ΜΑΘΗΜΑΤΙΚΑ.— **On the completion of ordered fields,** *by Carla Massaza**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Κ. Παπαϊωάννου.

INTRODUCTION

The principal aims of the present paper, which follows two preceding Notes ([**5**], [**6**]) on ordered fields and is a prepublication copy of a forthcoming Note, are to construct an ordered field which is the completion of a given one, and to prove that:

The completion of a maximal ordered field[1], *in the uniform topology induced by its order, is still maximal ordered.*

To this aim, given an ordered field K, I take, in the set Π(K) of K's sections (n. 1), the subset D(K) of its Dedekind's sections, and I prove that it is possible to introduce in D(K) one and only one structure of ordered field, which is an extension of K's one; moreover, D(K) is the maximal ordered field in which K is dense.

Having proved that D(K) is the completion $\hat{K}$ of K in the natural topology of K (n. 2), I reach the proposition by proving that, if M is a maximal ordered field and $\beta \in \overline{\hat{M}}(i)$ a root of a polynomial with coefficients in $\hat{M}(i)$ $(i = \sqrt{-1})$, every neighbourhood of β has a nonempty intersection with M(i).

Then, I observe that $\hat{\bar{K}}$ is the least overfield of K which is maximal ordered and complete, and I show in an exemple that, in the general case, we have: $\hat{\bar{K}} \neq \bar{K}$ $\hat{\bar{K}} \neq \hat{K}$; this fact enables us to construct nonarchimedean fields, which are not maximal ordered but are dense in their ordered closure, and solve, in this way, the problem proposed in [**5**], pag. 15.

After reaching these results, I knew that S. P. Zervos had already obtained that theorem, in a paper of 1961. His method of completing an ordered field consists in considering ordered fields as topological fields, and in showing that the complete ring of an ordered field K is an order-

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1. A «maximal ordered field» is what they call, sometimes, in English, a «real closed field».

ed overfield of K. From a certain point of view, his method can be considered as a generalization of Cantor's method for the construction of the real field, while I followed Dedekind's method. S. P. Zervos reaches the main result by using his theorem about the continuity of the roots of a polynomial as functions of its coefficients.

The different method which I followed, and some other propositions which can be found here, lead me to publish this Note.

1. Overfield of Dedekind's sections.

I. Definition. *We say that a subset* ξ *of an ordered field* K *is a section if:*

P. 1) $\xi \neq \varnothing, \quad \xi \neq K$.

P. 2) $\exists\, a \in \xi \,/\, b < a \Rightarrow b \in \xi$.

P. 3) ξ *has not a maximum.*

In particular, we set: $\xi_\alpha = \{x \in K : x < a\}$.

We call a section ξ *a «Dedekind's section» if it verifies the additional property:*

P. 4. D.) $\forall\, \varepsilon \in K^+, \quad \exists\, a \in \xi \,/\, a + \varepsilon \notin \xi$.

We define $f : K \to \Pi(K)$ by $f(a) = \xi_\alpha$ and denote by $\bar{f}$ the map $K \to f(K)$, where $\bar{f}(a) = f(a)$, for all $a \in K$; one can than, obviously, transport through $\bar{f}$ the structure of an ordered field possessed by K on $f(K)$; then, $\bar{f}$ becomes an isomorphism of ordered fields.

We say that a subset H of $\Pi(K)$, containing $f(K)$, admits a structure of ordered overfield of K, if it admits a structure making it an ordered overfield of $f(K)$, such that the order relation is the restriction to H of the inclusion relation defined in the set $P(K)$ of K's subsets.

While the set of sections of an ordered field K, which is not archimedean, *does not* admit a structure of overfield of K, we have that:

II. *The set* $D(K)$ *of Dedekind's sections of an ordered field* K *admits one and only one structure of ordered overfield of* K.

Let us denote by $\tau(K)$ the «*natural topology* of K» or open-interval topology, a base of which is the following:

$$u = \{U_K(x, \varepsilon)\}_{(x, \varepsilon) \in K \times K^+}, \quad \text{where} \quad U_K(x, \varepsilon) = \{y \in K : |x - y| < \varepsilon\}.$$

In **[6]**, prop. I, we have seen that:

III. *Let* H *be an ordered overfield of* K: *the restriction of* $\tau(H)$ *to* K *is* $\tau(K)$ *iff we cannot find positive elements of* H *which are less than every positive element of* K.

In particular:

IV. *If* H *is an ordered algebraic extension of* K, $\tau(H)$ *induces* $\tau(K)$ *on* K.

Using this property, we show that:

V. *If* K *is an ordered subfield of* H, *and* $\tau(H)$ *induces* $\tau(K)$ *on* K, *the map:*

$$j : \xi \in D(K) \to j(\xi) = \{h \in H : \exists k \in \xi / h \leqslant k\},$$

is an imbedding of the ordered field D(K) *into the ordered field* D(H).

In this situation, we say that D(K) is an ordered subfield of D(H), by identifying D(K) with its image jD(K).

According to [**5**], prop. III, we define:

$$\varphi : \alpha \in D(K) \to \xi_\alpha \in D(D(K)).$$

Then we have:

VI. *The set* D(D(K)) *coincides with* $\varphi(D(K))$; *so, it can be identified, in a natural way, with* D(K).

From V and VI we deduce that:

VII. $K \leqslant H \leqslant D(K) \to D(H) = D(K)$.

Furthermore, the field D(K) can be caracterized as follows:

VIII. D(K) *is the greatest ordered field in which* K *is dense.*

2. Completion of an ordered field.

Another characteristic property of the ordered field D(K) is the following:

IX. D(K) *is the completion of the ordered field* K, *in the uniform topology of* K.

In particular, if K satisfies the first axiom of countability, we have these propositions:

X. D(K) *is the completion of* K *with respect to Cauchy's sequences.*

This result agrees with the well known construction of the completion for an archimedean field. Also, of course,

XI. *The series* $S = \sum a_n$, *whose elements belong to a non-archimedean field* K, *satisfies the Cauchy condition iff lim* $a_n = 0$.

XII. *In a complete, non-archimedean field, a series converges iff its general term tends to zero.*

3. Ordered fields which are maximal and complete.

Let K(i) be the algebraic extension of the ordered field K, by means of $i = \sqrt{-1}$; we call *natural topology of* K(i) the product topology of the natural topology of K. The following facts are true:

XIII. M(i) *is complete iff* M *is complete.*

XIV. *Let* $B(x) = \sum_{h=0}^{n} b_h x^h \; (b_n > 0)$ *be a polynomial whose coefficients belong to* $\hat{M}(i)$, *where* M *is maximal ordered. If* β *is a root of* B(x) *in the algebraic closure* $\overline{\hat{M}}(i)$ *of* $\hat{M}(i)$ *and* ε *is a positive element of* M, *the circle* $V(\beta, \varepsilon)$, *centered in* β *with radius* ε, *has a nonempty intersection with* M(i).

As easy consequences of prop. XIV and VIII, we find that:

XV. *Each maximal ordered field* M *is dense in its overfield* $\hat{\overline{M}}$.

XVI. *The completion of a maximal ordered field is maximal ordered.*

Hence:

XVII. $\hat{\overline{K}}$ *is the least* K's *overfield, which is maximal ordered and complete and whose topology induces* $\tau(K)$ *on* K.

4. Non-archimedean fields, which are dense in their ordered closure.

In general, we have: $\hat{\overline{K}} \neq \overline{K}$, $\hat{\overline{K}} \neq \hat{K}$. In fact, if $K = Q(x)$ is ordered by setting x greater then every rational, $\widehat{Q(x)}$ is not maximal ordered, for otherwise it should contain an ordered closure of K, while $\sqrt{x} \notin \widehat{Q(x)}$; moreover, $\overline{Q(x)} \neq \overline{\widehat{Q(x)}}$, because the series $e^{1/x} = \sum_{n=0}^{\infty} (n!\,x^n)^{-1}$ belongs to $\widehat{Q(x)}$, but is transcendent over Q(x).

Using this fact, we can construct non-archimedean fields, which are dense in their ordered closure, solving a problem rised in **[5]**, pag. 15. More generally, we can say that:

XVIII. *The ordered non maximal fields, which are dense in their ordered closure, are the non maximal fields whose comoletion is maximal ordered.*

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ΠΕΡΙΛΗΨΙΣ

Ἀποδεικνύεται, ἐδῶ, τὸ θεώρημα: Ἡ πλήρωσις «μεγίστου διατεταγμένου σώματος», ὡς πρὸς τὴν uniform τοπολογίαν τὴν ἐπαγομένην ὑπὸ τῆς διατάξεως αὐτοῦ, εἶναι καὶ αὐτὴ «μέγιστον διατεταγμένον σῶμα».

Τοῦτο τὸ θεώρημα εἶχε, τὸ πρῶτον, ἀποδειχθῆ, ἀπὸ ἄλλης σκοπιᾶς, κατὰ τὸ ἔτος 1961, ὑπὸ τοῦ Σ. Π. Ζερβοῦ.

★

Ἐπὶ τῆς ἀνακοινώσεως τῆς δεσποινίδος Carla Massaza «Περὶ τῆς πληρώσεως τῶν διατεταγμένων σωμάτων» ὁ Ἀκαδημαϊκὸς κ. **Κ. Παπαϊωάννου** εἶπε τὰ ἑξῆς :

Ἡ Ἰταλὶς μαθηματικὸς δεσποινὶς Carla Massaza ἔχει δημοσιεύσει ἐνδιαφερούσας μαθηματικὰς ἐργασίας ἐπὶ τῶν διατεταγμένων σωμάτων εἰς Ἰταλικὰ μαθηματικὰ περιοδικά.

Ἡ παροῦσα ἐργασία της ἀποτελεῖ περίληψιν μακρᾶς ἐργασίας της, ἡ ὁποία θὰ δημοσιευθῇ εἰς τὴν Ἰταλίαν. Εἰς αὐτὴν γενικεύεται εἰς τὰ τυχόντα μέγιστα διατεταγμένα σώματα τὸ καλούμενον «θεμελιῶδες θεώρημα τῆς Ἀλγέβρας». Ὡς ἀναφέρει, μετὰ τὴν σύνταξιν τῆς ἐργασίας της διεπίστωσεν ὅτι τὸ θεώρημά της εἶχεν ἤδη εὑρεθῆ καὶ ἀποδειχθῆ ἀπὸ τοῦ ἔτους 1961 ὑπὸ τοῦ Σ. Π. Ζερβοῦ εἰς ἀνακοίνωσίν του εἰς τὴν Ἀκαδημίαν Ἀθηνῶν. Ἡ διαφορὰ ἀρχικῆς ἀντιμετωπίσεως τοῦ θέματος κάμνει τὴν ἐργασίαν της νὰ παρουσιάζῃ καὶ αὐτὴ πολὺ ἐνδιαφέρον.