*πρωτοβουλίας κατέβαλεν ἀπὸ τετραετίας περίπου μεγάλους κόπους διὰ τὴν μελέτην τοῦ εἰρημένου θέματος.*

*Δηλῶ δὲ πρὸς τοὺς κ. Συναδέλφους ὅτι προτίθεμαι νὰ ὑποβάλω πρὸς τὴν 'Ακαδημίαν 'Αθηνῶν πρότασιν, ὅπως, συμφώνως πρὸς ὃ ἔχει δικαίωμα, εἰσηγηθῇ πρὸς τὴν Κυβέρνησιν τῆς χώρας τὴν σύστασιν εἰδικῆς Ἐπιτροπῆς, ἥτις νὰ προγραμματίσῃ τὰ ληπτέα μέτρα πρὸς ἀποφυγὴν τῶν εἰρημένων κινδύνων.*

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ΜΑΘΗΜΑΤΙΚΑ.—. **An analysis of the Euler-Lagrange equations***, *ὑπὸ*
*Max Herzberger***. 'Ανεκοινώθη ὑπὸ τοῦ 'Ακαδημαϊκοῦ κ. 'Ιωάνν. Ξανθάκη.

Dedicated to the memory of Professor Constantin Caratheodory, the author's revered teacher and friend.

## Abstract.

The Euler - Lagrange equations in different fields of mathematics and physics can be transformed into the form given in optics by adding one more variable. The author gives to these equations a normal form which lends itself to the determining of the extremals, if the point and the direction are given. It is emphasized that the form of the equations is independent of the choice of variables.

## I. Theory.

Let $\ell$ be a function of position and direction, $\ell\,(x_i, \dot{x}_i)$, where $\ell$ is homogeneous in the first order in the $\dot{x}_i$, so that $E = \int_{\text{curve}} \ell\, ds$ is independent of the curve parameter. This parameter can therefore be taken as the arc length along the path, which is equivalent to stating that $\dot{x}_i^2 = 1$***. The curves for which Euler's differential equations,

$$\frac{d}{ds}\left(\frac{\partial \ell}{\partial \dot{x}}\right) = \frac{\partial \ell}{\partial x_i} \tag{1.1}$$

hold will be called *world lines.*

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* *Communication No. 2227 from the Kodak Research Laboratories.*

** **MAX HERZBERGER : 'Ανάλυσις τῶν ἐξισώσεων Euler - Lagrange.**

*** Starting with this equation, we shall use the Einstein convention of summing over any index appearing twice in the formula and leaving out the summation sign.

Now let us introduce the following vector notation:

$$\vec{\dot{a}}: \quad \text{components } \dot{x}_i = \frac{dx_i}{ds}$$

$$\vec{g}: \quad \text{components } \frac{\partial \ell}{\partial x_i}$$

$$\vec{p}: \quad \text{components } \frac{\partial \ell}{\partial \dot{x}_i}$$

and choose as parameter the arc length along the world lines.

Since $\ell$ is homogeneous in the first order in the $\dot{x}_i$, we have the following equation:

$$\Sigma \frac{\partial \ell}{\partial \dot{x}_k} \dot{x}_k = \frac{\partial \ell}{\partial \dot{x}_k} \dot{x}_k = \ell = \vec{p}\ \vec{\dot{a}}. \tag{1.2}$$

Differentiation of this equation with respect to the $\dot{x}_i$ and $x_i$, respectively, gives

$$\frac{\partial^2 \ell}{\partial \dot{x}_k \partial \dot{x}_i} \dot{x}_i = 0$$
$$\frac{\partial^2 \ell}{\partial \dot{x}_k \partial x_i} \dot{x}_k = \frac{\partial \ell}{\partial x_i}. \tag{1.3}$$

Since $\ell$ is a function of the $x_i$ and $\dot{x}_i$, the differentiation in equation (1.1) can be carried out and the equation can be written

$$\frac{\partial^2 \ell}{\partial \dot{x}_i \partial \dot{x}_k} \ddot{x}_k + \frac{\partial^2 \ell}{\partial \dot{x}_i \partial x_k} \dot{x}_k = \frac{\partial \ell}{\partial x_i}. \tag{1.4}$$

By using the second equation of (1.3), equation (1.4) finally takes the form:

$$\frac{\partial^2 \ell}{\partial \dot{x}_i \partial \dot{x}_k} \ddot{x}_k = \left( \frac{\partial^2 \ell}{\partial \dot{x}_k \partial x_i} - \frac{\partial^2 \ell}{\partial \dot{x}_i \partial x_k} \right) \dot{x}_k. \tag{1.5}$$

We shall consider this as the normal form of the Euler differential equations. The reader should notice that it connects two tensors important for the problem, namely, a symmetric one, given by the values of $\left( \frac{\partial^2 \ell}{\partial \dot{x}_i \partial \dot{x}_k} \right)$, and an antisymmetric one, given by

$$a_{ik} = - a_{ki} = \left( \frac{\partial^2 \ell}{\partial \dot{x}_k \partial x_i} - \frac{\partial^2 \ell}{\partial \dot{x}_i \partial x_k} \right). \tag{1.6}$$

The $a_{ik}$ are the components of the curl of $\vec{p}$, which in n-dimensional space is a bivector in the sense of Grassmann's theory, i. e., a vector with $\binom{n}{2}$ components*. In any case it is obvious that $\vec{A} \cdot \dot{\vec{a}} = 0$.

The solution of equation (1.5) cannot be attained by the normal method since the first equation in (1.3) shows that the determinant of the symmetric tensor $\left( \frac{\partial^2 \ell}{\partial \dot{x}_i \partial \dot{x}_k} \right)$ vanishes.

We introduce vectors $\vec{\ell_i}$ whose components are $\left( \frac{\partial^2 \ell}{\partial \dot{x}_i \partial \dot{x}_k} \right)$. Equations (1.3) and (1.5) then become

$$\left.\begin{aligned} (\vec{\ell_i}\, \dot{\vec{a}}) &= \vec{0} \\ (\vec{\ell_i}\, \ddot{\vec{a}}) &= \vec{A} \end{aligned}\right\}^{**} \quad \text{with the side conditions} \quad \left\{\begin{aligned} \dot{\vec{a}}^2 &= 1 \\ \dot{\vec{a}}\, \ddot{\vec{a}} &= 0. \end{aligned}\right. \tag{1.8}$$

We assume that the matrix of the $\frac{\partial^2 \ell}{\partial \dot{x}_i \partial \dot{x}_k}$ has the rank (n-1), i. e., that the n vectors $\vec{\ell_i}$ lie in an (n-1)-dimensional hyperplane. The vector $\dot{\vec{a}}$ is then the vector of unit length which is normal to this plane, and as such is well determined. The vector $\ddot{\vec{a}}$, on the other hand, is normal to $\dot{\vec{a}}$ and therefore lies in the hyperplane of the $\vec{\ell_i}$.

The analytical solution proceeds as follows: From (1.8) we derive (with arbitrary Lagrange multiplier λ)

$$((\vec{\ell_i} + \lambda \dot{\vec{a}}) \cdot \ddot{\vec{a}}) = \vec{A}, \tag{1.9}$$

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*The vector on the right-hand side of (1.5) can then be considered as the supplement of the outer product of the supplement of curl $\vec{p}$ with $\dot{\vec{a}}$; or, if the supplement is designated by the sign S, we find for the vector $\vec{A}$ on the right-hand side:

$$\vec{A} = S[\dot{\vec{a}}\ \ S(\text{curl}\ \vec{p})], \tag{1.7}$$

which in three-space is equivalent to the vector product

$$\vec{A} = \dot{\vec{a}} \times \text{curl}\ \vec{p}.$$

** This abbreviated way of writing indicates that the n dot-products on the left-hand side give the components of the vector on the right-hand side.

an equation which cannot have a vanishing determinant for all $\lambda$. We can show that the solution is independent of $\lambda$.

We shall illustrate the method of solving (1.9) by carrying out in detail the solution for the case of $n = 4$. Equation (1.9) shows that, for instance, the third component of $\vec{\ddot{a}}$, namely $\ddot{x}_3$, is given by the ratio

$$\ddot{x}_3 = \frac{\begin{vmatrix} \ell_{11}+\lambda\dot{x}_1 & \ell_{12}+\lambda\dot{x}_2 & A_1 & \ell_{14}+\lambda\dot{x}_4 \\ \ell_{12}+\lambda\dot{x}_1 & \ell_{22}+\lambda\dot{x}_2 & A_2 & \ell_{24}+\lambda\dot{x}_4 \\ \ell_{13}+\lambda\dot{x}_1 & \ell_{23}+\lambda\dot{x}_2 & A_3 & \ell_{34}+\lambda\dot{x}_4 \\ \ell_{14}+\lambda\dot{x}_1 & \ell_{24}+\lambda\dot{x}_2 & A_4 & \ell_{44}+\lambda\dot{x}_4 \end{vmatrix}}{\begin{vmatrix} \ell_{11}+\lambda\dot{x}_1 & \ell_{12}+\lambda\dot{x}_2 & \ell_{13}+\lambda\dot{x}_3 & \ell_{14}+\lambda\dot{x}_4 \\ \ell_{12}+\lambda\dot{x}_1 & \ell_{22}+\lambda\dot{x}_2 & \ell_{23}+\lambda\dot{x}_3 & \ell_{24}+\lambda\dot{x}_4 \\ \ell_{13}+\lambda\dot{x}_1 & \ell_{23}+\lambda\dot{x}_2 & \ell_{33}+\lambda\dot{x}_3 & \ell_{34}+\lambda\dot{x}_4 \\ \ell_{14}+\lambda\dot{x}_1 & \ell_{24}+\lambda\dot{x}_2 & \ell_{34}+\lambda\dot{x}_3 & \ell_{44}+\lambda\dot{x}_4 \end{vmatrix}}, \qquad (1.10)$$

where

$$\begin{aligned} A_1 &= \phantom{-a_{12}\dot{x}_1 - a_{23}\dot{x}_2} a_{12}\dot{x}_2 + a_{13}\dot{x}_3 + a_{14}\dot{x}_4 \\ A_2 &= -a_{12}\dot{x}_1 \phantom{- a_{23}\dot{x}_2} + a_{23}\dot{x}_3 + a_{24}\dot{x}_4 \\ A_3 &= -a_{13}\dot{x}_1 - a_{23}\dot{x}_2 \phantom{+ a_{23}\dot{x}_3} + a_{34}\dot{x}_4 \\ A_4 &= -a_{14}\dot{x}_1 - a_{24}\dot{x}_4 - a_{34}\dot{x}_3. \end{aligned} \qquad (1.11)$$

For $\lambda = 0$ both the numerator and the denominator vanish. When the determinants are developed with respect to $\lambda$, the coefficients of all powers except the linear ones vanish. Let $N_3$ and D represent the determinants resulting from setting $\lambda = 1$ in the numerator and denominator, respectively, of (1.10). Then we can write

$$\ddot{x}_3 = \frac{\lambda N_3}{\lambda D} = \frac{N_3}{D}, \qquad (1.12)$$

showing that $\ddot{x}_3$ is independent of $\lambda$.

We shall develop $N_3$ and D separately. If we represent a determinant, for brevity, by its general row, then we can write

$$\begin{aligned} D = \dot{x}_1 \mid 1, \ell_{i2}, \ell_{i3}, \ell_{i4} \mid + \dot{x}_2 \mid \ell_{i1}, 1, \ell_{i3}, \ell_{i4} \mid \\ + \dot{x}_3 \mid \ell_{i1}, \ell_{i2}, 1, \ell_{i4} \mid + \dot{x}_4 \mid \ell_{i1}, \ell_{i2}, \ell_{i3}, 1 \mid . \end{aligned} \tag{1.13}$$

We now multiply the first row of the first determinant in (1.13) by the factor $\dot{x}_1$ which precedes that determinant. Then we add to this row the sum of the other three rows, each multiplied by its corresponding $\dot{x}_i$. In like manner, we operate on the second row of the second determinant and so on for the third and fourth determinants. Then, remembering that $\ell_{ik}\dot{x}_k=0$, we have

$$\begin{aligned} D = \Bigg\{ \begin{vmatrix} \ell_{22} & \ell_{23} & \ell_{24} \\ \ell_{23} & \ell_{33} & \ell_{34} \\ \ell_{24} & \ell_{34} & \ell_{44} \end{vmatrix} + \begin{vmatrix} \ell_{11} & \ell_{13} & \ell_{14} \\ \ell_{13} & \ell_{33} & \ell_{34} \\ \ell_{14} & \ell_{34} & \ell_{44} \end{vmatrix} \\ + \begin{vmatrix} \ell_{11} & \ell_{12} & \ell_{14} \\ \ell_{12} & \ell_{22} & \ell_{24} \\ \ell_{14} & \ell_{24} & \ell_{44} \end{vmatrix} + \begin{vmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{12} & \ell_{22} & \ell_{23} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{vmatrix} \Bigg\} (\dot{x}_1 + \dot{x}_2 + \dot{x}_3 + \dot{x}_4). \end{aligned} \tag{1.14}$$

This means that, aside from the factor $(\dot{x}_1 + \dot{x}_2 + \dot{x}_3 + \dot{x}_4)$, D is the sum of the principal minors of the $\ell_{ik}$.

Analogously, we find

$$\begin{aligned} N_{33} = \dot{x}_1 \mid 1, \ell_{i2}, A_i, \ell_{i4} \mid + \dot{x}_2 \mid \ell_{i1}, 1, A_i, \ell_{i4} \mid \\ + \dot{x}_4 \mid \ell_{i1}, \ell_{i2}, A_i, 1 \mid . \end{aligned} \tag{1.15}$$

Performing the same operation on this equation, we obtain

$$\begin{aligned} N_3 = \Bigg\{ \begin{vmatrix} \ell_{22} & A_2 & \ell_{24} \\ \ell_{23} & A_3 & \ell_{34} \\ \ell_{24} & A_4 & \ell_{44} \end{vmatrix} + \begin{vmatrix} \ell_{11} & A_1 & \ell_{14} \\ \ell_{13} & A_3 & \ell_{34} \\ \ell_{14} & A_4 & \ell_{44} \end{vmatrix} \\ + \begin{vmatrix} \ell_{11} & \ell_{12} & A_1 \\ \ell_{12} & \ell_{22} & A_2 \\ \ell_{13} & \ell_{23} & A_3 \end{vmatrix} \Bigg\} (\dot{x}_1 + \dot{x}_2 + \dot{x}_3 + \dot{x}_4). \end{aligned} \tag{1.16}$$

Remembering formula (1.11), we can develop (1.16) as a linear function of the $a_{ik}$. We then find for the coefficients of $a_{12}$, $a_{13}$, $a_{14}$, $a_{23}$, $a_{24}$, and $a_{34}$, respectively, the results given in formula (1.17).

The method of finding the coefficients can be exemplified by determining the coefficient of $a_{23}$ in (1.15). This coefficient occurs only in $A_2$ and $A_3$, where it is multiplied by $\dot{x}_3$ and $-\dot{x}_2$, respectively. Going back to (1.10), we find the numerator of the coefficient of $a_{23}$ to be

$$\begin{vmatrix} \ell_{11}+\lambda\dot{x}_1 & \ell_{12}+\lambda\dot{x}_2 & 0 & \ell_{14}+\lambda\dot{x}_4 \\ \ell_{12}+\lambda\dot{x}_1 & \ell_{22}+\lambda\dot{x}_2 & \dot{x}_3 & \ell_{24}+\lambda\dot{x}_4 \\ \ell_{13}+\lambda\dot{x}_1 & \ell_{23}+\lambda\dot{x}_2 & -\dot{x}_2 & \ell_{34}+\lambda\dot{x}_4 \\ \ell_{14}+\lambda\dot{x}_1 & \ell_{24}+\lambda\dot{x}_2 & 0 & \ell_{44}+\lambda\dot{x}_4 \end{vmatrix}.$$

Multiplying the first row by $\dot{x}_1$, the second by $\dot{x}_2$, and the fourth by $\dot{x}_4$, and adding the sum of these to the third row multiplied by $\dot{x}_3$ gives

$$\begin{vmatrix} \ell_{11} & \ell_{12} & 0 & \ell_{14} \\ \ell_{12} & \ell_{22} & 1 & \ell_{24} \\ \dot{x}_1 & \dot{x}_2 & 0 & \dot{x}_4 \\ \ell_{14} & \ell_{24} & 0 & \ell_{44} \end{vmatrix} (\dot{x}_1+\dot{x}_2+\dot{x}_3+\dot{x}_4),$$

or

$$\begin{vmatrix} \dot{x}_1 & \dot{x}_2 & \dot{x}_4 \\ \ell_{11} & \ell_{12} & \ell_{14} \\ \ell_{14} & \ell_{24} & \ell_{44} \end{vmatrix} (\dot{x}_1+\dot{x}_2+\dot{x}_3+\dot{x}_4).$$

This leads to the equation

$$N_3 = \Bigg\{ a_{12}\begin{vmatrix} \dot{x}_1 & \dot{x}_3 & \dot{x}_4 \\ \ell_{13} & \ell_{33} & \ell_{34} \\ \ell_{14} & \ell_{34} & \ell_{44} \end{vmatrix} - a_{13}\begin{vmatrix} \dot{x}_1 & \dot{x}_2 & \dot{x}_4 \\ \ell_{12} & \ell_{22} & \ell_{24} \\ \ell_{14} & \ell_{24} & \ell_{44} \end{vmatrix}$$
$$+ a_{14}\begin{vmatrix} \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ \ell_{12} & \ell_{22} & \ell_{23} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{vmatrix} + a_{23}\begin{vmatrix} \dot{x}_1 & \dot{x}_2 & \dot{x}_4 \\ \ell_{11} & \ell_{12} & \ell_{14} \\ \ell_{14} & \ell_{24} & \ell_{44} \end{vmatrix} \tag{1.17}$$
$$- a_{24}\begin{vmatrix} \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{vmatrix} + a_{34}\begin{vmatrix} \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \\ \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{12} & \ell_{22} & \ell_{23} \end{vmatrix} \Bigg\} (\dot{x}_1+\dot{x}_2+\dot{x}_3+\dot{x}_4).$$

Analogous formulae can be derived for $N_1$, $N_2$, and $N_4$. We see that the right-hand side for each $\ddot{x}_i$ is a linear function of the $a_{ik}$, which are components of curl $\vec{p}$. The method also works for the equations in n-dimensional space. Knowing $\ddot{\vec{a}}$ as a function of $\vec{a}$ and $\dot{\vec{a}}$, we can calculate $\dddot{\vec{a}}$ and higher-order derivatives and thus get all the analytical solutions of the problem either in closed form or, if calculating machines are used, by approximation. Since the Euler equations are relations between two gradients, one with respect to $x_i$, and one with respect to $\dot{x}_i$, the equation is independent of the choice of variables, as can also be shown explicitly.

The equations can be used to determine the world lines for special forms of $\ell$, for instance, if the problem in question obeys certain conditions of symmetry. As an example, we treat the case that the space is isotropic, i. e, that $\ell$ does not depend on the $\dot{x}_i$.

Fulfilling the requirement that $\ell$ be homogeneous in the first order in the $\dot{x}_i$, we can write

$$\ell = n\sqrt{\dot{x}_i^2}, \tag{1.18}$$

where n is a function of position only, since $\sqrt{\dot{x}_i^2} = 1$. In this case

$$\vec{p} = \left(\frac{\partial \ell}{\partial \dot{x}_i}\right) = \left(\frac{n\dot{x}_i^2}{\sqrt{\dot{x}_i^2}}\right) = n\,\vec{a}$$

$$\frac{\partial^2 \ell}{\partial \dot{x}_i \partial \dot{x}_k} = n(\delta_{ik} - \dot{x}_i\dot{x}_k), \tag{1.19}$$

where

$$\delta_{ik} = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k. \end{cases}$$

Equation (1.5) then gives in this special case

$$\frac{\partial^2 \ell}{\partial \dot{x}_i \partial \dot{x}_k}\,\dot{x}_k = \left(\frac{\partial n}{\partial x_i}\,\dot{x}_k - \frac{\partial n}{\partial x_k}\,\dot{x}_i\right)\dot{x}_k,$$

and equation (1.5) reduces to

$$n\ddot{x}_i = \left(\frac{\partial n}{\partial x_i}\,\dot{x}_k - \frac{\partial n}{\partial x_k}\,\dot{x}_i\right)\dot{x}_k. \tag{1.20}$$

Introducing the vector $\vec{g} = \left(\frac{1}{n} \frac{\partial n}{\partial x_i}\right)$, which is the logarithmic gradient of n, we obtain the vector equation

$$\ddot{\vec{a}} = \vec{g}\,(\dot{\vec{a}}^2) - \dot{\vec{a}}\,(\vec{g}\ \dot{\vec{a}}) = S\,[\,S\,(\vec{g}\ \dot{\vec{a}})\ \dot{\vec{a}}], \tag{1.21}$$

which is the Grassmann generalization of the three-dimensional formula

$$\ddot{\vec{a}} = \dot{\vec{a}} \times (\vec{g} \times \dot{\vec{a}}). \tag{1.22}$$

## II. Applications.

The solution of the Euler differential equations is of importance in many problems of mathematics and physics.

a) *The general problem of the calculus of variations.* Given a function $L(X_i, T, \frac{dX_i}{dT})$, where L can be general (in this paper we shall assume that it is twice continuously differentiable, except for singular points). The integral

$$\int L(X_i, T, \frac{dX_i}{dT})\, dT \tag{2.1}$$

is then not independent of the curve parameter and we have not assumed homogeneity with respect to the $\frac{dX_i}{dT}$. However, if the problem is considered in (n + 1) – dimensional space, i. e., if we set

$$\begin{aligned} x_i(s) &= X_i \\ x_{n+1}(s) &= T, \end{aligned} \tag{2.2}$$

where s is an arbitrary parameter, we find

$$X_i' = \frac{dX_i}{dT} = \frac{\dot{x}_i}{\dot{x}_{n+1}}, \tag{2.3}$$

the points designating differentiation with respect to s. We find, moreover,

$$dT = \dot{x}_{n+1} ds,$$

and therefore

$$\int L dT = \int \dot{x}_{n+1} L ds. \tag{2.4}$$

Inserting

$$\dot{x}_{n+1} L = \ell, \tag{2.5}$$

we find that $\ell$ is a function of $x_i$ and $\dot{x}_i$ alone, not containing any other parameter, and that $\ell$ is, with respect to the $\dot{x}_i$, homogeneous in zero order. The function $\ell$ given by equation (2.5) is thus homogeneous in the first order in the $\dot{x}_i$. This brings the general problem within the class of problems treated in the previous section.

In particular, we have the identities

$$\frac{\partial \ell}{\partial \dot{x}_k} = \frac{\partial L}{\partial X'_k}$$

for $$k \neq n+1 \tag{2.6}$$

$$\frac{\partial \ell}{\partial \dot{x}_{n+1}} = L - \frac{\partial L}{\partial X'_i} X'_i = H.$$

The last component vanishes only if L is homogeneous in the first order in the $X'_i$.

Equation (1.1) gives the corresponding n equations

$$\frac{d}{dT}\left(\frac{\partial L}{\partial X'_k}\right) = \frac{\partial L}{\partial X_k} \tag{2.7}$$

and adds the equation

$$\frac{d}{dT}(H) = \frac{\partial L}{\partial T}. \tag{2.8}$$

If the determinant of the $\dfrac{\partial^2 L}{\partial X'_i \partial X'_k}$ is different from zero, the determinant $\dfrac{\partial^2 \ell}{\partial \dot{x}_i \partial \dot{x}_k}$ with $(n+1)$ rows and columns has the rank n.

Without further detailed derivation, we may mention a few other problems which lead to the basic equation (1.1):

(b) the characteristics of a partial differential equation ;

(c) the theory of conformal mapping (and therefore the theory of analytical functions);

(d) the theory of geodetic lines in Euclidean and non-Euclidean geometry;

(e) differential geometry;

(f) contact transformations ;

(g) all of classical physics and much of quantum mechanics.

And finally we may point out that $\partial L/\partial x_i$ and $\partial L/\partial \dot{x}_i$ are gradient functions which keep their form for generalized coordinates.

ΠΕΡΙΛΗΨΙΣ

Αἱ ἐξισώσεις Euler - Lagrange εἰς διάφορα πεδία τῶν Μαθηματικῶν καὶ τῆς Φυσικῆς δύνανται νὰ μετασχηματισθοῦν εἰς μορφήν, ἡ ὁποία δίδεται εἰς τὴν ὀπτικὴν διὰ προσθήκης μιᾶς ἐπὶ πλέον μεταβλητῆς.

Ὁ συγγραφεὺς δίδει εἰς τὰς ἐξισώσεις αὐτὰς μίαν κανονικὴν μορφήν, ἡ ὁποία, δεδομένων τοῦ σημείου καὶ τῆς διευθύνσεως, ὁδηγεῖ εἰς τὸν προσδιορισμὸν τῶν ἀκροτάτων τιμῶν. Τονίζεται ὅτι ἡ μορφὴ τῶν ἐξισώσεων εἶναι ἀνεξάρτητος τῆς ἐκλογῆς τῶν μεταβλητῶν.

* * *

Ὁ Ἀκαδημαϊκὸς κ. *Ἰωάνν. Ξανθάκης* ἀνακοινῶν τὴν ὡς ἄνω μελέτην εἶπεν τὰ ἑξῆς:

*Ὡς γνωστὸν αἱ διαφορικαὶ ἐξισώσεις τῶν Euler - Lagrange ἐνέχουν σπουδαιοτάτην σημασίαν εἰς τὴν λύσιν πολλῶν προβλημάτων τῶν Μαθηματικῶν καὶ τῆς Φυσικῆς. Ὁ κ. Herzberger, ὅστις διετέλεσε μαθητὴς καὶ συνεργάτης τοῦ ἀειμνήστου Κωνσταντίνου Καραθοδωρῆ εἰς μνήμην τοῦ ὁποίου καὶ ἀφιερώνει τὴν ἀξιόλογον ταύτην ἐργασίαν του, ἀποδεικνύει ὅτι αἱ ἐξισώσεις αὗται ἐπὶ διαφόρων πεδίων τῶν Μαθηματικῶν καὶ τῆς Φυσικῆς δύνανται νὰ μετασχηματισθοῦν εἰς μορφήν, ἡ ὁποία εἰς τὴν ὀπτικὴν δίδεται διὰ τῆς προσθήκης μιᾶς ἐπὶ πλέον μεταβλητῆς. Ἡ κανονικὴ μορφὴ τῶν ἐξισώσεων τούτων ὁδηγεῖ, δεδομένων τοῦ σημείου καὶ τῆς διευθύνσεως, εἰς τὸν προσδιορισμὸν τῶν ἀκροτάτων τιμῶν. Ἰδιαιτέρως δὲ τονίζεται ὅτι ἡ ἐν λόγῳ μορφὴ τῶν ἐξισώσεων εἶναι ἀνεξάρτητος τῆς ἐκλογῆς τῶν μεταβλητῶν.*