the «co-existence of the species». Second, the point $(\bar{x}, \bar{y})$ may be identical with the origin, when one can speak about the «extinction of the species». Third, one of $\bar{x}$ and $\bar{y}$ may be zero and the other positive, and this case corresponds to the «principle of competitive exclusion», a principle much used in ecology, but which has been much criticized.

We remark that if the variables x and y of the model (14) are numbers of individuals of the populations they are restricted to be (positive) integers, when they are «step functions» of time, and the functions f_1 and f_2 of (14) are restricted to assume values according to permitted values of x and y. The functions f_1 and f_2 in this case have no properties, as mentioned above, which make the model (14) a «well-posed» one. In this case, the model (14) is not a «continuous system», but a «discrete system». If we assume that x and y in the model (14) are the masses of the populations, we can remove the above restriction of x and y and the function f_1 and f_2 regain the properties needed in order for the model to be a «well-posed» one.

All the above remarks and results can be applied to different social problems, if the competitive species and the limiting resources are appropriately specified.

To apply the above in the field of economics, the variables x and y must denote the size or extent of two commercial enterprises competing for common sources and for a common market.

6th Application **Modern Physics, Dynamic Meterology : 1**

The classical procedure, discussed in the preceding, and especially the step to find the «well-posed-model», combined with numerical analysis and the use of high-speed computers, gave and may give much success in the investigation of problems of great contemporary interest.

The «Synchrotron» and the «weather prediction» can be used as examples.

a. *Synchrotron.* The recently discovered «strong-focusing-principle» is the basis for the study of the multibillion-volt proton accelerators. This principle is related to the stability of solutions of ordinary linear differential equations of second order with periodic coefficients. The actual orbits, because of unavoidable imperfections of magnets and other causes, follow, approximately, linear periodic differential equations, and a modified nonlinear model is not possible. Experimental studies, under various assumptions, the use of computing and mathematical analysis, give encouragement to the designers for success.

b. *Weather Prediction.* According especially to Bjerkness, one may formulate the laws of atmospheric phenomena by models which are partial differential equations. Based on the today's data and using the Bjerkness model as a «well-posed» one, the prediction of tomorrow's weather would require qualified computer men with desk computing machines for much time.

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ΠΕΡΙΛΗΨΙΣ

Εἰς προηγουμένην ἐργασίαν **3(α)** ἀνεπτύχθη μέθοδος ἐρεύνης «φυσικῶς δεκτῶν» μαθηματικῶν λύσεων εἰς φυσικὰ καὶ κοινωνικὰ συστήματα καὶ ὑπεδείχθησαν δυσκολίαι ἐφαρμογῆς τῆς μεθόδου, ὅπως καὶ τὰ πλεονεκτήματά της. Εἰς τὴν παροῦσαν ἐργασίαν δίδονται ἐφαρμογαὶ τῆς μεθόδου εἰς διάφορα πεδία ἐρεύνης, ὡς, λ. χ., εἰς τὴν θερμοδυναμικήν, Ἀστροδυναμικήν, μή-γραμμικὴν μηχανικήν, βιολογίαν, κ.λ.π.

★

Ὁ Ἀκαδημαϊκὸς κ. **Ἰω. Ξανθάκης** κατὰ τὴν ἀνακοίνωσιν τῆς ἀνωτέρω ἐργασίας εἶπε τὰ κάτωθι :

Ἔχω τὴν τιμὴν νὰ παρουσιάσω εἰς τὴν Ἀκαδημίαν Ἀθηνῶν τὸ δεύτερον μέρος τῆς ἐργασίας τοῦ κ. Δημητρίου Μαγείρου, ὑπὸ τὸν τίτλον :

«Δεκταὶ Μαθηματικαὶ Λύσεις Φυσικῶν Προβλημάτων».

Εἰς τὴν προηγηθεῖσαν ἀνακοίνωσίν του ὁ κ. Μάγειρος παρουσίασε μίαν μέθοδον ἐρεύνης μαθηματικῶν λύσεων φυσικῶν καὶ κοινωνικῶν συστημάτων «φυσικῶς ἀποδεκτῶν». Ὑπέδειξε δὲ τὰς δυσκολίας ἐφαρμογῆς τῆς μεθόδου ταύτης, ὅπως καὶ τὰ πλεονεκτήματά της, εἰς τὰς περιπτώσεις καθ' ἃς δύναται νὰ ἐφαρμοσθῇ.

Εἰς τὴν παροῦσαν ἀνακοίνωσιν παρέχονται αἱ ἐφαρμογαὶ τῆς ἐν λόγῳ μεθόδου εἰς διάφορα πεδία ἐρεύνης, ὅπως λ. χ. εἰς τὴν θερμοδυναμικήν, εἰς τὴν ἀστροδυναμικήν, ἐπὶ τοῦ προβλήματος τῆς κινήσεως τεχνητοῦ δορυφόρου ὑπὸ τὴν ἐπίδρασιν μιᾶς κεντρικῆς δυνάμεως πληρούσης τὸν νόμον τοῦ Νεύτωνος καὶ μιᾶς ὠστικῆς τοιαύτης ἐπενεργούσης ἐπί τι χρονικὸν διάστημα, καθὼς καὶ ἐπὶ προβλημάτων μή-γραμμικῶν Μηχανικῆς καὶ Βιολογίας.