ΦΥΣΙΚΗ.— **Eigenvalue Bounds to the Boltzmann Equation**, *by C. Syros**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Κ. Ἀλεξοπούλου.

The derivation of the time dependent Boltzmann distributions requires the knowledge of the eigenvalue spectrum of the Boltzmann operator.

In this note an arbitrary but bounded kernel is considered depending on the space and velocity coordinates.

In what follows a spectral property is establisned giving a lower and an upper bound for the eigenvalues. The physical system has an arbitrary convex shape of any number of dimensions.

**Definition.**

1°. $x \in R$, where R is the set of all p-dimensional vectors, $R \subset R^p$.

2°. The diameter of R is finite.

3°. The convex surface of R is $S\,(S \subset R)$ and $d\vec{S}$ is the vector surface-element on S. n is the outwards pointing normal on S.

4°. $v \in U$, where U is the velocity space $U \subseteq V^p$.

5°. $\{\psi_\lambda(x, v),\ \lambda \,|\, \lambda \in \Lambda\}$ are the solutions and the eigenvalues. All $\psi_\lambda(x, v)$ are scalar functions of x and satisfy the equation

$$v \cdot \nabla_x \psi_\lambda(x, v) + \upsilon\, \Sigma_t(x, \upsilon)\, \psi_\lambda(x, v) = \lambda\, \varkappa(x, v)\, \psi_\lambda(x, v), \tag{1}$$

where $\upsilon = |v|$,

6°. $\nabla_x$ is the p-dimensional gradient operator acting on functions of x.

7°. $M^{ij}_{\mu\lambda} = \int_U d\upsilon'^p \int_R dx'^p\, \psi^{(i)}_\mu \int_U d\upsilon^p \int_R dx^p\, \varkappa\, \psi^{(j)}_\lambda; \quad \psi^{(1)}_\lambda = \mathrm{Re}\,\psi_\lambda,$

$\psi^{(2)}_\lambda = \mathrm{Im}\,\psi_\lambda, \quad i, j = 1, 2.$

The various quantities in Eq. (1) are specified in the following:

**Assumptions.**

1°. $\psi_\lambda(x, v) \in H$; $(\forall \lambda \,|\, \lambda \in \Lambda)$, where H is the space of all $L^2$-summable functions defined on $R \otimes U$.

* ΚΩΝΣΤ. ΣΥΡΟΣ, **Φραγμοὶ ἰδιοτιμῶν τῆς ἐξισώσεως Boltzmann.**

2°. The linear bounded operator, $\varkappa(x, v)$, is the scattering kernel defined on H by

$$\varkappa(x, v)\, \psi(x, v) = \int_R \int_U dx^p\, dv^p\, \varkappa(x, x'; v, v')\, \psi(x', v').$$

3°. The total cross-section $\Sigma_t(x, \upsilon)$ is supposed such that

$$\left|\left(\psi_\mu(x, v),\, \upsilon\, \Sigma_t(x, \upsilon)\, \psi_\lambda(x, v)\right)\right| < \infty\, ; \; (\forall \lambda,\ \forall \mu \mid \lambda, \mu \in \Lambda).$$

With these definitions and assumptions we shall prove the following:

**Theorem.**

Let $\psi_\lambda(x, v)$, $\psi_\mu(x, v)$ be solutions of Eq. (1) satisfying the boundary condition $\psi_\lambda(x, v) = \psi_\mu(x, v) = 0\, ;\ \{(\forall x \mid x \in S) \wedge (\forall n, v \mid n \cdot v < 0)\}$. Let furtner $\lambda_1$, $\lambda_2$, $G_{\lambda\mu}$ be given numbers.

Then, the spectrum of the linear operator defining Eq. (1) satisfies:

1°. $$\lambda_1 \leqslant |\lambda| \leqslant \lambda_2$$

where

2°. $$\lambda_1 = \inf_{\lambda \in \Lambda} \left\{ \left[ \langle \upsilon\, \Sigma_t \rangle_{\lambda\bar{\lambda}} + \frac{1}{2} G_{\lambda\bar{\lambda}} \right] \Big/ \left[ (M^{11}_{\lambda\bar{\lambda}} + M^{22}_{\lambda\bar{\lambda}})^2 + (M^{12}_{\lambda\bar{\lambda}} - M^{21}_{\lambda\bar{\lambda}})^2 \right]^{\frac{1}{2}} \right\},^{(*)}$$

3°. $$\lambda_2 = \sup_{\lambda \in \Lambda} \left\{ \left[ |\langle \upsilon\, \Sigma_t \rangle_{\lambda\lambda}| + \frac{1}{2} |G_{\lambda\lambda}| \right] \Big/ \left[ (M^{11}_{\lambda\lambda} - M^{22}_{\lambda\lambda})^2 + (M^{12}_{\lambda\lambda} + M^{21}_{\lambda\lambda})^2 \right]^{\frac{1}{2}} \right\}$$

and

4°. $$G_{\lambda\mu} = \int_S \int_U dv^p\, v \cdot dS\, \psi_\lambda(x, v)\, \psi_\mu(x, v).$$

Proof:

From Eq. (1) it follows that

$$v \cdot \nabla_x (\psi_\lambda \psi_\mu) + 2\upsilon\, \Sigma_t(x, \upsilon)\, \psi_\lambda \psi_\mu = \lambda\, \psi_\mu \varkappa \psi_\lambda + \mu\, \psi_\lambda \varkappa \psi_\mu . \tag{2}$$

Since $\psi_\lambda$, $\psi_\mu$ are scalar functions of x it follows by integration that:

$$\lambda\, (\bar{\psi}_\mu, \varkappa \psi_\lambda) + \mu\, (\bar{\psi}_\lambda, \varkappa \psi_\mu) = 2 \langle \upsilon\, \Sigma_t(x, \upsilon) \rangle_{\lambda\mu} + G_{\lambda\mu} . \tag{3}$$

In Eq. (3) $\langle \upsilon\, \Sigma_t \rangle_{\lambda\mu}$ is given by

(*) $\bar{X}$ is the complex conjugate of X.

$$\langle v \Sigma_t(x, v) \rangle_{\lambda\mu} = \int_R d^p x \int_U d^p v\, \psi_\mu\, v\, \Sigma_t(x, v)\, \psi_\lambda \tag{4}$$

and $\psi_\lambda(x, v)\, \psi_\mu(x, v)$ being scalar function of x satisfies the theorem:

$$\int_U dv^p \int_R dx^p\; v \cdot \nabla_x (\psi_\lambda \psi_\mu) = \int_U dv^p \int_S v \cdot dS\, \psi_\lambda \psi_\mu . \tag{5}$$

If now is put $\mu = \bar{\lambda}$ in Eq. (3), it follows that the expectation value of $v\, \Sigma_t(x, v)$ is positive,

$$\langle v\, \Sigma_t(x, v) \rangle_{\lambda\bar{\lambda}} > 0. \tag{6}$$

On the other hand, since $\psi_\lambda(x, v) = 0$ for all x on the boundary surface S and for all velocities satisfying $v \,.\, n < 0$, it follows that the inetgrand in Eq. (5) is everywhere on S semi-positive. Therefore $G_{\lambda\bar{\lambda}}$ is also positive,

$$G_{\lambda\bar{\lambda}} > 0. \tag{7}$$

Moreover, it follows from Eq. (3) for $\mu = \bar{\lambda}$ after some calculations that

$$|\lambda| \geqslant \inf_{\lambda \in \Lambda} \left\{ \left[ \langle v \Sigma_t \rangle_{\lambda\bar{\lambda}} + \frac{1}{2} G_{\lambda\bar{\lambda}} \right] \Big/ \left[ (M^{11}_{\lambda\bar{\lambda}} + M^{22}_{\lambda\bar{\lambda}})^2 + (M^{12}_{\lambda\bar{\lambda}} - M^{21}_{\lambda\bar{\lambda}})^2 \right]^{\frac{1}{2}} \right\} \tag{8}$$

and for $\mu = \lambda$ we get from Eq. (3)

$$|\lambda| \leqslant \sup_{\lambda \in \Lambda} \left\{ \left[ |\langle v \Sigma_t \rangle_{\lambda\lambda}| + \frac{1}{2} |G_{\lambda\lambda}| \right] \Big/ \left[ (M^{11}_{\lambda\lambda} - M^{22}_{\lambda\lambda})^2 + (M^{12}_{\lambda\lambda} + M^{21}_{\lambda\lambda})^2 \right]^{\frac{1}{2}} \right\} \tag{9}$$

Q. E. D.

**Remark I.** If $\varkappa(x, v)$ is a self-adjoint operator, then it follows from Eq. (3) that,

$$\lambda = \left[ \langle v \Sigma_t \rangle_{\lambda\bar{\lambda}} + \frac{1}{2} G_{\lambda\bar{\lambda}} \right] \Big/ \langle \varkappa \rangle_{\lambda\bar{\lambda}} . \tag{10}$$

C o r o l l a r y : The values of λ for which solution of equation (1) exists are restricted on the annulus defined by $\lambda_1$ and $\lambda_2$.

## Π Ε Ρ Ι Λ Η Ψ Ι Σ

Ὑπὸ τὴν προϋπόθεσιν τῆς ἀθροιστικότητος - $L^2$ τῶν ἰδιοσυναρτήσεων τῆς γραμμικοποιημένης ἐξισώσεως Boltzmann προκύπτει μία φασματικὴ ἰδιότης. Δίδεται ἀπόδειξις δι' οἱονδήποτε ἀριθμὸν διαστάσεων $p \geqslant 1$ τοῦ $R^p$.

S U M M A R Y

A spectral property of the linearized Boltzmann equation is deduced from the $L^2$-summability assumption of the eigenfunctions. The proof is given for any number of dimensions $p \geqslant 1$ of $R^p$.

★

Ὁ Ἀκαδημαϊκὸς κ. Κ. **Ἀλεξόπουλος** λαβὼν τὸν λόγον εἶπε τὰ ἑξῆς:

Κύριε Πρόεδρε,

Λαμβάνω τὴν τιμὴν νὰ παρουσιάσω εἰς τὴν Ἀκαδημίαν Ἀθηνῶν ἐργασίαν τοῦ κ. Κωνσταντίνου Σύρου μὲ τίτλον «Χαρακτηριστικαὶ τιμαὶ τῶν γραμμικῶν ἐξισώσεων τοῦ Boltzmann».

Ὁ κ. Σῦρος εἶναι φυσικὸς ἐργαζόμενος εἰς τὴν Εὐρωπαϊκὴν Κοινότητα Ἀτομικῆς Ἐνεργείας ἐν Βρυξέλλαις, ἀπὸ ἐτῶν δὲ ἀσχολεῖται μεταξὺ ἄλλων μὲ τὴν ἐξίσωσιν Boltzmann. Ἡ ἐξίσωσις αὕτη ἐπιτρέπει τὴν μελέτην τῶν διαδοχικῶν μορφῶν, τὰς ὁποίας λαμβάνει ἡ κατάστασις ἑνὸς συνόλου σωματίων. Ἡ ἐξέλιξις τῶν θέσεων καὶ τῶν ταχυτήτων τῶν σωματίων εἶναι δυνατὸν νὰ μελετηθῇ, ὅταν εἶναι γνωστὴ ἡ ἀλληλεπίδρασις μεταξὺ τῶν σωματίων. Ὁ ὅρος ἀλληλεπίδρασις θὰ ἠδύνατο νὰ δηλοῖ τὰς δυνάμεις, αἱ ὁποῖαι ἐμφανίζονται κατὰ τὴν σύγκρουσιν δύο σωματίων. Ἀναλόγως, λοιπόν, τοῦ τύπου τῶν δυνάμεων ἐμφανίζονται διάφοροι μορφαὶ τῆς ἐξισώσεως Boltzmann.

Πρὸ ἑξαμήνου ἀνεκοίνωσα ἐνώπιον ὑμῶν λύσιν διὰ τὴν περίπτωσιν νετρονίων. Εἰς ἐκείνην τὴν περίπτωσιν ὁ κ. Σῦρος ἐμελέτησε τὸ στατικὸν πρόβλημα. Εἰς τὴν παροῦσαν ἀνακοίνωσιν μελετᾶται, ἀντιθέτως, τὸ πρόβλημα κατανομῆς Boltzmann μεταβαλλομένης χρονικῶς. Τὸ πρόβλημα τοῦτο ἀναφαίνεται, ὁσάκις μελετᾶται δέσμη νετρονίων μεταβλητῆς ἐντάσεως. Τὰ συμπεράσματα τῆς ἐργασίας διατυποῦνται ὑπὸ μορφὴν θεωρήματος, τὰς λεπτομερείας τοῦ ὁποίου θὰ πρέπει νὰ ἀναζητήσῃ ὁ ἐνδιαφερόμενος εἰς τὰ Πρακτικά.