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ΑΝΑΚΟΙΝΩΣΙΣ ΜΗ ΜΕΛΟΥΣ

ΜΑΘΗΜΑΤΙΚΑ.— **Actual Mathematical Solutions of Problems Posed by Reality, I. (A classical Procedure)***, *by D. G. Magiros***. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ἰω. Ξανθάκη.

INTRODUCTION

We discuss here phenomena or situations posed by reality which are changes of variable quantities that influence and interact to each other in an organized behavior. We call such a situation a «real system». A real system, either physical or social, corresponds to a «physical or social problem», mathematically expressed.

To describe quantitatively and explain the entire spectrum of the functional behavior of a real system, a «theory» of the system is needed, that is a set of statements concerning the behavior of the objects of the system. This theory, in general, implies a mathematical expression of the relationships between certain quantities of the system, that is a «mathematical model» of the system, associated with the «data» of the system.

The main requirement that the solution of a model is expected to satisfy is: «to interpret the real system in an adequate way». Such a

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solution of the model we call an «actual solution» of the system. There are non-actual solutions, called «formal solutions», which only satisfy the equations of the model and the data.

The above requirement would be fulfilled by the actual solution if this solution satisfies certain mathematical restrictions, called «Hadamard's restrictions». If this is so, the correspondent model must have an appropriate mathematical structure. This structure of the model shows the existence of an actual solution, although this solution is unknown.

Models with actual solutions are called: «well-posed», and with formal solutions: «non-well-posed».

In this paper we discuss a procedure by which «well-posed-models» may be constructed, and «actual solutions» be determined. Three important phases in the procedure can be distinguished: (a) the formulation of a theory concerning the system, which will lead to the construction of a model of the system; (b) the selection of a «well-posed-model», by applying the «Hadamard's restrictions»; and (c) the application of mathematical methods to find the solution of the «well-posed-model». The discussion, interpretation and evaluation of the results will complete the cycle of the research.

THE CLASSICAL PROCEDURE

1. Theory and models of real systems. (3), (4), (9).

The construction of a mathematical model is, in many systems, the most important step of the study of the systems. It presupposes a «theory» of the system, that is, a set of statements some of which are verified by the experimental observations, while some others may be postulated. The objects and the formulation of the statements are the two constituents of the theory.

For a real system, the investigator needs a detailed knowledge of the observational and experimental facts, the pertinent laws, a penetrating insight, a mature judgment.

The data of the system, that is the results of observation, experiments and measurements related to the system, must be completely stated and known approximately within an accepted error, thus called «admissible data».

The domain on which the system operates must be known. This is related to the selection of the «major» variables, among the variables of the system, which define the process of the motion of the system, that is the main subject of the theory. The other variables of the system, the «minor» variables, are either «parameters» of the system, thus defining the «environment» of the system, or a «noise». A good theory of the system depends on the appropriate selection of the major and minor variables of the system.

All the above considerations, and many others of special interest, constitute the theory of a real system, and a satisfactory theory of the system makes the system «correctly stated».

Correctly stated systems can be represented in a mathematical form, called «model of the system», which gives the functional behavior of the system in a quantitative way.

The «strong interactions» between the variables of the system can be expressed explicitly in a mathematical form and give the basis of a completely deterministic (non-statistical) model. The «weak interactions» are not represented explicitly in the equations, but they are an essential part of the system related to statistical mechanics.

Arbitrary assumptions, decisions and choices in developing a theory and the model of the system, have as a result the construction of various models to the same real system.

The model summarizes the data of the system, and if one repeats an experiment or gets numbers, by using the model, and these results agree with the data assumed in constructing the model, the model is acceptable.

The applicability of a model depends on the possibility of estimating their parameters from the data.

Models, parts of which cannot explain given experimental facts of the system, are not acceptable. But if these parts of the model are cancelled, then every feature of it is related in some way to the experimental data and the model becomes acceptable.

Usually, the models are «boundary and/or initial value problems».

2. Hadamard's restrictions, well-posed-models.

Among the possible models of a real system correctly stated, one can distinguish the «well-posed» ones of which the solutions are «actual solution» of the system (7).

As we know, if the equations of the system and its data are such that :

a. the model has a solution corresponding to the data,

b. the solution is unique, and

c. the solution depends continuously on the data,

we say that the solution satisfies these three «Hadamard's restrictions», and that the equations of the model and the data give a «well-posed» or «reasonable» problem, and we have a definition of «correctness» of the problem in the sense of Hadamard (2a).

If the solution fails to satisfy even one of the «Hadamard's restrictions», it is a formal solution, and the problem a «non-well-posed» or «non-reasonable», or «improperly posed» problem.

Much attention has been given to such problems in recent years (6), (8).

Appropriate continuity and differentiability properties of the mathematical expression of the «well-posed-problem», by means of known existence, uniqueness and continuity theorems, assure the existence of a solution which satisfies the Hadamard's restrictions.

It remains now to see that the solutions which satisfy the Hadamard's restrictions are actual solutions of the real problem.

We remark that (3), (5) :

First : The well-posed-problem must necessarily possess a solution.

The existence of a solution and its determination are different concepts, and we try to determine a solution only if we know it exists.

Second : The solution must be just one.

The existence and uniqueness properties of the solution express our belief «in causality» or «in determinism», a principle according to which one can repeat experiments with the expectation to get consistent results.

Third : The continuous dependence of the solution on the data has as a result that small changes in the data imply small changes in the solution.

The data, as results of observation, experiments and measurements, are given with small errors, when the solution has an uncertainty and its estimation an error (**2b**). If the above continuity property of the solution holds, then «the smaller the error in the data, the smaller the error in the estimation of the solution», and «admissible solutions» correspond to «admissible data» (**1**). The above continuity property holds for finite time. If it holds for any time, this property becomes a «stability property» of the solution in the sense of Liapunov in a «parameter-space», which is here the «data-space» (**5**).

The above remarks make clear that solutions which satisfy the Hadamard's restrictions are actual solutions of real systems.

3. The determination of the actual solution.

Having now found the «well-posed-problem» corresponding to the real system, we try, next, to determine its solution. Heuristic scientific reasoning toward the ultimate solution can be frequently used. The «principle of approximation» can be introduced to achieve the solution needed. The continuity property of the Hadamard's restrictions helps to apply this principle.

To the «well-posed-problem» M we try to find an appropriate «approximate problem» M_n, of which the solution A_n, containing the index n, will be determined. The limit A of the solution A_n as $n \to \infty$, $A = \lim\limits_{n \to \infty} A_n$, is the solution of the well-posed-problem, that is the actual solution of the real system (**2a**).

4. Summary and Conclusion.

Summarizing the preceding we see that the research for finding

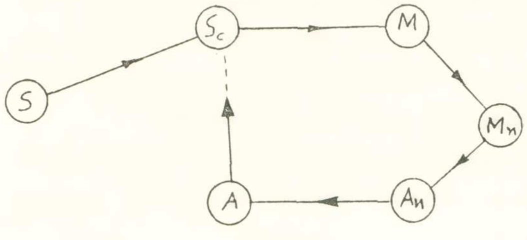

Figure 1.

S : real system
S_c : system correctly stated
M : well-posed-model
M_n : approximate problem to M
A_n : solution of M_n
$A = \lim\limits_{n \to \infty} A_n$: solution of M

actual solutions of real systems is constituted by a set of steps, which makes a complete cycle, shown schematically in Figure 1.

(i) Step : $S \rightarrow S_c$; We need the theory of the system, which will make the system correctly stated and the modeling possible.

(ii) Step : $S_c \rightarrow M$; We use the Hadamard's restrictions, which will help to get the well-posed-model of the system.

(iii) Step : $M \rightarrow M_n$; We select the appropriate approximation to « w - p - m ».

(iv) Step : $M_n \rightarrow A_n$; We find the solution A_n the problem M_n.

(v) Step : $A_n \rightarrow A$; We find the solution required of the system, by using the limiting process, as $n \rightarrow \infty$.

The step (i) is a decisive step. It requires ingenuity, experience and knowledge of the field to which the real system belongs;

The step (ii) helps to get the reasonable model on which the whole investigation is based;

The steps (iii) and (iv) need a constructive imagination and a complete knowledge of mathematics to be applied;

The step (v) is an application of the principle of approximation, which can be succeeded by applying the continuity property of the Hadamard's restrictions.

5. Remarks on the classical procedure.

The classical procedure for finding actual solutions of real systems shows a general orientation of thought on the problem. It is based on the Hadamard's definition of «correctness», of «well-poseness».

It was found that the Hadamard's definition of «correctness» rules out as «non-well-posed-problems» important real problems, as, e.g., problems of geophysics, and some authors at the present time present different notions of «correctness» and various approaches to the formulation and investigation of the «non-well-posed-problems» (**6**).

There are difficulties in the application of the classical procedure in some scientific fields. Problems of modern physics especially present these difficulties (**2b**).

The exact laws governing the behavior of the system under consideration may not be known, and, in this case, a complete theoretical description of the system is impossible.

Problems of fundamental particle structure and reaction present the difficulties in an acute form.

More often the situation is analogous to that of problems of atomic and molecular structure, where the interactions are known, but where the structure in a particular problem may be complicated.

In problems of classical dynamics of compressible fluids, the differential equations supplemented by boundary conditions are not always a sufficiently complete framework for an adequate description of physical reality.

The above remarks must be considered, when one tries to present a mathematical description of real systems.

Selected applications from different scientific fields should show the importance of the above procedure and remarks.

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Π Ε Ρ Ι Λ Η Ψ Ι Σ

Εἰς τὴν παροῦσαν ἐργασίαν περιγράφεται μέθοδος ἐρεύνης ἡ ὁποία ἀφορᾷ εἰς τὴν εὕρεσιν «φυσικῶς δεκτῶν» μαθηματικῶν λύσεων εἰς φυσικὰ καὶ κοινωνικὰ προβλήματα. Αἱ κύριαι φάσεις τῆς μεθόδου εἶναι: (α) Ἡ ἔρευνα ὅπως καθιερωθῇ μία «θεωρία τοῦ συστήματος», ἡ ὁποία νὰ καθιστᾷ τὸ σύστημα ἀφ᾿ ἑνὸς μὲν καλῶς ἐκπεφρασμένον, ἀφ᾿ ἑτέρου δὲ νὰ ὑποβοηθῇ εἰς τὴν εὕρεσιν μαθηματικῶν προτύπων τοῦ συστήματος· (β) Ἡ ἐκλογὴ ἑνὸς «καλῶς τεθειμένου προτύπου» τοῦ συστήματος, τὸ ὁποῖον νὰ ὁδηγῇ εἰς ἓν «καλῶς τεθειμένον μαθηματικὸν πρόβλημα», τοῦ ὁποίου νὰ προσδιορισθῇ ἡ λύσις· καὶ (γ) Ἡ εὕρεσις τῆς λύσεως τοῦ ἀρχικοῦ συστήματος, βάσει τῆς εὑρεθείσης λύσεως τοῦ καλῶς τεθέντος μαθηματικοῦ προβλήματος.

Ὑπάρχουν πεδία ἐρεύνης, ὅπου ἡ μέθοδος εἶναι ἢ δύσκολον ἢ ἀδύνατον νὰ ἐφαρμοσθῇ, ἀλλ᾿ ὅμως ὅπου αὕτη δύναται νὰ ἐφαρμοσθῇ τὰ ἀποτελέσματα δύνανται νὰ ἑρμηνεύσουν τὴν φυσικὴν πραγματικότητα κατὰ πολὺ ἱκανοποιητικὸν τρόπον.

Εἰς ἑπομένην ἐργασίαν θὰ δοθοῦν ἐφαρμογαὶ — ἀπὸ διάφορα πεδία ἐρεύνης — διὰ τῶν ὁποίων θὰ διασαφηνίζεται ἡ πορεία τῆς μεθόδου καὶ θὰ καταδεικνύεται ἡ σημασία της.

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Ὁ Ἀκαδημαϊκὸς κ. **Ἰω. Ξανθάκης** κατὰ τὴν ἀνακοίνωσιν τῆς ἐργασίας τοῦ κ. Δ. Μαγείρου εἶπε τὰ ἑξῆς:

Ἔχω τὴν τιμὴν νὰ παρουσιάσω εἰς τὴν Ἀκαδημίαν ἐργασίαν τοῦ κ. Δημ. Μαγείρου ὑπὸ τὸν τίτλον: «Δεκταὶ Μαθηματικαὶ λύσεις φυσικῶν προβλημάτων». Εἰς τὴν ἐργασίαν ταύτην ὁ συγγραφεὺς ἐκθέτει μίαν μέθοδον ἐρεύνης ἀφορῶσαν εἰς τὴν εὕρεσιν μαθηματικῶν λύσεων φυσικῶν προβλημάτων ἀποδεκτῶν ἀπὸ φυσικῆς ἀπόψεως. Αἱ κύριαι φάσεις τῆς μεθόδου εἶναι αἱ ἑξῆς:

α) Ἡ διατύπωσις μιᾶς «θεωρίας τοῦ συστήματος» ἡ ὁποία νὰ καθιστᾷ ἀφ᾿ ἑνὸς μὲν καλῶς ἐκπεφρασμένον τὸ σύστημα, ἀφ᾿ ἑτέρου δὲ νὰ ὑποβοηθῇ εἰς τὴν εὕρεσιν μαθηματικῶν προτύπων τοῦ συστήματος.

β) Ἡ ἐκλογὴ ἑνὸς καταλλήλου προτύπου τοῦ συστήματος, τὸ ὁποῖον νὰ ὁδηγῇ εἰς ἓν καλῶς τεθειμένον μαθηματικὸν πρόβλημα καὶ

γ) Ἡ εὕρεσις τῆς λύσεως τοῦ ἀρχικοῦ συστήματος βάσει τῆς εὑρεθείσης λύσεως τοῦ καλῶς τεθέντος μαθηματικοῦ προβλήματος.

Ὁ συγγραφεὺς ἀναφέρει ὅτι ὑπάρχουν πεδία ἐρεύνης, ὅπου ἡ μέθοδος αὕτη εἶναι εἴτε δύσκολον εἴτε ἀδύνατον νὰ ἐφαρμοσθῇ. Ὅπου ὅμως αὕτη δύναται νὰ ἐφαρμοσθῇ τὰ ἀποτελέσματα, κατὰ τὸν συγγραφέα, δύνανται νὰ ἑρμηνεύσουν τὴν φυσικὴν πραγματικότητα κατὰ τρόπον λίαν ἱκανοποιητικόν.

Τέλος ὁ κ. Δ. Μάγειρος προτίθεται εἰς προσεχῆ ἀνακοίνωσίν του νὰ παρουσιάσῃ ἐφαρμογὰς ἐπὶ διαφόρων πεδίων ἐρεύνης διὰ τῶν ὁποίων θὰ διαγράφεται ἡ πορεία τῆς μεθόδου καὶ θὰ καταδεικνύεται ἡ σημασία της.