ΜΑΘΗΜΑΤΙΚΑ.— **On the Generalized Poincaré Conjecture and the Structure of Manifolds,** *by George M. Rassias**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φ. Βασιλείου.

1. Introduction. H. Poincaré was the first who invented the connection between the topology of a space and the number of critical points of a function defined on it.

Then, M. Morse continued the work of Poincaré and he created a new theory known as "Morse theory" studying the connection between the topological structure of a manifold M and the Morse functions defined on M.

S. Smale [**14**, **15**], verified the importance of Morse theory in attacking long-standing unsolved problems of the field of Topology. S. Smale proved a fundamental theorem on the structure of manifolds the so called (h-cobordism theorem) which is one of the deepest and most important theorems in the fields of Topology and Geometry.

He also proved the generalized Poincaré conjecture, (that is, any homotopy n-sphere, $n > 4$ is the n-sphere), and also the generalized Schoenflies conjecture.

Through that work, he generalized the well-known construction of any closed orientable surface in the form of a sphere with handles, by inventing the construction of attaching handles to higher dimensional manifolds.

2. The connection between the topological structure of a manifold M and the number and type of critical points of Morse functions defined on M, is expressed by the following Morse inequalities. Let M be a closed C^{∞} differentiable manifold, and f a Morse function defined on M. Denote by c_i the number of critical points of index i of f, and by R_i the

---

* Γ. Μ. ΡΑΣΣΙΑΣ, **Ἡ γενικευμένη εἰκασία Poincaré καὶ ἡ δομὴ τῶν πολλαπλοτήτων.**

i-th Betti number of M, i.e., R_i is the rank of the homology group $H_i(M)$. Note $n = \dim M$. The Morse inequalities are the following:

$$c_0 \geqq R_0$$
$$c_1 - c_0 \geqq R_1 - R_0$$
$$c_2 - c_1 + c_0 \geqq R_2 - R_1 + R_0$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$c_n - c_{n-1} + \ldots + (-1)^n c_0 = R_n - R_{n-1} + \ldots + (-1)^n R_0 .$$

The last of the Morse inequalities in fact is an equality and equals to the Euler characteristic of M, a topological invariant of M.

A corollary of these inequalities is that $c_i \geqq R_i$.

The advantage of the Morse inequalities is that they are expressed in terms of the Betti numbers of the manifold which are topological invariants and can be calculated using Algebraic Topological (homology) techniques. See, R. Bott [**1**].

The following theorem expresses the relation between the Morse functions on a manifold M and the topology of M.

Theorem (1). *Let* f *be a Morse function on* M, *i.e. a* C^∞ *function on* M, *constant on* ∂M *with only non-degenerate critical points* $P_1, P_2, \ldots, P_m$ *such that*

$$f(P_1) < f(P_2) < \ldots < f(P_m).$$

Let $M_i \subseteq M$ *be the submanifold*

$$M_i = \{ x \in M / f(x) \leqq c_i \}$$

where c_i *is a real number where* $f(P_i) < c_i < f(P_{i+1})$.

Then $$M_1 \subset M_2 \subset \ldots \subset M_n = M$$

and $$M_i \approx M_{i-1} \bigcup_{\Phi_i} D^{K_i} x D^{n-K_i}$$

where K_i *is the index of* f *at the critical point* P_i *and*

$$\Phi_i : \partial D^{K_i} x D^{n-K_i} \to \partial M_{i-1} \subseteq M_{i-1}$$

is a differentiable imbedding (i. e., a critical point of index κ_i corresponds to attaching a thickened cell of dimension κ_i).

Therefore, any Morse function f on M induces a specific geometric decomposition of M

$$M \approx D^{K_0} x D^{n-K_0} \underset{\Phi_1}{\cup} D^{K_1} x D^{n-K_1} \underset{\Phi_2}{\cup} \ldots\ldots \underset{\Phi_m}{\cup} D^{K_m} x D^{n-K_m}$$

which plays an important role for the understanding of the topological structure of M.

3. The Generalized Poincaré Conjecture is that every closed n-manifold which has the homotopy type of the n-sphere S^n, is in fact homeomorphic to the n-sphere.

We are going to indicate the main theorems of S. Smale that gave a proof to the Generalized Poincaré Conjecture for $n \geqq 5$.

Theorem (1). *Let* M^n *be a closed* (C^∞) *differentiable manifold which has the homotopy type of the n-sphere* S^n, $n \geqq 5$. *Then* M^n *is homeomorphic to* S^n.

Definition. Let M^n be a compact manifold with boundary ∂M, and consider D^K is the κ-disk.

Assume $f : (\partial D^K) x D^{n-K} \to M$ is an imbedding.

Then, $H(M; f; \kappa)$, is defined by imposing a differentiable structure on

$$M \underset{f}{\cup} D^K x D^{n-K}$$

and identifying points under f.

Similarly, if

$$f_i : (\partial D_i^K) x D_i^{n-K} \to M, \quad i = 1, \ldots, m$$

are imbeddings with disjoint images, $H(M; f_1, \ldots, f_m; \kappa)$ can be defined.

In the case where M is a disk, then $H(M; f_1, \ldots, f_m; \kappa)$ is called a "handlebody".

T h e o r e m (2). *Let* $n \geqq 2\varkappa + 2$, *or if* $\varkappa = 1$, $n \geqq 5$, M^{n-1} *be a simply connected,* $(\varkappa - 1)$ *- connected closed manifold and* $\mathscr{C}_M(n, m, \varkappa)$ *the set of all manifolds which have presentation of the form,*

$$(M \times [0,1], \; M \times 1; \; f_1, \ldots, f_m; \; \varkappa).$$

Now, let $C \in \mathscr{C}_M(n, m, \varkappa)$, $N = \partial C - M \times O$,

$$U = H(C, N; h_1, \ldots, h_\nu; \varkappa + 1)$$

and assume that the map $\pi_\varkappa(M \times O) \rightarrow \pi_\varkappa(U)$ *is an isomorphism. Assume, also, if* $\varkappa = 1$, *that*

$$\pi_1(H(C, N; h_1, \ldots, h_{\nu-m}; 2)) = 1.$$

Then, $U \in \mathscr{C}_M(n, \nu - m, \varkappa + 1)$.

By taking M to be the (n — 1)-sphere, we have the so called "handlebody theorem".

A consequence of the handlebody theorem is the following theorem which implies the e v e n dimensional part of the Generalized Poincaré Conjecture (for $n \geqq 5$).

T h e o r e m (3). *Let* M *be a closed* $(\varkappa - 1)$ *- connected* C^∞ $2\varkappa$*-manifold,* $\varkappa \neq 2$. *Then, there exist a Morse function on* M *whose type numbers equal the Betti numbers of* M.

D e f i n i t i o n. Two closed (i.e. compact without boundary) C^∞ differentiable manifolds M and N are h-c o b o r d a n t if there exists a C^∞ differentiable manifold W whose boundary $\partial W = M - N$, and each of M, N is a deformation retract of W.

The o d d - dimensional part of the Generalized Poincaré Conjecture follows from the following restatement of the so called h-cobordism theorem.

T h e o r e m (4). *Let* M, N *be closed simply-connected manifolds of dimension* $n \geqq 5$, *which are h-cobordant. Then,* M, N *are diffeomorphic.*

This theorem has many important consequences in Topology, such as the following corollaries.

C o r o l l a r y (1). *Two simply-connected, closed manifolds of dimension* $n \geqq 5$ *which are h-cobordant, are diffeomorphic.*

C o r o l l a r y (2). *Let* C^n, $n \geqq 6$ *be a compact, simply-connected manifold with simply-connected boundary. Then, the following are equivalent, (a)* C^n *is diffeomorphic to the* n*-disk* D^n, *(b)* C^n *has the homotopy type of a point, (c)* C^n *has the homology of a point.*

R e m a r k. The above corollary remains true for the case $n = 5$ provided ∂C^n is diffeomorphic to the 4-sphere.

C o r o l l a r y (3). *Let* M *be an imbedded* (n-1)*-sphere in* S^n. *If* $n \geqq 5$, *then the closure of each component of* $S^n - M$ *is diffeomorphic to* D^n.

4. In this section an application is made of the Morse theory to prove theorems of Algebraic Topology.

T h e o r e m (5). *The Euler characteristic of any closed* odd-*dimensional manifold* M *equals to zero i. e.* $x(M) = 0$.

P r o o f: Making use of the last (equality) of the Morse inequalities, namely,

$$\sum_{i=0}^{n} (-1)^i c_i(f) = \sum_{i=0}^{n} (-1)^i R_i \left(= x(M) \right).$$

let $f: M \to R$ be a Morse function of type $(c_0, \ldots, c_n)$, and $a < f(x) < b$, for all $x \in M$. Thus we assume that $f: M \to [a, b]$ is a Morse function such that

$$\partial M = f^{-1}(a) \cup f^{-1}(b)$$

where a, b are regular values.

Applying the following

$$x(M) = \sum_{i=0}^{n} (-1)^K c_K(f)$$

to f and $-$f, we obtain

$$x(M) = \sum_{K=0}^{n} (-1)^{K} c_K (f) = \sum_{K=0}^{n} (-1)^{K} c_{n-K} (-f) =$$
$$= (-1)^n \sum (-1)^{n-K} C_{n-K} (-f) = -x(M).$$

Hence, $x(M) = 0$ Q. E. D.

T h e o r e m (6). *The Euler characteristic of the boundary of any compact topological manifold is an even natural number.*

P r o o f. If $\dim M = 2k$, then $\dim(\partial M) = 2k - 1$, where ∂M is the boundary of M. Therefore $x(\partial M) = 0$ because of the previous theorem.

If the dimension of M is odd, then we consider the «double» manifold of M,

$$N = M \bigcup_{\partial M} M$$

which is compact without boundary.

Then, $$x(N) = 2\,x(M) - x(\partial M)$$

and therefore

$$x(\partial M) = 2\,x(M)$$

because the dimension of N is odd, and so $x(N) = 0$.

Hence, $x(\partial M)$ is even.

Q. E. D.

T h e o r e m (7). *Let* M *and* N *be closed, orientable manifolds. Then,*

$$x(M x N) = x(M) \cdot x(N)$$

P r o o f. Let $f : M \to R^+$ be a positive Morse function defined on M, and let $x_1, \ldots, x_r$ be the critical points of f.

Also, let $c_i = f(x_i)$, be the critical values corresponding to f, for $i = 1, \ldots, r$.

Similarly, let $g : M \to R^+$ be a positive Morse function defined on N, and let $y_1, \ldots, y_s$ be the critical points of g.

Also, let $d_j = g(y_j)$. be the critical values corresponding to g, for $j = 1, \ldots, s$.

It may be assumed that $c_i d_j \; (= f(x_i) \; g(y_j))$ are pairwise distinct.

Then the function defined by

$$h(x, y) = f(x) \, g(y)$$

is a Morse function on the product MxN.

The critical points of h are the pairs (x_i, y_j), and if x_i is a critical point of index p of f, and if y_j is a critical point of index q of g, then (x_i, y_j) is a critical point of index $p + q$ of $h = fxg$.

Consequently, if C_p is the number of critical points of index p of f, and if D_q is the number of critical points of index q of g, then the number of critical points of index r of $h = fxg$ is the following:

$$B_r = \sum_{p+q=r} C_p \cdot D_q$$

Thus,

$$x(MxN) = \sum_{r=0}^{m+n} (-1)^r \cdot B_r = (\sum_{p=0}^{m} (-1)^p C_p) \cdot (\sum_{q=0}^{n} (-1)^q C_q) = x(M) \cdot x(N).$$

Hence, $x(MxN) = x(M) \cdot x(N)$.

Q. E. D.

ΠΕΡΙΛΗΨΙΣ

Ἡ παροῦσα ἐργασία ἀναφέρεται εἰς τὴν θεωρία τοῦ Morse ὡς καὶ εἰς τὰς ἐφαρμογάς της εἰς τὸ θεμελιῶδες πρόβλημα τῆς ταξινομήσεως τῶν πολλαπλοτήτων ὅσον ἀφορᾶ τὴν δομὴ αὐτῶν.

Εἰδικῶς ἀναφέρεται εἰς τὴν ἀπόδειξιν τοῦ θεωρήματος τοῦ h-συμφραγμοῦ, τῆς Γενικευμένης Εἰκασίας τοῦ Poincaré ὡς καὶ ὡρισμένων ἀξιολόγων θεωρημάτων τῆς Ἀλγεβρικῆς Τοπολογίας.

REFERENCES

1. R. Bott, «Lectures on Morse theory», Bonn, 1960.
2. J. Cerf, «Sur les difféomorphismes de la sphère de dimension trois ($\Gamma_4 = 0$)», Springer, Math. no. 53 (1968).
3. S. Eilenberg - N. Steenrod, «Foundations of Algebraic Topology», Princeton Univ Press, 1952.

4. M. Kervaire and J. W. Milnor, «Groups of homotopy spheres», Ann. of Math. 77 (1963).
5. B. Mazur, «Relative neighborhoods and the theorems of Smale», Ann. of Math. 77 (1963).
6. J. Milnor, «On manifolds homeomorphic to 7-sphere», Ann of Math. 64 (1956).
7. G. M. Rassias, «On the Morse-Smale characteristic of a differentiable manifold», Bulletin of the Australian Math. Society, vol. 20 (1979), no. 2.
8. —, «Counter-examples to a conjecture of René Thom», Springer-Verlag Lecture Notes in Math. no. 792, Berlin - New Υork 1980.
9. —, «Covering of differentiable manifolds with open disks», Archiv. der Mathematik, 3 (1981), Germany.
10. —, «On the nondegenerate critical points of differentiable functions», Tamkang Journal of Mathematics, vol. 20, no. 1 (1979), Taiwan, Republic of China.
11. —, «Algebraic and Differential Topology-Global Differential Geometry, Teubner-Texte zur Mathematik, BSB B. G. Teubner Publishers, Leipzig, 1984, DDR.
12. G. De Rham, «Varietes differentiables», Paris, Hermann, 1955.
13. S. Seifert and W. Threlfall, «Lehrbuch der Topologie», N. Υ., Chelsea (1947).
14. S. Smale, «Generalized Poincaré conjecture in dimensions greater than four», Ann. of Math. (2) 74 (1961).
15. —, «On the structure of manifolds», Amer. J. Math. vol. 84, (1962).