ΘΕΩΡΙΑ ΑΡΙΘΜΩΝ.— **On sums of like powers of the numbers less than N and prime to N,** *by G. S. Kazandzidis.** Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ὅ. Πυλαρινοῦ.

1. Algebra of C-generalized subsets.

Let C be the ring of common integers and let E be any nonempty set. Iff (if and only if) A is a subset of the cartesian product $C \times E$ exhibiting for every $e \in E$ a unique corresponding pair (α, e) as element, we take A as defining what we call a «C-generalized subset» B of E, where every $e \in E$ appears as many «times» as an element, as the corresponding integer α, whether $\alpha = 0$, $\alpha = 0$, or $\alpha < 0$ and we call this integer α the multiplicity of e in B.

Note: From the point of view of the construction that follows, we consider A as identical with the set of those of its pairs (α, e) for which $\alpha \neq 0$, but the presence of pairs $(0, e)$ provides comfort in exposition; because of this, even when one comes to a set of pairs (α, e) whose second elements e do not cover the whole of E, one thinks of it enriched with the set of pairs $(0, e)$ for the lacking e's.

Let **A** be the set of all subsets of $C \times E$ of the kind considered above and let **B** be the set of the C-generalized subsets of E that they define.

For any given $A \in \mathbf{A}$, any given $\lambda \in C$, and (-this provided E be multiplicative with right cancellation-law) any given $u \in E$, we define

(1) the left scalar product λA

(2) the right scalar product Au

respectively as

(1′) the set of pairs $(\lambda\alpha, e)$

(2′) the set of pairs (α, eu)

taken from the pairs (α, e) of A on multiplication

(1″) of the first elements α by λ

(2″) of the second elements e by u

The left C- and right E-scalar product

$$\lambda A u = \lambda(Au) = (\lambda A)u$$

becomes clear.

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* Γ. Σ. ΚΑΖΑΝΤΖΙΔΟΥ, **Ἀθροίσματα ὁμοβαθμίων δυνάμεων τῶν φυσικῶν ἀριθμῶν τῶν μικροτέρων φυσικοῦ ἀριθμοῦ $N > 1$ καὶ πρώτων πρὸς τὸν N.**

We define addition and multiplication in **A** as additions and multiplications of the first elements of the pairs with common second element: For every $e \in E$, the corresponding pairs in $A + A'$ and AA' are, by our definition, respectively $(\alpha + \alpha', e)$ and $(\alpha\alpha', e)$ where (α, e), $(\alpha' e)$ are the corresponding pairs in A and A′.

We write $A^{(n)}$ for the set of pairs (α, e^n) taken from the pairs (α, e) of A on raising the second elements to the n^{th} power.

We transfer the above operations and notations (conservatively of the correspondence $A \longleftrightarrow B$) into the set **B** of the C-generalized subsets of E and we have this **B** automatically as a left C-algebra and (— this, provided E be a ring without right null-divisors) a right E-modul (linear space).

Finally we define union and intersection in **B** as follows: The multiplicities of any given $e \in E$ in $B \cup B'$ and $B \cap B'$ shall be respectively the maximum and the minimum of its multiplicities in B, B′.

The «structure» **B** thus obtained is an extension of the structure P(E) of all ordinary subsets of E and offers comforts that the power-set P(E) lacks; samples thereof were given in [**1**] and we are about to give here another.

Since every C-algebraic operation in **B** (addition, multiplication, C-scalar multiplication) is meant solely as an operation on multiplicities,

1* every C-algebraic relation

(i) $f(B_1, B_2, \ldots, B_\nu) = B$

in **B** stands for the set of similar relations

(ii) $f(\alpha_1, \alpha_2, \ldots, \alpha_\nu) = \alpha$

in C, each among the multiplicities in $B_1, B_2, \ldots B_\nu, B$ of an $e \in E$. Hereby, since the multiplicity of e^n in $B^{(n)}$ is the multiplicity of e in B,

2* the C-algebraic relation (i) entails

(iii) $f(B_1^{(n)}, B_2^{(n)}, \ldots, B_\nu^{(n)}) = B^{(n)}$

We characterize an element B of **B** as *finite* or *infinite* according as the number of elements of non-zero multiplicities in B is finite or infinite.

Whenever it has a meaning and is unambiguous, we speak of the sum of products αe of the elements by their multiplicities in a $B \in \mathbf{B}$, symbolically of

$$\sum_B \alpha e$$

as of the *sum of elements of* B. If the set E is a C-modul itself, in particular if E is a ring, there is always a definite sum of elements of every finite B and, moreover, then by 1*,

3 every C-modul (linear over C) relation

(iv) $\lambda_1 B_1 + \lambda_2 B_2 + \ldots + \lambda_\nu B_\nu = B$

among finite elements $B_1, B_2, \ldots B_\nu, B$ of **B**

entails the similar relation

$$\text{(v)} \quad \lambda_1 \sum_{B_1} \alpha_1 e + \lambda_2 \sum_{B_2} \alpha_2 e + \ldots + \lambda_\nu \sum_{B_\nu} a_\nu e = \sum_{B} \alpha e$$

among their sums of elements.

Note : Obviously, by defining

1) the left scalar sum $\lambda + A$ as the set of pairs $(\lambda + \alpha, e)$ and (in case E is a ring).

2) the right scalar sum $A + u$ as the set of pairs $(\alpha, e + u)$, where the (α, e) are the pairs of A, one makes of **B** also what we can call a *C-left ring* and an *E-right ring.*

2. Sums of like powers of the numbers $t(N)$.

In what follows

1) N is any given natural > 1 and $p_1, p_2, \ldots, p_\nu$ are the distinct prime divisors of N;

2) $T(1) = \varnothing$, $T(N) = \{1, 2, \ldots, N-1\}$;

3) $t(1) = \varnothing$, $t(N)$ is the set of naturals less than N and prime to N;

4) $[N]_n$ is the sum of the n^{th} powers of the numbers $t(N)$.

We are about to establish the general formula

$$1^* \quad [N]_n = \frac{1}{n+1} \sum_{\varkappa=0}^{n} \left(1 - p_1^{\varkappa-1}\right)\left(1 - p_2^{\varkappa-1}\right)\ldots\left(1 - p_\nu^{\varkappa-1}\right)\binom{n+1}{\varkappa} b_\varkappa N^{n+1-\varkappa}$$

where $b_0 = 1$ and $b_1, b_2, \ldots$ are the «Bernoulli-numbers of small type», i. e. those defined by the recurrent

$$1 + \binom{n+1}{1} b_1 + \binom{n+1}{2} b_2 + \ldots + \binom{n+1}{n} b_n = 0 \quad (n = 1, 2, \ldots).$$

Proof. We take as set E of § 1 the ring C itself and, in the algebra **B** of the C-generalized subsets of C, we consider the set

$$B = \sum_{d/N} \mu(d)\, T(N/d)\, d$$

where $\mu(d)$ is the Möbius function and, by the notations adopted in § 1, $\mu(d)\, T(N/d)\, d$ is that set, say B_d, of **B**, in which the elements of $T(N/d)\, d$ (the products by d of the elements of $T(N/d)$) have multiplicities $\mu(d)$, in particular B_N is the zero-set of **B**:

(i) Every integer $t \notin T(N)$ has in B multiplicity zero, in particular B is finite as is also every summand $\mu(d)\, T(N/d)\, d$.

(ii) Let $t \in T(N)$ and let the greatest common divisor $(t, N) = \delta$; then $t \in T(N/d)\, d$ iff (if and only if) d/δ and so the multiplicity of t in B equals the sum

$$\sum_{d/\delta} \mu(d) = \begin{Bmatrix} 1, & \text{if} \quad \delta = 1 \\ 0, & \text{if} \quad \delta > 1 \end{Bmatrix}.$$

Thus

$$B = t(N).$$

Hence, by theorem (1, 2*),

$$\{t(N)\}^{(n)} = \sum_{d/N} \mu(d) \{T(N/d)\, d\}^{(n)} = \sum_{d/N} \mu(d) \{T(N/d)\}^{(n)} d^n .$$

Now for every natural x, the sum of elements of $\{T(x)\}^{(n)}$ is expressed by the Bernoulli polynomial

$$s_n(x) = \frac{1}{n+1} \sum_{\varkappa=0}^{n} \binom{n+1}{\varkappa} b_\varkappa\, x^{n+1-\varkappa}$$

$[\{T(1)\}^{(n)} = \varnothing$ and $s_n(1) = 0]$; and so, by theorem (1, 3*), the sum fo elements of $\{t(N)\}^{(n)}$, i. e. our

$$[N]_n = \sum_{d/N} \mu(d) \left\langle \frac{1}{n+1} \left\{ \sum_{\varkappa=0}^{n} \binom{n+1}{\varkappa} b_\varkappa (N/d)^{n+1-\varkappa} \right\} d^n \right\rangle$$

$$= \frac{1}{n+1} \sum_{\varkappa=0}^{n} \left\{ \sum_{d/N} \mu(d)\, d^{\varkappa-1} \right\} \binom{n+1}{\varkappa} b_\varkappa\, N^{n+1-\varkappa}$$

whilst trivially

$$\sum_{d/N} \mu(d)\, d^{\varkappa-1} = (1-p_1^{\varkappa-1})(1-p_2^{\varkappa-1}) \ldots (1-p_\nu^{\varkappa-1}).$$

Formula 1* has been established.

REFERENCES

1) G. S. Kazandzidis : Algebra of subsets and Möbius pairs, Mathematische Annalen 152 (1963) pp. 208 - 225.

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Ὁ Ἀκαδημαϊκὸς κ. **Ὄθων Πυλαρινὸς** κατὰ τὴν ἀνακοίνωσιν τῆς ὡς ἄνω ἐργασίας τοῦ κ. Γ. Καζαντζίδη εἶπε τὰ κάτωθι :

Εἰς τὴν ἐργασίαν ταύτην ἀποδεικνύεται ὅτι τὸ ἄθροισμα τῶν δυνάμεων μὲ ἐκθέτην τὸν τυχόντα φυσικὸν ἀριθμὸν n τῶν φυσικῶν ἀριθμῶν τῶν μικροτέρων δοθέντος φυσικοῦ ἀριθμοῦ $N>1$ καὶ πρώτων πρὸς τὸν N εἶναι δυνατὸν νὰ τεθῇ ὑπὸ μορφὴν πολυωνύμου ὡς πρὸς N βαθμοῦ $n+1$, τοῦ ὁποίου οἱ συντελεσταὶ εἶναι πλήρως καθωρισμέναι συναρτήσεις τοῦ n, τῶν διακεκριμένων πρώτων διαιρετῶν τοῦ N καὶ τῶν $n+1$ πρώτων ἀριθμῶν τοῦ Bernoulli μικροῦ τύπου. Συγκεκριμένως ἀποδεικνύεται ὅτι, ἐὰν εἶναι $p_1, p_2, \ldots p_\nu$ οἱ διακεκριμένοι πρῶτοι διαιρέται τοῦ N καὶ $b_0 = 1, b_1, b_2, \ldots b_n$, οἱ $n+1$ πρῶτοι ἀριθμοὶ τοῦ Bernoulli μικροῦ τύπου, τὸ ἐν λόγῳ ἄθροισμα εἶναι ἴσον πρὸς

$$\frac{1}{n+1} \sum_{\varkappa=0}^{\varkappa=n} (1-p_1^{\varkappa-1})\ (1-p_2^{\varkappa-1})\ \ldots\ (1-p_\nu^{\varkappa-1}) \binom{n+1}{\varkappa} b_\varkappa N^{n+1-\varkappa}.$$