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ΕΦΗΡΜΟΣΜΕΝΑ ΜΑΘΗΜΑΤΙΚΑ.— **The stability of a class of helicoid precessions in the sense of Liapunov and Poincaré,** *by Demetrios G. Magiros** Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ἰω. Ξανθάκη.

Introduction

In a previous paper, Ref. 1, we discussed the stability of a helicoid precession in case of constant torque, whereby, by employing different stability concepts, we found for this precession different stability situations. The concept of this helicoid precession was successfully applied in problems of current interest in Astrodynamics, treated in papers Ref. 2, 3.

In the present paper, we discuss the stability of a «class of helicoid precessions», of which the helicoid precession of the paper Ref. 1 is only a member.

The concepts of stability in the sense of Liapunov and Poincaré, Ref. 4, are employed.

We found that all the members of the class of precessions are

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unstable in Liapunov sense; but in Poincaré sense the stability of a member S of the class is either stable, or asymptotically stable, or unstable, if the limit value of the pitch distance of S is either a constant, or zero, or infinite, respectively.

There are reasons which suggest that the stability situation of the above class of the helicoid precessions in the sense of Poincaré is close to practical stability, then it is preferred.

1. The class of the helicoid precessions

The rotational motion of a rigid body around its symmetry axis, is governed by the Euler's equations:

$$\dot{\omega}_1 = \frac{L_1}{I_1}, \quad \dot{\omega}_2 = \frac{L_2}{I} - \frac{I_1 - I}{I}\,\omega_1\,\omega_3, \quad \dot{\omega}_3 = \frac{L_3}{I} + \frac{I_1 - I}{I}\,\omega_1\,\omega_2 \tag{1}$$

where $\underline{\omega} = (\omega_1, \omega_2, \omega_3)$ is the angular velocity; $\underline{L} = (L_1, L_2, L_3)$ the external resultant torque acting on the body, I_1, $I_2 = I_3 = I$ the moments of inertia about the coordinate axis $0_1\,\omega_1\,\omega_2\,\omega_3$, 0_1 the center of the mass of the boby.

In the case where the torque is:

$$L_1 = L_1(t), \quad L_2 = 0, \quad L_3 = 0 \tag{2}$$

the solution of (1) is:

$$\left.\begin{aligned} \omega_1(t) = \frac{1}{I_1}\int L_1(t) + c, \quad \omega_2(t) = A\cos Q(t), \quad \omega_3(t) = A\sin Q(t) \\ Q(t) = \frac{I_1 - I}{I}\int \omega_1(t)\,dt \end{aligned}\right\} \tag{3}$$

c and A are constants to be determined from the initial angular velocity:

$$\underline{\omega_0} = (\omega_{10}, \omega_{20}, \omega_{30}), \quad \text{and} \quad A = (\omega_{20}^2 + \omega_{30}^2)^{1/2}.$$

In case $\omega_1(t)$ is increasing function of and tends to infinity with time t, the solution (3) is a helicoid curve S on the surface of an orthogonal circular cylinder of radius A and gives a helicoid precession corresponding to the specified function $L_1(t)$, and so (3) gives a «class of helicoid precessions», each member of which is determined by the specification of $L_1(t)$. We remark that if $\omega_1(t)$ does not satisfy the above requirement, the corresponding precession is not helicoid, as, e.g., for $L_1 = 0$, when we have the «regular precession» and its «precessional curve» is circumference on the surface of the cylinder; or for $L_1 = \sin t$, when the precessional curve is closed curve on the surface of the cylinder. But, for $L_1 =$ constant, $\omega_1(t) \rightarrow \infty$ as $t \rightarrow \infty$, and we have a helicoid precession, the simplest one of the class (3).

We discuss here the stability of the class of helicoid precessions (3) in the sense of Liapunov and Poincaré.

2. Stability in Liapunov sense

The vector $\underline{\omega}_{23} = (\omega_2, \omega_3)$ on the ω_2, ω_3 — plane, Fig. 1(a), of which the components are given by (3), is periodic in t, but with period dependent on t, then the motion of the end point of this vector on the circumference with radius A and center 0_1, is unstable in Liapunov sense, Ref. 4, and, as a consequence, *«the class of helicoid precessions is «unstable» in Liapunov sense»*.

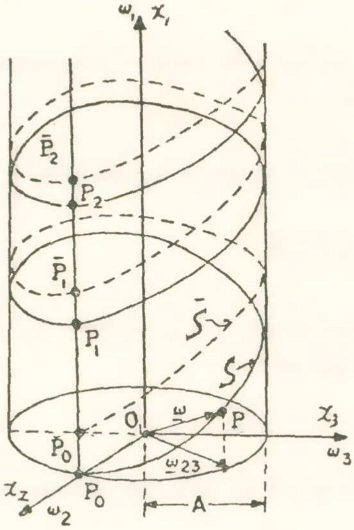
3. Stability in Poincaré sense (orbital stability)

The orbital stability of any member of the above class of precessions depends upon the structure of the corresponding function $L_1(t)$. Some auxiliary distances and their properties, shown below, will help to create a criterion for the orbital stability.

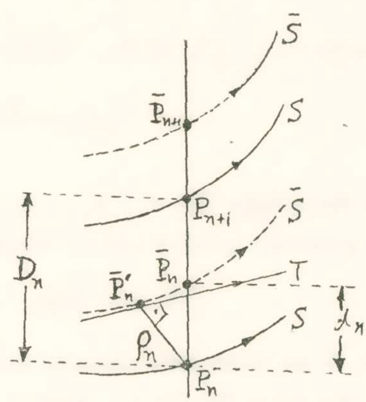
3.1 Some auxiliary distances and their properties.

Let us take two helicoid precessional curves S and $\overline{S}$ belonging to the same family, that is corresponding to the same function $L_1(t)$, but starting from different points $P_0(0, \omega_{.0}, 0)$ and $\overline{P}_0(\omega_{10}, \omega_{20}, 0)$, respectively, Fig. 1(a).

The generator of the cylinder through P_0 intersects S into the points: $P_0, P_1, P_2, \ldots, P_n, \ldots$, and $\overline{S}$ into the points: $\overline{P}_0, \overline{P}_1, \overline{P}_2, \ldots, \overline{P}_n, \ldots, S_0$, at the point P_n of S, we can define, as shown in Fig. 1(b), the pitch distances: $D_n = P_n P_{n+1}$, $d_n = P_n \bar{P}_n$. A third distance at P_n is defined by the plane through P_n perpendicular to $\bar{S}$ at $\bar{P}'_n$, the distance $\varrho_n = P_n \bar{P}'_n$.



(a)



(b)

Fig. 1.

The above distances have properties very useful for the discussion of the orbital stability of the helicoid precessions. These properties are given by the following theorem.

Theorem 1. *The distances* D_n, d_n, ϱ_n *have the properties*:

. (a): $D_n > d_n > \varrho_n$

. (b): *The limit distance:* $\lim_{n\to\infty} D_n = \bar{D}$ *is either a constant,* or zero, *or infinite, when the* $\lim_{n\to\infty} d_n = \bar{d}$, $\lim_{n\to\infty} \varrho_n = \bar{\varrho}$ *are either constant, or zero, or infinite, respectively.*

Proof . (a): The point $\bar{P}_n$ is a point of the segment $P_n P_{n+1}$, Fig. 1 (b), which means that: $D_n > d_n$.

The plane through P_n and perpendicular to $\bar{S}$ at $\bar{P}'_n$ intersects perpendicularly the tangent $\bar{P}'_n T$ of $\bar{S}$ at $\bar{P}'_n$, so this tangent is perpendicular to the distance ϱ_n. The plane through $\bar{P}'_n$ and perpendicular to ϱ_n contains the tangent $\bar{P}'_n T$ and divides the whole space into two parts, one of which contains all the points of $\overline{P}'_n \overline{S}$, and the other contains the point P_n, so the distance $P_n \overline{P}_n$ is bigger than the distance $P_n \overline{P}'_n$, that is $d_n > \varrho_n$.

. (b): For the second part of the theorem the calculation of the limit D is needed.

The curve S starts from $P_0(0, \omega_{20}, 0)$ at $t_0 = 0$, and corresponds to $c = 0$ in the formulae (3). Its equations are :

$$\left.\begin{array}{c} \omega_1(t) = \frac{1}{I_1}\int L_1(t)\,dt, \quad \omega_2(t) = \omega_{20}\cos Q(t), \quad \omega_3(t) = \omega_{20}\sin Q(t) \\ Q(t) = \frac{I_1 - I}{I}\int dt\int L_1(t)\,dt \end{array}\right\} \quad (4)$$

For the points $P_0, P_1, P_2, \ldots, P_n$, we have $\omega_2(t) = \omega_{20}$, $\omega_3(t) = 0$, then the quantity $Q(t)$ of (4) for these points must be:

$$Q(t_n) = 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots \quad (5)$$

and, if we take into account some restrictions of Q and the nature of n, we can solve (5) for t_n, when :

$$t_n = \overline{Q}(n) \ldots \quad (6)$$

Inserting (6) into the first of (4), we can get the value of $\omega_1(t)$ corresponding to the point P_n: $\omega_1(t_n) = P_0 P_n = (\omega_1)_n$, when the distance D_n at P_n is :

$$D_n = P_0 P_{n+1} - P_0 P_n = P_n P_{n+1} = (\omega_1)_{n+1} - (\omega_1)_n \quad (7)$$

As $n \to \infty$, the points P_n and P_{n+1} go to infinity, the distances $P_0 P_n$ and $P_0 P_{n+1}$ tend to infinity, and the $\lim\limits_{n\to\infty} D_n = \overline{D}$ tends to get the undetermined form $(\infty - \infty)$, which may be either a constant, or zero or infinite, and, then the limits $\bar{d}$ and $\bar{\varrho}$ may be either constants, or zero, or infinite, respectively.

3.2 Orbital stability criterion

Based on the above properties of the distances D_n, d_n, ϱ_n, we can formulate a criterion for the orbital stability of the helicoid precessions (3), expressed by the :

Theorem 2. *«Any member of the class of the helicoid precessions (3), corresponding to a given function $L(t)$, is orbitally either stable, or asymptoti-*

cally stable, or unstable, if, respectively, the limit distance $\bar{D}$ is either a constant, or zero, or infinite».

Proof : We first see that, for sudden perturbation when the initial conditions are only perturbed, the curve $\bar{S}$ can be considered as the perturbed of S and the distance ϱ_n, defined above, is the «Poincaré distance» of S at P_n, Ref. 4.

If the number $\bar{D}$ is a constant, given $\varepsilon > 0$, we can really find a $\delta > 0$ such that, if the Poincaré distance initially is $\varrho_0 < \delta$, then inequality $\varrho_n < \varepsilon$, for all n, can be implied, since we can select $\delta = \varepsilon$, and $d_0 = \delta$, when $\varrho_0 < \delta$ implies $\varrho_n < \varepsilon$, and S is «orbitally stable».

If $\bar{D} = 0$, then $\bar{\varrho} = 0$, and S is «orbitally asymptotically stable».

If $\bar{D} = \infty$, S is «orbitally unstable».

3. 3 Example. As an example, we mention the case $L = \bar{L}_1 =$ constant, treated in Ref. 1.

The corresponding helicoid precession is in this case given by:

$$\left.\begin{aligned} \omega_1(t) = \frac{\bar{L}_1}{I_1} t, \quad \omega_2(t) = \omega_{20}\cos Q_1(t), \quad \omega_3(t) = \omega_{20}\sin Q_1(t) \\ Q_1(t) = (I_1 - I)\bar{L}_1 t^2 / 2 I I_1 \end{aligned}\right\} \quad (8)$$

This precession, due to the form of $Q_1(t)$, is «Liapunov unstable»; but it is «orbitally asymptotically stable», since the distance D_n is given by: $D_n = \alpha\left(\sqrt{n+1} - \sqrt{n}\right)$, $\alpha =$ constant, and of which the limit, as $t \to \infty$, is $\bar{D} = 0$.

For this example, we can determine the region of the permitted deviations of the precessional curve, the «ε-region», and the corresponding region of the initial points, the «δ-region», for which regions the helicoid precession is orbitally asymptotically stable, when this stability situation of (8) has a practical importance.

Given a point $(\varpi_{10}, \varpi_{20}, \varpi_{30})$ on the surface of the cylinder as a starting point of a helicoid precessional curve (8), the coordinates $\omega_1, \omega_2, \omega_3$ of any point of this curve are related to $\omega_{10}, \omega_{20}, \omega_{30}$ by:

$$\omega_2^2 + \omega_3^2 = \omega_{20}^2 + \omega_{30}^2 = \text{constant} \qquad (8.1)$$

$$(\omega_1)_n \leqslant \varpi_{10} \leqslant (\omega_1)_{n+1} \qquad (8.2)$$

The inequality (8. 2), by using $(\omega_1)_n = \alpha\sqrt{n}$, leads to:

$$n \leqslant \bar{\omega}_{10}/\alpha \leqslant n+1 \tag{8 3}$$

from which the integer n can be determined, when, as a result, $D_n = \alpha(\sqrt{n+1} - \sqrt{n})$ is known. This distance D_n is the upper limit of δ and ε.

We remark that we can calculate the «ε, δ-regions» of any member S of the helicoid precessions (3), if the distance $\bar{D}$ of S is zero or finite, when the orbital stability situation of S, and not its Liapunov stability situation, has a practical meaning.

3. 4 Remarks. We saw above that, for the same phenomenon, we have different stability situations, if we apply different stability concepts.

There arises the problem of the selection of the stability concept appropriate to the phenomenon, that is of the selection of the stability situation, which interprets the reality in an adequate way, and it is more close to «practical stability» of the phenomenon.

The possibility of the determination, by using a physical situation, of the region of the permitted deviations of motion and orbit, of the corresponding region of the initial points, and of the appropriate region of the perturbation, in case of persistent perturbations, Ref. 5, makes the stability results practically important and physically accepted.

Stability investigations, which may satisfy mathematical curiosities or needs, will become useful if they are oriented towards «practical usefulness».

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ΠΕΡΙΛΗΨΙΣ

Εἰς προηγουμένην ἐργασίαν, ἀνακοινωθεῖσαν εἰς τὴν Ἀκαδημίαν τῶν Παρισίων (1969), ἐμελετήθη ἡ εὐστάθεια μιᾶς ἑλικοειδοῦς μεταπτώσεως εἰς τὴν περίπτωσιν σταθερᾶς ἐξωτερικῆς ροπῆς μὲ χρησιμοποίησιν διαφόρων ὁρισμῶν εὐσταθείας, καὶ εὑρέθησαν διαφορετικαὶ καταστάσεις εὐσταθείας, ὑπεδείχθη δὲ ποία ἐκ τῶν καταστάσεων εὐσταθείας τῆς ἑλικοειδοῦς ἔχει πρακτικὴν ἀξίαν.

Εἰς τὴν παροῦσαν ἐργασίαν μελετᾶται ἡ κατάστασις εὐσταθείας μιᾶς κλάσεως ἑλικοειδῶν μεταπτώσεων, ποὺ περιέχει ὡς ἕνα μέλος της τὴν ἑλικοειδῆ μετάπτωσιν τῆς προηγουμένης ἐργασίας.

Χρησιμοποιοῦνται δύο ὁρισμοὶ εὐσταθείας, οἱ κατὰ Liapunov καὶ Poincaré. Τὰ συμπεράσματα τῆς παρούσης ἐργασίας εἶναι :

α. Ὅλα τὰ μέλη τῆς κλάσεως τῶν ἑλικοειδῶν μεταπτώσεων εἶναι εἰς ἀσταθῆ κατὰ Liapunov κατάστασιν.

β. Ἡ κατὰ Poincaré κατάστασις εὐσταθείας οἱουδήποτε μέλους S τῆς κλάσεως ἐξαρτᾶται ἀπὸ τὴν ὁριακὴν τιμὴν τοῦ βήματος τῆς ἑλικοειδοῦς S, καὶ ὅταν ἡ ὁριακὴ τιμὴ εἶναι σταθερὰ ἢ μηδὲν ἢ ἄπειρον, τότε ἡ ἑλικοειδὴς εἶναι εὐσταθής, ἢ ἀσυμπτωτικὰ εὐσταθὴς ἢ ἀσταθής, ἀντιστοίχως.

γ. Εἰς τὴν περίπτωσιν ποὺ ἡ ἑλικοειδὴς S εἶναι ἀσυμπτωτικὰ εὐσταθής, τότε ἡ κατάστασις αὐτὴ καὶ μόνον ἔχει πρακτικὴν ἀξίαν.

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Ἔχω τὴν τιμὴν νὰ παρουσιάσω εἰς τὴν Ἀκαδημίαν Ἀθηνῶν τὴν ἐργασίαν τοῦ κ. Δημητρίου Μαγείρου, Ἐπιστημονικοῦ Συμβούλου τῆς General Electric τῶν Η.Π.Α. ὑπὸ τὸν τίτλον «Ἡ Εὐστάθεια μιᾶς Κλάσεως Ἑλικοειδῶν Μεταπτώσεων κατὰ Liapunov καί Poincaré».

Ὁ κ. Μάγειρος εἰς προηγουμένην ἐργασίαν του, ἀνακοινωθεῖσαν εἰς τὴν Ἀκαδημίαν Παρισίων, ἐμελέτησε τὴν εὐστάθειαν μιᾶς Ἑλικοειδοῦς Μεταπτώσεως εἰς τὴν περίπτωσιν σταθερᾶς ἐξωτερικῆς ροπῆς.

Εἰς τὴν παροῦσαν ἐργασίαν μελετᾶται ἡ κατάστασις εὐσταθείας μιᾶς κλάσεως ἑλικοειδῶν μεταπτώσεων εἰς τὴν ὁποίαν ἓν ἐκ τῶν μελῶν της εἶναι καὶ ἡ ἑλικοειδὴς μετάπτωσις τῆς ἀναφερθείσης ἤδη προηγουμένης ἐργασίας.

Χρησιμοποιοῦνται πρὸς τοῦτο οἱ ὁρισμοὶ εὐσταθείας κατὰ Liapunov καὶ Poincaré, τὰ δὲ ἀντίστοιχα πορίσματα τῆς ἐρεύνης εἶναι τὰ κάτωθι :

α) Ὅλα τὰ μέλη τῆς κλάσεως τῶν ἑλικοειδῶν μεταπτώσεων εὑρίσκονται εἰς ἀσταθῆ κατὰ Liapunov κατάστασιν.

β) Ἡ κατὰ Poincaré κατάστασις εὐσταθείας οἱουδήποτε μέλους τῆς ἐξεταζομένης κλάσεως ἐξαρτᾶται ἀπὸ τὴν ὁριακὴν τιμὴν τοῦ βήματος τῆς ἑλικοειδοῦς. Οὕτω, ὅταν ἡ ὁριακὴ τιμὴ εἶναι σταθερὰ ἢ μηδὲν ἢ ἄπειρος, τότε ἡ ἑλικοειδὴς εἶναι ἀντιστοίχως εὐσταθής, ἀσυμπτωματικὰ εὐσταθής, ἢ ἀσταθής.

γ) Εἰς τὴν περίπτωσιν ὅπου ἡ ὁριακὴ τιμὴ εἶναι μηδέν, ὁπότε ἡ ἑλικοειδὴς εἶναι ἀσυμπτωματικὰ εὐσταθής, τότε καὶ μόνον τότε ἡ κατάστασις αὕτη ἔχει πρακτικὴν ἀξίαν.