ΜΑΘΗΜΑΤΙΚΑ.— **A generalized approach to Morse theory and Plateau's problem,** *by Themistocles M. Rassias* *. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φίλωνος Βασιλείου.

1. It is Euclid who asked the fundamental question :

Find the shortest line which may be drawn from a point to a given line, and it is Apollonius of Perga who in the fifth book of conics posed the problem about *the determination of the shortest line which may be drawn from a point to a given conic section.* It is thus seen that a sort of theory of maxima and minima was known long before the discovery of the differential calculus and it may be justified in saying that efforts to develop this theory influenced the discovery of calculus.

Fermat, for example, after making numerous restorations of two books of Apollonius often cites this old geometer in his *«method for determining maximum and minimum»*, 1638, a work which in some instances is so closely related to the calculus that Lagrange, Laplace, Fourier, and others wished to consider Fermat as the discoverer of the calculus.

This is something he probably would have done if he had started from a somewhat more general point of view, as in fact was done by Newton (*Opuscula Newton*, I, 86 - 88). Descartes has already remarked, in a letter of March 1, 1638, that Fermat's rule for finding maxima and minima was imperfect, and we know that many imperfections still existed for a long time after the invention of the calculus by Newton. Weierstrassin his lectures in the University of Berlin (published as a Bulletin of the University of Cincinnati in 1903) explained that to a considerable degree these inaccuracies are due to one of the greatest mathematicians, Lagrange, and they have been diffused in the French school by Bertrand, Serret, and others.

The theory of equilibrium points or critical points of functions appear in fragmentary form in the work of Poincaré, Maxwell and Kronecker.

Birkhoff introduced the minimax principle and applied it in dynamics. Marston Morse initiated a systematic study of critical points of

* ΘΕΜΙΣΤΟΚΛΗ ΡΑΣΣΙΑ, **Γενίκευσις θεωρίας τοῦ Morse καὶ τὸ πρόβλημα τοῦ Plateau.**

functions on n variables in 1922. In 1924 the Variational Analysis in the large was introduced and developed as an extension of the theory of critical points of n-variables. The integrals used were ordinary and were regarded as functions of the curves along which they were evaluated.

In 1937 the theory was put on a more general basis with the function defined on a abstract metric space. In this general theory one is free from any limitation of dimension. It does not matter whether the independent variable is a point in a Euclidean space, a curve, a surface or a more general configuration. When the function is an integral in the variational Analysis for example, the integral of length on a surface or the area of a surface bounded by a curve, a critical point (curve or surface) is one satisfying the Euler - Lagrange equations.

The unification which appears in the general theory is made possible by a topological definition of a critical point, a definition which is independent of the particular case at hand. The years 1929 - 1936 saw a rapid development of the Theory in the large of integrals depending on a curve. There remained the multi-dimensional variational problems which are very important in Mathematical Physics.

2. As it had been remarked in T. Rassias [**3**] one of the most difficult problems in Global Variational Analysis is Plateau's problem. This is the problem of determining the surfaces of minimum area spanned in a given curve or subject to other boundary conditions. Lagrange had posed the problem of finding a minimal surface for a given contour and Plateau gave a physical realization by means of physical experiments.

R. Palais and S. Smale (cf. [**1**], [**2**], [**7**]) have found an extension of Morse theory of critical points to a certain class of functions on Hilbert manifolds. This theory is applicable to some variational problems and partial differential equations (systems) on vector bundles.

It was a problem if the Palais — Smale theory on Hilbert manifolds can be applied to the Plateau's problem. In [**3**] the author had proved that this theory is not applicable for the Plateau's problem and a new theory was proposed as a generalization of the Morse - Palais - Smale theory for Hilbert (or Banach) manifolds. It was stated in T. Rassias [**3**] that if the conjecture in [**3**; p. 369] is true then the problem of applying the Morse theory on Hilbert manifolds to the Plateau's problem is completely solved. This conjecture was finally proved and *the proof of it was* *announced by the author in the international Congress of Mathematicians in Helsinki, August 1978.* This way one obtains a global analytic proof of the Morse-Tompkins and Shiffman [6] theorem which states that the existence of two relative minima for the energy functional implies the existence of a third unstable minimal surface. Thus one gets a special theory for the problem of unstable minimal surfaces bounded a given closed curve in a Euclidean space.

3. Let Ω be an open set in R^n and consider a C^2 immersion $F:\Omega \rightarrow R^{n+k}$.

Definition. F is a «*minimal immersion*» if and only if F satisfies the system

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial F}{\partial x^j}\right) = 0 \tag{1}$$

where $$g = \det\left((g_{ij})\right), \quad \left((g^{ij})\right) = \left((g_{ij})\right)^{-1}$$

and $$g_{ij} = \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle .$$

It is easy to prove that the above is equivalent to the requirement that $F(\Omega)$ have mean curvature identically zero.

Definition. The immersion F is said to be «*non-parametric*» if it has the form $F(x) = (x, f(x))$ for some function $f:\Omega \rightarrow R^k$.

Then (1) has the form:

$$\left.\begin{aligned} &\sum_{i=1}^{n} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij}\right) = 0 ; \quad j = 1, 2, \ldots, n \\ &\sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial x^j}\right) = 0 \end{aligned}\right\} \tag{2}$$

where g and (g^{ij}) are defined above, and in this case

$$g_{ij} = \delta_{ij} + \left\langle \frac{\partial f}{\partial x^i}, \frac{\partial f}{\partial x^j} \right\rangle .$$

Definition. The *generalized parametric minimal surface* in R^n having a given Jordan curve Γ as boundary is defined as a map

$F : D \to R^n$, where $D = \{z = x + iy \in \mathbf{C} : |z| \leqslant 1\}$, with the following properties being satisfied:

$$F \varepsilon C^0(D) \cap C^\omega(\overset{\circ}{D}), \tag{1}$$

$$\Delta F = 0, \tag{2}$$

$$\langle F_x, F_x \rangle = \langle F_y, F_y \rangle, \quad \langle F_x, F_y \rangle = 0, \tag{3}$$

$$F|_{\vartheta D} : \vartheta D \to \Gamma \quad \text{is a homeomorphism.} \tag{4}$$

Theorem. Consider Γ to be a Jordan curve in R^{n+k} and suppose that Γ can be expressed as the graph of a continuous function $f : \vartheta\Omega \to R^k$ where Ω is abounded, convex domain in R^k. Then every generalized parametric minimal surface with boundary Γ has a one-to-one, nonsingular projection onto Ω, i.e. every such surface can be expressed as the graph of a function $\Psi \varepsilon C^0(\bar{\Omega}) \cap C^\omega(\Omega)$ and Ψ satisfies the relation (2) in Ω, $\Psi : \bar{\Omega} \to R^k$.

The above theorem generalizes the following very interesting theorem due to T. Rado.

Theorem. A Jordan curve Γ whose orthogonal projection on some plane is a simply corered convex curve bounds a unique area minimizing surface.

By applying techniques from Morse theory one can also prove the following:

Theorem. Let $\Gamma \subset C^{3,\alpha}$ be a Jordan curve in R^3 with total curvature $K(\Gamma) < 4\pi$. Then these exists a unique solution of Plateau's problem for Γ.

Theorem. Let Γ be an extreme Jordan curve in R^3 with $K(\Gamma) \leqslant 4\pi$. Then there exists an embedded solution of Plateau's problem for Γ.

Remark. It can be easily proved that embedded surfaces are open in the C^1-topology.

4. In 1900, Liebmann proved the theorem that every *convex* surface of constant mean curvature is a sphere. The problem was also proved by Jelled in 1853 for starshaped surfaces. It is clear that neither theorem answers *the question about the shape* of a general closed surface of constant

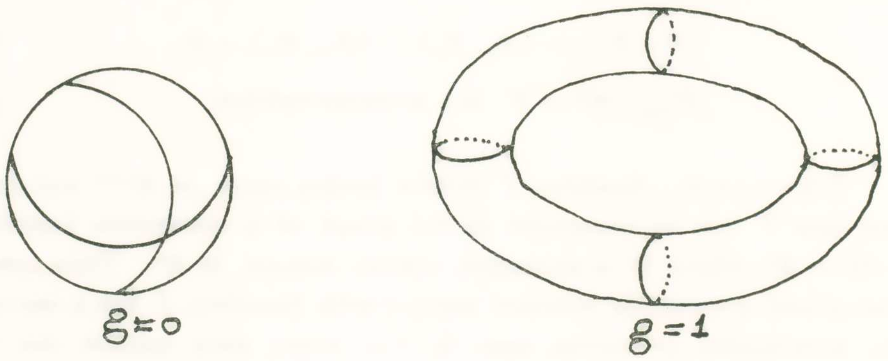

Fig. 1.

mean survature. Aside from the sphere (of genus $g = 0$) could there be such ringtype surfaces (of genus $g = 1$) or pretzel - type surfaces (of genus $g = 2$) e.t.c.?

In 1951, H. Hopf proved that a sphere-like surface, having constant mean curvature must in fact be a sphere. A few years later the Russian

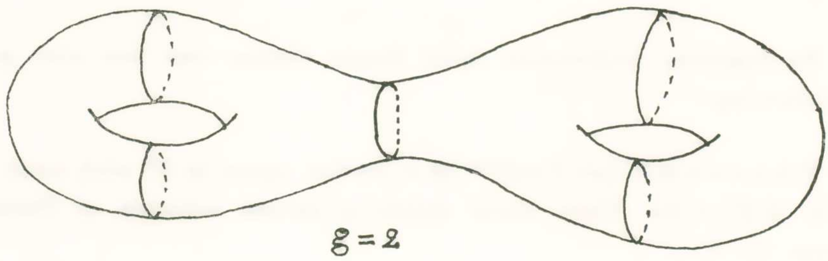

Fig. 2.

mathematician A. D. Alexandrov proved that any physical surface of constant mean curvature and of *arbitrary* genus must be a sphere. By a physical surface he meant the surface that appears as boundary of a domain, i. e. as the interface separating a quantity of matter from its outside, and thus obviously cannot possess self-intersections. The long-standing problem whether there are closed surfaces of constant mean curvature other than the sphere remained to be open despite the continuous efforts of some of the greatest mathematicians. A partial answer of this problem is obtained in the framework of our generalization to Morse theory on Hilbert (or Banach) manifolds.

The proofs of the above theorems will appear elsewhere, because they are very lengthy to be whitten in the present paper.

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ΠΕΡΙΛΗΨΙΣ

Ἡ παροῦσα ἐργασία ἀναφέρεται εἰς μίαν γενίκευσιν τῆς θεωρίας τοῦ Marston Morse εἰς τὸ πρόβλημα τοῦ Plateau. Τὸ πρόβλημα αὐτὸ παρέμενεν ἄλυτον παρὰ τὰς προσπαθείας πολλῶν Μαθηματικῶν εἰς διαφόρους χώρας. Ὁ συγγραφεὺς εἶχεν ἀναπτύξει μίαν πρωταρχικὴν θεωρίαν εἰς τὰς δύο προηγουμένας του ἀνακοινώσεις εἰς τὴν Ἀκαδημίαν, ὁδηγούσας τὴν λύσιν τοῦ προβλήματος ἐφαρμογῆς τῆς θεωρίας τοῦ Morse εἰς τὸ πρόβλημα τοῦ Plateau εἰς μίαν εἰκασίαν, τῆς ὁποίας ἡ ἀλήθεια θὰ ἐπέλυε γενικὰ τὸ πρόβλημα. Εἰς τὴν παροῦσαν ἐργασίαν του ὑποστηρίζεται ὅτι ἡ ἐν λόγῳ εἰκασία ἀπεδείχθη ἀληθής, πρᾶγμα τὸ ὁποῖον ὁ συγγραφεὺς ἀνεκοίνωσεν εἰς τὸ Διεθνὲς Συνέδριον Μαθηματικῶν εἰς τὸ Ἐλσίνκι τῆς Φινλανδίας τὴν 18ην Αὐγούστου, 1978.

Ἡ ἀνωτέρω πορεία σκέψεως ὡδήγησεν ἐπίσης τὸν συγγραφέα εἰς μίαν γενίκευσιν τοῦ θεωρήματος τοῦ T. Rado σχετικὰ μὲ τὸν ἀριθμὸν λύσεων εἰς τὸ πρόβλημα τοῦ Plateau διὰ μίαν δοθεῖσαν καμπύλην τοῦ Jordan εἰς τὸν Εὐκλείδειον χῶρον τῶν τριῶν διαστάσεων.

Ἡ ἐργασία τελειώνει μὲ ὡρισμένας παρατηρήσεις σχετικὰ μὲ τὸ πρόβλημα: Ἐὰν ὑπάρχουν κλεισταὶ ἐπιφάνειαι σταθερᾶς μέσης καμπυλότητος (constant mean curvature) διαφορετικαὶ ἀπὸ τὴν σφαῖραν. Τὸ πρόβλημα αὐτὸ ἤρχισεν νὰ ἀποτελῇ πρόβλημα ἐρεύνης ἀπὸ τὸ 1900, τὴν ἐποχὴν τοῦ Liebmann.