ΜΑΘΗΜΑΤΙΚΑ.— **On a convenient category of topological algebras, II: Applications,** *by Anastasios Mallios**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φίλωνος Βασιλείου.

The following constitutes the Second Part of a longer paper, the First Part of which has been appeared under the title «On a convenient category of topological algebras, I: General theory», within Vol. 50, 1975 of this journal. The respective literature of this part of the paper is already included in that of the First Part (ibid.). The Sections herewith are also consecutively numbered in continuation to those of the First Part, the reference to which becomes also clear from the context. A supplementary bibliography referring to this Part of the paper is added, the references being indicated by small latin numbers.

It is hopped that there will be a Third Part as well of the present discussion, referred to «sectional representations», which have been alludded to at the Introduction of the First Part of this study.

5. Cohomology and homotopy in the spectra of topological algebras. The topological algebras which we are dealt with in the sequel are mostly such that they satisfy the conditions of Corollary 4.2 in the foregoing. Thus, in order to fix the terminology applied, we single their respective class out by putting the following:

Definition 5.1. By a *topological algebra of type* (A) (or, for short, an (A)-*algebra*) we shall mean a commutative, complete, finitely generated, spectrally barrelled, locally m-convex algebra, having an identity element and compact spectrum.

Thus, by the preceding definition and Corollary 4.2 in the foregoing, one concludes that *the spectrum of an* (A)-*algebra is homeomorphic to a compact polynomially convex subset of* $\mathbf{C}^n$, where n denotes the number of generators of the algebra under consideration. Furthermore, *it is equicontinuous, and the algebra itself is a Q-algebra, and hence a bounded algebra* as well (Corollary 3.1).

The preceding class of algebras contains, of course, that of finitely generated and commutative Banach algebras with an identity element

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(: *Banach algebras of type* (A)), and even more that of *Fréchet locally m-convex algebras of type* (A), i. e. commutative, finitely generated, with an identity element and compact spectrum (cf., for instance, Ref. [**10**]).

Now, it is our main objective in the following lines to point out that *a whole number of recent results obtained within the class of Banach algebras of type* (A), *or* more generally, of *their «colimits»* (cf., for instance, [**16**], [**48**], [**28**]) *are actually valid for the much wider class of topological algebras described by the preceding Definition 5.1, or their «colimits»,* more generally (cf. Definition 5. 2 below). However, since the present discussion is rather of a preliminary and mainly informative nature, we are content below with only giving indications of the results, which could be obtained, omitting thus the respective proofs. On the other hand, the latter are mostly based on the corresponding ones in the case of Banach algebras, by using essentially for the present case the preceding Corollary 4. 2, plus standard argumentation in topological algebras theory. Besides, in connection with the terminology applied, this is the analogous, within the present context, of the Banach algebras theory one, so that we refer to the pertinent cited works and the references given therein.

In this concern, we also note that a possible extension of relevant results for the case of Banach algebras to more general classes of topological algebras, however different in nature from those considered herein, has also been alludded to, although in a somewhat sceptic manner as to its significance, regarding applications, in Ref. [**57**].

Thus, we first have the following extended form, in our case, of a previous result, concerning the «*cohomology* (with complex coefficients) *of the spectrum* (: maximal ideal space)» *of a Banach algebra of type* (A), in the preceding terminology, which is initially due to A. Browder [**11**], and has also recently been discussed by O. Forster in Ref. [**16** ; p. 5, Satz 1], in the more general case of an arbitrary abelian «group of coefficients». The respective result in our case, as it is stated below, can be proved by an appropriate adaptation to the present context, of the argumentation applied in Ref. [**16**], as well as of that one in Ref. [**22**], used for the proof of the initial Browder's theorem (ibid. ; p. 67, Corollary 3. 1. 16), by taking into account Corollary 4. 2 in the foregoing. (In this concern, cf. also the comments in Ref. [**38** ; p. 160]). Thus, we have:

Theorem 5.1. *Let E be a topological algebra of type (A), which is n-generated* (cf. Definition 5.1), *and whose spectrum is $M(E)$. Moreover, let G be an abelian group. Then, one has the relation:*

$$\check{H}^q(M(E), G) = 0, \tag{5.1}$$

for every integer $q \geqslant n$. ∎

Now, it has been observed in Ref. [**38**; p. 160, Theorem 5.1] that *a commutative, complete, spectrally barrelled locally m-convex algebra with an identity element and compact spectrum is actually a pseudo-Banach algebra,* in the sense of Ref. [**2**]. On the other hand, *a locally convex inductive limit of Banach algebras is,* certainly, *a spectrally barrelled* (locally convex) *algebra* (cf., for instance, Ref. [**38**; p. 157, Lemma 4.2], an extended form of which has been given by Proposition 3.2 in the foregoing. In this connection, cf. also a relevant recent study in Ref. [**7**]). Besides, the well-known Arens-Royden Theorem has been given in Ref. [**2**] for the class of pseudo-Banach algebras (ibid.; p. 68, Theorem 4.5).

Furthermore, a «holomorphic functional calculus» is, of course, valid for the class of locally m-convex algebras (cf., for instance, [**8**; p. 412, Théorème 3]), hence one can obtain, within the same class of algebras, an extended version of the previous result of Arens-Royden, a form of which has already been given for Banach algebras by R. Arens in Ref. [**6**]. On the other hand, the extended form in question of the said result, constitutes essentially a strengthening, within the present context, of a similar discussion of O. Forster in Ref. [**16**; p. 9, Satz 5], in the framework of Banach algebras theory, as well as of a previous analogous one by M. E. Novodvorskii in Ref. [**43**].

In the sequel we will apply the corresponding terminology of the cited works above, without further comments, referring thus to those papers for more details or else to the standard literature for the notation involved.

Thus, given the topological spaces X and Y, we shall denote by $[C_c(X, Y)]$ $\big($resp. $\pi_0(C_c(X, Y))\big)$ the set of connected (resp. path-connected) components of the space $C_c(X, Y)$ of all continuous maps of X into Y, equipped with the «compact-open topology», while $[X, Y]$ will denote the set of homotopy classes of (continuous) maps of X into Y. In this respect, we first, need the following.

Lemma 5.1. *Let X be a topological space which is first countable or locally compact and Hausdorff, and let Y be a topological space in such a way that the space $C_c(X, Y)$ is locally path connected. Then,*

$$[C_c(X, Y)] = [X, Y], \tag{5.2}$$

or, in other words, one has the relation:

$$[C_c(X, Y)] \equiv \pi_0\big(C_c(X, Y)\big) = [X, Y], \tag{5.3}$$

where, in both the preceding relations, equality means set-theoretic isomorphism (: bijection). ∎

As an immediate consequence of the preceding Lemma 5.1, one now obtains the following corollary, which specializes to the first part of Theorem 1 in Ref. [**43**; p. 487]. That is, we have:

Corollary 5.1. *Let E be a topological algebra whose spectrum M(E) is a locally compact (Hausdorff) space, and let G be an open subset of a locally convex (topological vector) space F. Then, one has the relation:*

$$\pi_0\big(C_c(M(E), G)\big) = [M(E), G], \tag{5.4}$$

within a bijection. ∎

Remark.— Concerning the topological space G in the preceding Corollary 5.1, one could formulate a more general statement by considering instead an *ANR of a* given *locally convex space F,* or even *a complex (homogeneous) manifold modelled on a metrizable locally convex space.* (In this respect, cf. also Ref. [**44**]). A variant of the latter case will also be considered in the following.

Now, consider a topological algebra E, *whose spectrum* $M(\mathrm{E})$ *is a k-space*, and for which *the respective Gel'fand map* $g: \mathrm{E} \to C_c(M(\mathrm{E}))$ *is continuous.* Moreover, let F be *a complete locally convex* (topological vector) *space.* Then, by using the preceding map g, one obtains a continuous map:

$$\psi \equiv g \,\hat{\otimes}_{\varepsilon}\, \mathrm{id}_{\mathrm{F}} : \mathrm{E} \,\hat{\otimes}_{\varepsilon}\, \mathrm{F} \to C_c\big(M(\mathrm{E}), \mathrm{F}\big) \tag{5.5}$$

(cf. also Ref. [**34**; p. 478, Lemma 4.1]). Thus, if G is an open subset of F, one has a natural map:

$$(5.6) \qquad \pi_0(\psi) : \pi_0\left((E \hat{\otimes}_\varepsilon F)_G\right) \to \pi_0\left(C_c(M(E), F)_G\right),$$

where by $(E \hat{\otimes}_\varepsilon F)_G$ ($\equiv E_G$ by extension of the notation applied in Ref. [**43**]) one denotes *the set of those elements in* $Im(\psi)$, as in (5.5) above, *whose ranges are contained in G*, the set $\mathrm{Im}(\psi)$ being topologized as a subspace of the range of ψ; the analogous notation is applied concerning the range of the map $\pi_0(\psi)$.

In particular, we are interested in applying the foregoing to the case $F = \mathbf{C}^n$, in order *to get an expression of the respective relation to (5.4) above in terms of the domain of the corresponding map* $\pi_0(\psi)$. In this concern, Lemma 1 in Ref. [**43**; p. 491] might be the appropriate motivation. We omit the pertinent details for another treatment. On the other hand, the holomorphic functional calculus applied to the case considered herewith, plus a result of K. J. Ramspott, concernig *homotopy equivalence of holomorphic functions on a Stein manifold* (cf., for instance, Ref. [**45**; p. 58, Satz. 1]), provides the following basic result, this being besides one of the main conclusions of this Section, and which combined with Corollary 5.1 above constitutes, in our case, the extended version of Theorem 1 in Ref. [**43**; p. 487]. That is, one has the following:

Theorem 5.2. *Let E be a topological algebra of type (A)* (Definition 5.1), *whose spectrum is M(E). Moreover, let G be an open subset of* $\mathbf{C}^n$ *on which a complex analytic Lie group acts holomorphically and transitively. Then, one has the following relation:*

$$(5.7) \quad \pi_0(E_G) \equiv [E_G] = \pi_0\left(C_c(M(E), G)\right) \equiv \left[C_c(M(E))_G\right] = [M(E), G],$$

the respective equality relations holding true within a set-theoretic isomorphism (: bijection). ∎

Scholium 5.1. The conditions set forth in the hypothesis of the preceding Theorem 5.2. imply already that *the given algbra E is actually a Q-*(locally m-convex) *algebra, having continuous inversion.* In this concern, cf. also Ref. [**37**; p. 108, Corollary 3.3] and Ref. [**9**; p. 15, and p. 31]. Besides, Théorème 2 in p. 45 of the last reference should be compared with Theorem 5.2. above, as well as Theorem 5.3 in the sequel, in the sense that it also provides possible extensions of the relevant results obtained herewith, concerning the space G, with regard to the extension of the Arens - Calderón Theorem, considered by V. Ya. Lin in Ref. [**28**], and/or G. R. Allan in Ref. [**1**]. However, for simplicity's sake we only discussed, at this place, the case given by the preceding Theorem 5.2, which also corresponds to that one, studied by M. E. Novodvorskii, for Banach algebras, in Ref. [**43**].

We come now to a substantial extension of the preceding Theorem 5.2, which is essentially motivated by, and based on, Lemma 3 in Ref. **43**; p. 492]. Similar arguments are also applied in Ref. [**9**; p. 44] for the case considered therein. However, the kind of topological algebras which is dealt with in the sequel, although different in nature, is to a certain extend more general than that considered in the latter reference (cf. the definition, which follows).

Thus, we first have the following.

Definition 5.2. A complete locally m-convex algebra E is said to be a *topological algebra of type* (LA), or for short, an (LA) - *algebra,* whenever E is (algebraically) the limit of an inductive system $(E_\alpha)_{\alpha \in I}$ of sub-algebras of type (A) (Definition 5.1), in such a way that one has $M(E) = \varinjlim M(E_\alpha)$, within a homeomorphism.

Now, the proof of the folowing lemma can easily be supplied by applying standard reasoning. Its content will be useful for the sequel. That is, we have.

Lemma 5.3. *Suppose that a given algebra E is the limit of an inductive system $(E_\alpha)_{\alpha \in I}$ of topological algebras, each of which has an identity element and a compact spectrum $M(E_\alpha)$, $a \in I$. Moreover, suppose that E is endowed with the respective inductive limit vector space topology. Then, E is a topological algebra (with a separately continuous multiplication), having an identity element and a (non-void) compact spectrum M(E), this latter space being homeomorphic to the projective limit of the spectra of the given algebras E_α, $a \in I$, defining the algebra E.* ∎

It is quite evident that *the preceding lemma has,* in particular, *a special bearing on every* (LA) - *algebra,* in the sense of the above Definition 5.2. On the other hand, the consideration of a «final» vector space topology (not necessarily locally convex one) on the limit algebra of a given inductive system of topological algebras, which might be locally convex, or even locally m-convex ones, is also of a particular significance for applications of topological algebras theory to several complex variables. This was, of course, already applied by L. van. Hove [**23**], and it has also recently been considered by W. R. Zame, in connection with his work on the subject (cf., for instance, Ref. [**62**; p. 6, Definition 15, as well as p. 9, Remark 1.15]).

Thus, we now have the following.

Theorem 5.3. *Let E be an (LA)-algebra, whose spectrum is $M(E)$, and let G be a homogeneous complex analytic manifold, homotopically equivalent to an open subset of* $\mathbf{C}^n$. *Then, by applying analogous notation to that of the previous Theorem 5.2., one has the relation :*

$$[E_G] = [M(E), G] = [C_c(M(E), G)], \tag{5.8}$$

within a bijection. ∎

We are now in position to state our next basic result of this section which actually constitutes the topological algebra theory analogon of the corresponding one for Banach algebras, given by M. E. Novodvorskii in Ref. [**43**; p. 490, Theorem 2], and independently by O. Forster in Ref. [**16**; p. 12, Satz 6]. The same result, within the framework of Banach algebras theory, has also been considered by J. Wagner in Ref. [**58**] and called «Forster-Narashimhan Theorem» (ibid.; p. 166, Proposition). Thus, we have :

Theorem 5.4. *Let E be an (LA)-algebra, whose spectrum is $M(E)$. Then, the functor*

$$P \to P \otimes_E C(M(E)) \tag{5.9}$$

defines an equivalence between the category $\mathcal{P}(E)$ of projective, finitely generated E-modules and the category $\mathcal{P}(C(M(E)))$ of projective, finitely generated $C(M(E))$-modules. ∎

Now, a byproduct of the proof of the preceding Theorem 5.4 *is the following isomorphism :*

$$\mathscr{P}\big(C(M(\mathrm{E}))\big) = \mathscr{P}(\mathrm{E}), \tag{5.10}$$

so that this, combined with Swan's Theorem (cf., for instance [**iii**; p. 375, XIV, Theorem 3.1]) and the Homotopy Classification Theorem (cf. [**v**; p. 33, Theorem 7.2], *gives rise to the relation :*

$$[M(\mathrm{E}), G_n(\mathbf{C}^\infty)] = \mathrm{Vect}_n(M(\mathrm{E})) = \mathscr{P}\big(C(M(\mathrm{E}))\big) = \mathscr{P}(\mathrm{E}), \tag{5.11}$$

which is valid within an isomorphism of the sets involved. In this respect, we still denote by $\mathscr{P}(\mathrm{E})$ the set (actually *semiring*, with respect to the operations $\oplus$ and $\otimes$) of equivalence classes of projective, finitely generated E-modules, *with regard to the given (LA)-algebra E,* whose spectrum is M(E). The analogous notation is applied, concerning the first member of (5.10).

As a particular application of the foregoing one now obtains the following theorem, which might be considered as a *strengthened positive answer* to the well-known «Šilov's program», however yet in a rather complicated way, to the extent, at least, (algebraic and/or topological) K-Theory is (!). That is, one has.

Theorem 5.5. *Let $(\mathscr{L}A)$ be the category whose objects are (LA)-algebras, and morphisms the topological algebra ones. Then, the image of the K_0-functor (of algebraic K-Theory) is uniquely determined, within a homotopy equivalence, by the respective «topological algebra spectrum-functor» M, restricted to the given category $(\mathscr{L}A)$ (i.e., $M: (\mathscr{L}A) \to Top: E \to M(E)$).* ∎

In a less technical language, we may express the preceding, by saying that (LA)-*algebras having homotopic spectra are indistinguishable with regard to K_0-functor.* Thus (LA)-algebras, having *contractible spectra* are within the «K_0-class» (the class in $(\mathscr{L}A)$, as in Theorem 5.5 above, determined by the K_0-functor) 1-dimensional algebras, i.e. one has the relation :

$$\mathrm{K}_0(\mathrm{E}) = \mathrm{K}_0(\mathbf{C}) = \mathbf{Z}, \tag{5.12}$$

the last equality holding true within an isomorphism of the respective groups. (In this connection, cf. also Corollary 6.1 in the sequel).

Furthermore, as a byproduct of the preceding Theorem 5.5, one obtains, by applying «Bott periodicity», the following relation (5.13), which may be considered as *the topological - algebraic analogue in our case of «Künneth formula»*. That is, we have:

$$K_0(E \hat{\otimes}_{\tau} F) = K_0(E) \otimes K_0(F) \tag{5.13}$$

for (suitable) topological algebras E and F of type (LA) and a suitable completed topological tensor (product) algebra $E \hat{\otimes}_{\tau} F$.

Besides, by applying again (5.10), *an analogous «stability theorem»* with a respective one of algebraic K-theory, actually, corresponding to a convenient form of the «K_0-Stability Theorem», could also be obtained *within the present topological-algebraic context.*

On the other hand, another application of the preceding Theorem 5.4 is the following result, the analogon in our case of the respective one for Banach algebras, given by O. Forster in Ref. [**16**; p. 18, Satz 8]. That is, one obtains:

Theorem 5.6. *Let E be an (LA)-algebra. Then, one has the relation:*

$$\check{H}^2(M(E), \mathbf{Z}) = Pic(E), \tag{5.14}$$

within an isomorphism of the respective groups, where by Pic(E) is denoted the corresponding P i c a r d g r o u p *of the ring E, underlying the given algebra.* ∎

6. Matrices depending continuously on parameters. We are discussing in the sequel still another domain of applications of the class of topological algebras dealt with in the foregoing, by considering matrices with values in a given topological algebra of the type under discussion, which may be replaced by its corresponding Gel'fand transform (: function) algebra. The results obtained extend to our case recent analogous considerations of V. Ya. Lin in Ref. [**28**], who has worked with commutative, semi-simple Banach algebras having an identity element (ibid.; p. 127, Theorem 3). A similar treatment has also independently been given by N. Sibony and J. Wermer, within the same context of Banach algebras theory, in Ref. [**48**]. In the sequel we mainly follow the argumentation applied by Lin (ibid.), whose paper was besides the initial motivation to the material of this section, by extending it, according to the preceding, to the present more general context. As a byproduct, we also obtain a strengthening, within this framework, of some relevant results of M. B. Šubin in Ref. [**vi**] which are related to previous ones by R. Arens in Ref. [**i**] and [**ii**] (cf. Lemma 6.1 below and the Scholium following it).

To start with, we first fix the terminology applied in the sequel. Thus, suppose we are given a topological algebra E, whose spectrum is $\mathcal{M}(E)$. Now, if x is an element of E, we denote by $Z(\hat{x})$ the «zero-set» of the Gel'fand transform of x, i.e. one has, by definition, the relation:

$$Z(\hat{x}) = \{f \in \mathcal{M}(E) : \hat{x}(f) = 0\}. \tag{6.1}$$

On the other hand, suppose in particular that the given algebra E is commutative with an identity element, and let $(x_1, \ldots, x_n)$ be a given finite sequence of elements of E. Then, the set of all elements of E of the form $\sum_{i=1}^{n} x_i y_i$, with $y_i \in E$ $(i = 1, \ldots, n)$ is an ideal of the algebra E, containing the given sequence $(x_1, \ldots, x_n)$ and besides is the smallest one with this propetry, i.e. it coincides with the ideal generated by the sequence $(x_1, \ldots, x_n)$. We denote it by $\mathcal{J}(x_1, \ldots, x_n)$, so that one has the relation:

$$\mathcal{J}(x_1, \ldots, x_n) = \left\{\sum_{i=1}^{n} x_i y_i : y_i \in E ;\ i = 1, \ldots, n\right\}. \tag{6.2}$$

We are thus in a position to state the following:

Lemma 6.1. *Let E be a commutative topological algebra with an identity element, and whose spectrum is M(E). Moreover, let F be an «inverse-closed», dense subalgebra of E, which also contains the constants, and let $(x_1, \ldots, x_n)$ be a given finite sequence of elements of F. Furthermore, consider the following two assertions:*

1) The ideal in F generated by the given sequence $(x_1, \ldots, x_n)$ coincides with F.

2) The following relation holds true:

$$\bigcap_{i=1}^{n} Z(\hat{x}_i) = \emptyset. \tag{6.3}$$

Then, 1) $\Rightarrow$ 2). In particular, suppose that E is a commutative, complete, semi-simple, locally m-convex Q-algebra, Then, the two preceding assertions are equivalent.

Proof: 1) $\Longrightarrow$ 2): Since, by hypothesis, the unit element $1 \in E$ is also contained in F, one concludes that there exist elements $y_i \in F$ $(i = 1, \ldots, n)$ such that one has

$$\sum_{i=1}^{n} x_i y_i = 1, \tag{6.4}$$

so that one obtains, for every $f \in \mathcal{M}(E)$, the relation:

$$\sum \hat{x}_i(f)\hat{y}_i(f) = 1, \tag{6.5}$$

which proves the assertion. Now, with the supplementary hypothesis for the algebra E, we prove that *2)* $\Longrightarrow$ *1):* Namely, under the supposition that (6.3) is satisfied, one proves that the given elements $x_i \in F$ $(i = 1, \ldots, n)$ generate with respect to E, the whole algebra E. Indeed, otherwise they would be contained in some maximal ideal of E, which, since it is closed, by hypothesis for E, determines an element in $\mathcal{M}(E)$, by which the relation (6.3) would be contradicted. Therefore, there exist elements $y_i \in E$ $(i = 1, \ldots, n)$ for which (6.4), and hence (6.5) as well, is satisfied. Now, by hypothesis for F, we can approximate the elements y_i by corresponding elements $z_i \in F$ $(i = 1, \ldots, n)$, in such a way that we still have the relation:

$$\sum_{i=1}^{n} \hat{x}_i(f)\hat{z}_i(f) \neq 0, \tag{6.6.}$$

for every $f \in \mathcal{M}(E)$. Hence, the element $\sum_i x_i z_i \in F$ is a regular element of the algebra E (cf. [**40**; p. 22, Theorem 5.4]), so that by hypothesis for F, one obtains that $(\sum_i x_i z_i)^{-1} \in F$. Therefore, if

$$\alpha_i = z_i(\sum x_i z_i)^{-1}, \tag{6.7}$$

one concludes that $\alpha_i \in F$ $(i = 1, \ldots, n)$, in such a way that the following relation is also satisfied:

$$\sum_{i=1}^{n} \hat{x}_i(f)\hat{\alpha}_i(f) = 1, \tag{6.8}$$

for every $f \in \mathcal{M}(E)$, i.e., one obtains

$$\widehat{(\sum_{i=1}^{n} x_i z_i)}(f) = 1, \tag{6.9}$$

for every $f \in M(E)$, so that by the semi-simplicity of E, one gets the relation :

$$\sum_{i=1}^{n} x_i z_i = 1 , \tag{6.10}$$

and this proves the assertion, and the proof is finished. ∎

Scholium 6.1. The preceding lemma has a special bearing on an analogous result of M. A. Šubin for algebras of holomorphic functions, the above being its extended version within the context of abstract topological algebras theory (cf. [**vi**; p. 68, Proposition 1.1]). Both results may be compared with previous relevant considerations of R. Arens in Ref. [**i**], [**ii**]: The elements x_i in cond. 1) of the preceding Lemma might well be understood as corresponding to the «rational generators» referred to by Arens in [**i**; p. 178]. Within the same context, we finally remark that the commutativity for the algebra considered, concerning the first part of the Lemma, has been set only for simplicity's sake, although it is not necessary in general. (Cf. also the pertinent terminology of Arens in Ref. [**i**; p. 172, § 3), for «right regular systems» in a given ring with unit).

Now, we still suppose that we are given a *commutative topological algebra E with an identity element,* and whose spectrum is $M(E)$. Moreover, let

$$\alpha = (\alpha_{ij}) \tag{6.11}$$

be an $m \times n$ matrix, whose entries α_{ij}, $1 \leqq i \leqq m$ and $1 \leqq j \leqq n$, all belong to the given algebra E.

Now, admitting that $1 \leqq m \leqq n$, we shall say that *the matrix α,* as above, *is complemented in the algebra E,* whenever there exists an $n \times n$ matrix $\bar{\alpha}$, satisfying the following conditions:

(6.11.1) The entries of $\bar{\alpha}$ all belong to the algebra E.

(6.11.2) The matrix $\bar{\alpha}$ contains α, i.e., the «first $m \times n$ minor matrix» of $\bar{\alpha}$ coincides with α (: $\bar{\alpha}_{ij} = \alpha_{ij}$, with $i=1, \ldots, m$ and $j = 1, \ldots, n$).

(6.11.3) The determinant of $\bar{\alpha}$ (not.: $\det\bar{\alpha}$), being an element of the algebra E, does not belong to ker(f), for every $f \in \mathcal{M}(E)$, i.e. one has the relation:

(6.11.3.α) $$f(\det\bar{\alpha}) \neq 0,$$

for every $f \in \mathcal{M}(E)$.

On the other hand, given a matrix $\alpha = (\alpha_{ij})$, as in (6.11) above, one defines the *rank of* α *in E,* as the largest natural number k for which the minors of α of order k do not belong, all together, to ker(f), for every $f \in \mathcal{M}(E)$. We denote this by $r(\alpha, E) = k$.

In particular, if the $m \times n$ matrix $\alpha = (\alpha_{ij})$, as above, satisfies the relation

(6.12) $$r(\alpha, E) = m,$$

then we shall say that *the matrix a is of maximum rank.*

Thus, it is an easy consequence of the foregoing that: *if a given matrix a, as in (6.11) above, is complemented in the algebra E, then it is of maximum rank.* Now, the main result below provides sufficient conditions, regarding the given algebra E, in order that the converse statement to hold also true, extending thus the classical situation, which one has in case, for instance, of the topological algebra $\mathbf{C}^n$ (cf. Theorem 6.1, below and its Corrollary).

We first comment on the relevant notation which is needed for the sequel. We mostly follow the terminology applied by V. Ya. Lin in Ref. [**28**], by adapting it to the present more general framework.

Thus, suppose that E is a topological algebra, whose spectrum is $\mathcal{M}(E)$, and let X be a complex manifold. Now, we shall say (cf. [**28**; p. 122]) that a continuous map $\phi : \mathcal{M}(E) \to X$ is *spectral* (resp. *weakly spectral*), whenever for every (complex-valued holomorphic) function $h \in Hol(\phi(\mathcal{M}(E)))$, i.e. holomorphic on some open neighborhood U of Im φ in X, (resp., for every $h \in Hol(X)$), one has the relation:

(6.13) $$h \circ \phi = \hat{x},$$

for some $x \in E$.

In particular, *in case E is a commutative, complete locally m-convex algebra with an identity element, and whose spectrum is* $\mathcal{M}(E)$, *and if X is a*

Stein manifold, then every weakly spectral map $\phi: \mathcal{M}(E) \to X$ *is spectral:* This is, in our case, the generalized version of Šilov - Arens - Calderón theorem, given for Banach algebras by V. Ya. Lin (ibid; p. 122, Theorem 1). Its proof can be derived by applying Lin's argumentation (ibid.), and taking into account the «holomorphic functional calculus» for the case of locally m-convex algebras (cf., for instance, [**8**; p. 412, Théorème 3]).

Now, suppose in particular that E is an (LA)-algebra, and let $(E_\alpha)_{\alpha \in I}$ be a given family of topological algebras of type (A), defining the algebra E, i. e. one has by definition $E = \varinjlim E_\alpha$, with the respective inductive limit vector space topology (Definition 5. 2). Moreover, suppose that X is a Stein manifold, and let $h: \mathcal{M}(E) \to X$ be a given weakly spectral map.

Thus, assumming that X is regularly imbedded in some complex (numerical) space $\mathbf{C}^n$ (: Remmert Theorem; cf., for instance, [**iv**; p. 224, Theorem 10]), we shall say that *the preceding map h is subordinated by an index* $\alpha \in I$, whenever the respective topological algebra E_α (of type (A)) is n-generated (Definition 5. 1), in such a way that the map

$$h : \mathcal{M}(E) \to h(\mathcal{M}(E)) \subseteq U \subseteq X \underset{\to}{\subseteq} \mathbf{C}^n, \tag{6. 14}$$

with U denoting an open neighborhood of Imh in X, *is given by the n generators* $x_1, \ldots, x_n$ *of the algebra* E_α, i. e., one has the relation:

$$h(f) = (\hat{x}_1(f), \ldots, \hat{x}_n(f)), \tag{6. 15}$$

for every element $f \in \mathcal{M}(E)$.

We are now in position to state the following:

Lemma 6. 2. *Let E be an (LA)-algebra, whose spectrum is M(E), having besides the respective Gel'fand map continuous, and let* $E = \varinjlim E_\alpha$ *be a given canonical decomposition of E into topological algebras* E_α, $\alpha \in I$, *of type (A). Moreover, let* $h: M(E) \to X$ *be a weakly spectral map of the spectrum of E into a Stein manifold X, subordinated to a given index* $\alpha \in I$. *Then, the following two statements are equivalent:*

1) There exists an open neighborhood U of the image of h in X, and a *principal holomorphic fiber bundle* $\xi = (S, \pi, U)$, *admitting a continuous covering map*

$$\sigma : M(E) \to S$$

of the given map h.

2) The bundle ξ *as above, admits a spectral covering map*

$$\sigma_* : M(E) \to S$$

of the map h. ∎

Scholium 6.2. The preceding Lemma 6.2 constitutes the extended version in our case of the fundamental result of V. Ya. Lin's paper (cf. [**28**; p. 123, Theorem 2]). The proof of Lin's result is based on a series of several lemmas which are actually valid for suitable topological algebras with compact spectra. On the other hand, one applies for the proof of the present version of the cited result Corollary 4.2 in the foregoing, plus standard terminology of the general theory of topological algebras, in conjunction with Lemma 5.3 above.

We come now to the formulation and proof of the main result of this section. Its content has a special bearing on the respective result of Lin's paper (cf. [**28**; p. 127, Theorem 3]). Namely, we have:

Theorem 6.1. *Let E be an (LA)-algebra, whose spectrum is M(E) and which has the respective Gel'fand map continuous, and let* $E = \varinjlim E_\alpha$ *be a given canonical decomposition of E into topological algebras* E_α, $\alpha \in I$, *of type (A). Moreover, suppose that we are given an* $m \times n$ *matrix* $\alpha = (\alpha_{ij})$, *with entries from the algebra E, and in such a way that there exists an index* $\alpha \in I$, *such that the number of generators of the respective algebra* E_α *is* $m \cdot n$. *Then, the following two statements are equivalent:*

1) The given matrix a is complemented in the algebra E.

2) The Gel'fand transform matrix $\hat{a}$ *of a is complemented in the algebra* $C_c(M(E))$.

Proof: It is easy to prove, by the corresponding definitions, that *1)* $\Rightarrow$ *2)*. Conversely, *suppose that the statement 2) holds true,* and let $V_m(\mathbf{C}^n)$ be the set of all $m \times n$ matrices with entries in $\mathbf{C}$ which are complemented in $\mathbf{C}$, or equivalently, the set of all «m-framings» in $\mathbf{C}^n$, with $1 \leqq m \leqq n$. Then, one defines a principal holomorphic fiber bundle, by the map π: $GL(n, \mathbf{C}) \to V_m(\mathbf{C}^n)$, which assigns to each invertible matrix $\omega = (\omega_{ij}) \in GL(n, \mathbf{C})$ the m-tuple consisting of the first m rows of ω. Now, the matrix $\hat{\alpha}$ defines a continuous map

$$\hat{\alpha} : \mathcal{M}(E) \to \mathbf{C}^{m \cdot n}, \tag{6.16}$$

by the relation:

$$\hat{\alpha}(f) = (\hat{\alpha}_{ij}(f)), \tag{6.17}$$

for every $f \in \mathcal{M}(E)$. On the other hand, since by hypothesis $\hat{\alpha}$ is complemented in the algebra $C(\mathcal{M}(E))$, one concludes that $\mathrm{Im}(\hat{\alpha}) \subseteq V_m(\mathbf{C}^n)$, so that there exists a continuous covering map h of $\hat{\alpha}$ relative to the bundle π, i.e. one making the following diagram commutative:

$$\begin{tikzcd} & GL(n, \mathbf{C}) \arrow[d, "\pi"] \\ \mathcal{M}(E) \arrow[ur, "h"] \arrow[r, "\hat{\alpha}"'] & V_m(\mathbf{C}^n) \end{tikzcd} \tag{6.18}$$

Furthermore, by considering $\hat{\alpha}$ as a weakly spectral map, one concludes, by Lemma 6.2, that there exists a spectral covering map

$$h_* : \mathcal{M}(E) \to GL(n, \mathbf{C}) \tag{6.19}$$

of the map (6.16). Now, if $h_* = (h_{ij})$, with $1 \leqq i, j \leqq n$, is the matrix decomposition of the map h_*, since this is a spectral map, one concludes that $h_{ij} \in E^\wedge \subseteq C_c(\mathcal{M}(E))$ $(i, j = 1, \ldots, n)$, in such a way that one has;

$$h_{ij} = \hat{\alpha}_{ij}, \text{ with } 1 \leqq i \leqq m, \text{ and } 1 \leqq j \leqq n. \tag{6.20}$$

Therefore, if $b_{ij} \in E$, with $m < i \leqq n$, and $1 \leqq j \leqq n$, such that

$$g(b_{ij}) = h_{ij} = \hat{\alpha}_{ij}, \tag{6.21}$$

where $g : E \to C_c(\mathcal{M}(E))$ denotes the respective Gel'fand map of the algebra E, one can easily prove that *the matrix*

$$\bar{\alpha} = (\bar{\alpha}_{ij}), \text{ with } \begin{cases} \bar{\alpha}_{ij} = \alpha_{ij}, & \begin{array}{l} i = 1, \ldots, m \\ j = 1, \ldots, n \end{array} \\ \bar{\alpha}_{ij} = b_{ij}, & \begin{array}{l} i = m+1, \ldots, n \\ j = 1, \ldots, n \end{array} \end{cases} \tag{6.22}$$

defines a conplement in E of the given matrix a, and this finishes the proof of the theorem. ∎

In particular, one obtains the following result which provides, for the case considered, a necessary and sufficient condition that a given matrix, all of whose entries are elements of the algebra E, to be complemented in E. That is, one has :

Corollary 6.1. *Suppose that the hypotheses of the preceding Theorem 6.1 are sotisfied. Moreover, suppose that the spectrum $M(E)$ of the given algebra E is a contractible space. Then, the given $m \times x$ matrix a is complemented in the algebra E if, and only if, it is of maximum rank, i.e., one has the relation* $r(a, E) = m$. ∎

(Added in proof). Regarding the class of topological algebras considered above in Theorem 6.1 and its Corollary, a more general (and natural) setting has been given in a forthcoming paper, in which much of the material of Section 6 of the present paper is also discussed in detail.

APPENDIX

We conclude the present discussion by the following comments on certain recent results obtained in Ref. [**3**] and [**17**], and which may naturally be fitted within the framework of applications of the class of topological algebras considered in this study.

Thus, the problem of transferring a group of automorphisms of a given topological algebra to a similar one of its spectrum, and vice-versa, via the Gel'fand map, is studied in Ref. [**3**]. Spectrally barrelled topological algebras, in particular, those having locally compact, or equivalently, locally equicontinuous spectra [**38**; p. 154, Corollary 2.2.], play an important role. The well-known work of G. Šilov on «homogeneous function algebras» (cf. [**49**], as well as [**41**]) might be in this connection of a particular interest.

Finally, results extending the classical Bochner - Weil - Raikov theorem on integral representations of positive linear forms on Banach *- algebras to the case of suitable classes of topological algebras *without involution* have been given in Ref. [**17**]. The class of spectrally barrelled topological algebras is proved to be again the appropriate framework for this kind of results (ibid.; p. 25, Theorem 4.1, and p. 27, Remark 4.1), the initial motive in this regard being similar considerations of G. Lumer for the case of Banach algebras (cf., for instance, Ref. [**29**]), the involution therein being replaced by a suitable finite group of transformations of the algebra under consideration (ibid.).

ΠΕΡΙΛΗΨΙΣ

Ἡ παροῦσα ἐργασία ἀποτελεῖ τὸ δεύτερον μέρος μιᾶς ἐκτενεστέρας μελέτης, ἀναφερομένης εἰς τὴν κατηγορίαν τῶν «φασματικῶς κυλινδροειδῶν» (spectrally barrelled) ἀλγεβρῶν, τὰ βασικὰ στοιχεῖα τῆς γενικῆς θεωρίας τῶν ὁποίων, ἐδόθησαν εἰς τὸ πρῶτον μέρος τῆς ἐν λόγῳ μελέτης (περιληφθὲν ἤδη ἐντὸς τῶν αὐτῶν «Πρακτικῶν», Τόμος 50, 1975). Εἰς τὸ παρὸν 2ον Μέρος δίδονται χαρακτηριστικαὶ ἐφαρμογαὶ τῆς ὡς ἄνω κατηγορίας τῶν τοπολογικῶν ἀλγεβρῶν, ἀναφερόμεναι ἰδιαιτέρως εἰς τὰ θέματα: 1ον) *Συνομολογία καὶ ὁμοτοπία εἰς φάσματα τοπολογικῶν ἀλγεβρῶν* (§ 5): Αἱ θεωρούμεναι ἄλγεβραι εἶναι μεταθετικαί, πλήρεις, πεπερασμένως παραγόμεναι, φασματικῶς κυλινδροειδεῖς, τοπικῶς m-κυρταί, μὲ μοναδιαῖον στοιχεῖον καὶ συμπαγὲς φάσμα (*τοπολογικαὶ ἄλγεβραι τύπου* (A). Πρβλ. Definition 5.1). Γενικώτερον, ἐπαγωγικὰ ὅρια τοιούτων ἀλγεβρῶν (*τοπολογικαὶ ἄλγεβραι τύπου* (LA). Πρβλ. Definition 5.2). Αἱ τελευταῖαι ἄλγεβραι γενικεύουν οὐσιωδῶς τὰς μεταθετικὰς ἀλγέβρας Banach μὲ μοναδιαῖον στοιχεῖον. Ἀποδεικνύεται οὕτως ὅτι *εἰς τὴν κατηγορίαν τῶν ἀλγεβρῶν τύπου* (LA), *ἄλγεβραι ἔχουσαι ὁμοτοπικὰ φάσματα δὲν διαφέρουν ὡς πρὸς τὸν «τελεστὴν»* (functor) K_o (: ἀλγεβρικὴ K - Θεωρία. Πρβλ. Theorem 5.5). Τοῦτο ἔχει ὡς συνέπειαν, μεταξὺ τῶν ἄλλων, δι' ἐφαρμογῆς τοῦ «Θεωρήματος περιοδικότητος τοῦ Bott», *ἐν ἀνάλογον τοῦ «τύπου Künneth» διὰ τὴν θεωρουμένην ἐν προκειμένῳ περίπτωσιν* (πρβλ. (5.13)). 2ον) *Πίνακες ἐξαρτώμενοι ἀπὸ παραμέτρους* (§ 6): Σχετικῶς θεωροῦνται πίνακες (matrices), τῶν ὁποίων τὰ στοιχεῖα ἀνήκουν εἰς μίαν τοπολογικὴν ἄλγεβραν τύπου (LA) καὶ *ἐξετάζονται συνθῆκαι ὑπὸ τὰς ὁποίας τοιοῦτοι πίνακες εἶναι «συμπληρωματικοί»* (: «συμπληροῦνται» εἰς ἀντιστρεπτοὺς πίνακας ὡς πρὸς τὴν θεωρουμένην ἄλγεβραν. Πρβλ. σχέσεις (6.11: 1, 2, 3)). Τὸ πρόβλημα ἰσοδυναμεῖ, δι' ὡρισμένας καταλλήλους κατηγορίας ἀλγεβρῶν τύπου (LA), μὲ τὸ ἀνάλογον διὰ τὴν (ἁπλουστέραν) ἄλγεβραν ὅλων τῶν συνεχῶν μιγαδικῶν συναρτήσεων ἐπὶ τοῦ φάσματος τῆς θεωρουμένης ἀλγέβρας (Theorem 6.1). Δίδεται ἐπίσης *κριτήριον συμπληρωματικότητος πίνακος*, ὡς πρὸς δεδομένην ἄλγεβραν, ὡς προηγουμένως, εἰς τὴν περίπτωσιν κατὰ τὴν ὁποίαν, τὸ φάσμα τῆς ὑπ' ὄψιν ἀλγέβρας εἶναι, ἰδιαιτέρως, (συμπαγὴς τοπολογικὸς) χῶρος «συσταλτὸς» (contractible. Πρβλ. Corollary 6.1).

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(: *supplementary to the present Part of this study*)

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