ΘΕΩΡΗΤΙΚΗ ΜΗΧΑΝΙΚΗ.— **The finite deformation of an almost homogeneous elastic solid** *, *by A. D. Kydoniefs* **. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φ. Βασιλείου.

1. 1. Introduction

Owing to the non-linearity of the equations involved, the number of exact solutions to problems in the theory of finite deformations that have been obtained is very limited. Most of these solutions are given by Green and Zerna [1] and Green and Adkins [2] who also give references to original sources. Some further solutions can be obtained with the assumption of a simplified form for the strain-energy function. But even with this simplification the equations often remain intractable. There has therefore been considerable interest in the development of perturbation and approximation methods. Thus, for instance, various authors, some of whose work is described in [2], have employed successive approximation procedures ; Green, Rivlin and Shield [3] have developed the theory of small deformations superposed on large ; Adkins [4] noted the possibility of perturbing the shape at the boundary of either the deformed or the undeformed body ; Spencer and Kydoniefs [5] used series expansion of the solutions in terms of a geometrical parameter of the problem which could be regarded as small.

A further possibility is to perturb the strain-energy function $W(I_i)$, where I_i are the strain invariants to be defined later. The first of these types of problem was described by Spencer [6] who examined the case of a homogeneous body. In this paper we consider the effect on the deformation of the inhomogeneous perturbation term $\varepsilon W'(I_i, \vartheta_i)$ where I_i and ϑ_i are the strain invariants and the convected coordinates respectively and ε a constant number small compared with unity. The

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* Α. Δ. ΚΥΔΩΝΙΕΩΣ, Μεγάλαι παραμορφώσεις ἐντὸς σχεδὸν ὁμογενοῦς ἐλαστικοῦ στερεοῦ.

** Department of Theoretical Mechanics, University of Nottingham.

general theory is given and also applied to the case in which the original deformation is a uniform extension. Finally, as an example, we solve the problem in which the specified deformation is a uniform extension and the undeformed body an incompressible right circular cylinder with axis Ox_3 and $W' = W'(I_1, I_2, x_3)$.

The result of the perturbation of the strain-energy function is in general an additional small deformation superposed on the original finite deformation. If we know the original deformation, and suitable boundary conditions are given, the additional deformation can be determined by the equations of equilibrium and the stress-strain relations which we give in the general theory.

The strain invariants, stress-strain relations and equations of equilibrium are the same in form as in the theory of small deformations superposed on large deformations. The coefficients in the stress-strain relations are formally the same as in [6] but in this case the additional terms depend also on the convected curvilinear coordinates.

Though the equations of equilibrium are considerably reduced if we assume the original deformation to be a uniform extension and the undeformed body an incompressible right circular cylinder, they are still too complicated to be solved in general. In order to find a solution we further assumed a restricted form $W' = W'(I_1, I_2, x_3)$ for the perturbation of the strain-energy function. Under these assumptions the additional deformation is determined.

Throughout this paper the notation of Green and Zerna [1] has been used. For easy reference, a brief summary of the results and formulae necessary for the solution of the above problems is given in the Sections 1.2 to 1.4. The reader is referred to [1] for details of the notation and derivation of the formulae in these Sections.

1.2. **Finite Deformations**

The points of the undeformed and deformed body are referred to the rectangular cartesian coordinates (o, x_i) and (O, y_i) respectively, and to the system of general convected curvilinear coordinates ϑ_i. The base vectors for the undeformed and deformed body are denoted by $\mathbf{g}_i$ and $\mathbf{G}_i$ respectively.

The metric tensors for the unstrained body are

$$g_{ij} = \frac{\partial x^m}{\partial \vartheta^i} \frac{\partial x^m}{\partial \vartheta^j}, \qquad g^{ij} = \frac{\partial \vartheta^i}{\partial x^m} \frac{\partial \vartheta^j}{\partial x^m}, \qquad g_{im} g^{mj} = \delta_i^j, \quad (1.2.1)$$

and the metric tensors for the strained body are given by

$$G_{ij} = \frac{\partial y^m}{\partial \vartheta^i} \frac{\partial y^m}{\partial \vartheta^j}, \qquad G^{ij} = \frac{\partial \vartheta^i}{\partial y^m} \frac{\partial \vartheta^j}{\partial y^m}, \qquad G_{im} G^{mj} = \delta_i^j, \quad (1.2.2)$$

where δ_j^i are the Kronecker deltas.

The strain tensor is by definition

$$\gamma_{ij} = \frac{1}{2}(G_{ij} - g_{ij}), \qquad \gamma_j^i = g^{ik}\gamma_{kj} = \frac{1}{2}(g^{ik}G_{kj} - \delta_j^i), \quad (1.2.3)$$

and the strain invariants of γ_j^i are

$$I_1 = g^{rs}G_{rs}, \qquad I_2 = g_{rs}G^{rs}I_3, \qquad I_3 = Gg^{-1}, \quad (1.2.4)$$

where G and g are respectively the determinants of the covariant metric tensors of the deformed and undeformed body.

The contravariant stress tensor measured per unit area of the deformed body is denoted by τ^{ij} and the equations of motion are

$$\tau^{ij}//_i + \varrho F^j = \varrho f^j \quad (1.2.5)$$

where ϱ is the density of the deformed body, $F^j\,\mathbf{G}_j$ the body force per unit mass and $f^j\,\mathbf{G}_j$ the acceleration vector. The double line denotes covariant differentiation with respect to the coordinates ϑ_i and the metric tensor of the deformed body.

When the surface force $P^i\,\mathbf{G}_i$ is prescribed at a surface with unit outward normal vector $n_i\,\mathbf{G}^i$ then, at that surface

$$P^i = \tau^{ji} n_j. \quad (1.2.6)$$

For an homogeneous and isotropic body the strain-energy function W has the form $W = W(I_1, I_2, I_3)$ and the stress-strain relations are given by

$$\tau^{ij} = \Phi g^{ij} + \Psi B^{ij} + pG^{ij}, \qquad B^{ij} = I_1 g^{ij} - g^{ir} g^{js} G_{rs},$$

$$\Phi = 2I_3^{-\frac{1}{2}}(\partial W/\partial I_1), \quad \Psi = 2I_3^{-\frac{1}{2}}(\partial W/\partial I_2), \quad p = 2I_3^{\frac{1}{2}}(\partial W/\partial I_3). \quad (1.2.7)$$

For an incompressible solid $I_3 = 1$, the strain-energy function

depends on I_1 and I_2 only, and p is a scalar invariant function of the coordinates ϑ_i to be determined by the equations of motion and the boundary conditions.

1. 3. **Small deformation superposed on finite deformation**

We give now a brief summary of the theory of small deformations superposed on finite deformations. This theory has been described by Green, Rivlin and Shield [3].

The finite deformation of the body B_o to the body B is assumed to be completely determined and we consider a further deformation of B to B', this last deformation being such that the state of strain and stress at any time differs at most by terms of the order of ε from the state of the known finite deformation.

Let P_o, P, P' be the corresponding points in the bodies B_o, B and B' respectively. Because of the above conditions we can assume that

$$\mathbf{P_o P'} = \boldsymbol{\upsilon}\,(\vartheta_i, t) + \varepsilon\,\boldsymbol{\omega}\,(\vartheta_i, t)\,, \tag{1.3.1}$$

where $\boldsymbol{\upsilon} = \mathbf{P_o P}$ and ε is a constant number small enough compared with unity for its squares and higher powers to be neglected.

If we denote by $\mathbf{G}_i + \varepsilon \mathbf{G}'_i$ the covariant base vectors of the coordinate system ϑ_i at the points P' and refer $\boldsymbol{\omega}$ to the base vectors $\mathbf{G}_i$, $\mathbf{G}^i$ at the points P, we have

$$\boldsymbol{\omega} = \omega_m \mathbf{G}^m = \omega^m \mathbf{G}_m\,, \tag{1.3.2}$$

$$\mathbf{G}'_i = \boldsymbol{\omega}_{,i} = \omega_m //_i \mathbf{G}^m = \omega^m //_i \mathbf{G}_m\,. \tag{1.3.3}$$

The covariant and contravariant metric tensors at the points P' will be denoted by $G_{ij} + \varepsilon G'_{ij}$ and $G^{ij} + \varepsilon G'^{ij}$ respectively where

$$G'_{ij} = \omega_i //_j + \omega_j //_i\,, \qquad G'^{ij} = -\,G^{ir} G^{js} G'_{rs}\,, \tag{1.3.4}$$

and if $|\,G_{ij} + \varepsilon\, G'_{ij}\,| = G + \varepsilon\, G'$, then

$$G' = G\, G^{ij} G'_{ij}\,. \tag{1.3.5}$$

The strain invariants associated with the body B' are $I_i + \varepsilon I'_i$ with I'_i given by

$$I'_1 = g^{rs} G'_{rs}\,, \qquad I'_2 = g_{rs}(G'^{rs} I_3 + G^{rs} I'_3)\,, \qquad I'_3 = I_3 G^{rs} G'_{rs}\,. \tag{1.3.6}$$

The tensor B^{ij} in (1. 2. 7) becomes $B^{ij} + \varepsilon B'^{ij}$ for B' with

$$B'^{ij} = (g^{ij} g^{rs} - g^{ir} g^{js}) G'_{rs} = e^{irm} e^{jsn} g_{rs} G'_{mn} g^{-1} \qquad (1.3.7)$$

Similarly, the Christoffel symbols for B' are $\Gamma^r_{ij} + \varepsilon \Gamma'^r_{ij}$, where

$$\Gamma'^r_{ij} = \frac{1}{2} G^{rs} (G'_{si,j} + G'_{sj,i} - G'_{ij,s}) + \frac{1}{2} G'^{rs} (G_{si,j} + G_{sj,i} - G_{ij,s}). \qquad (1.3.8)$$

The stress tensor for the strained body B', referred to curvilinear coordinates ϑ_i in B', is $\tau^{ij} + \varepsilon \tau'^{ij}$ with

$$\tau'^{ij} = g^{ij} \Phi' + B^{ij} \Psi' + B'^{ij} \Psi + G^{ij} p' + G'^{ij} p, \qquad (1.3.9)$$

where Φ', Ψ' and p' are given by

$$\Phi' = AI'_1 + FI'_2 + EI'_3 - (2 I_3)^{-1} \Phi I'_3 \quad ,$$
$$\Psi' = FI'_1 + BI'_2 + DI'_3 - (2 I_3)^{-1} \Psi I'_3 \quad ,$$
$$p' = I_3 (EI'_1 + DI'_2 + CI'_3) + (2 I_3)^{-1} pI'_3 \quad ,$$

and the coefficients $A, B, \ldots, F$ by

$$A = 2 I_3^{-\frac{1}{2}} (\partial^2 W / \partial I_1^2), \qquad B = 2 I_3^{-\frac{1}{2}} (\partial^2 W / \partial I_2^2),$$
$$C = 2 I_3^{-\frac{1}{2}} (\partial^2 W / \partial I_3^2), \qquad D = 2 I_3^{-\frac{1}{2}} (\partial^2 W / \partial I_2 \partial I_3),$$
$$E = 2 I_3^{-\frac{1}{2}} (\partial^2 W / \partial I_3 \partial I_1), \qquad F = 2 I_3^{-\frac{1}{2}} (\partial^2 W / \partial I_1 \partial I_2). \qquad (1.3.10)$$

If the contravariant components of the body force and acceleration vectors for B', referred to base vectors $\mathbf{G}_i + \varepsilon \mathbf{G}'_i$, are $F^j + \varepsilon F'^j$ and $f^j + \varepsilon f'^j$ respectively, then the equations of motion for B' are

$$\tau'^{ij}//_i + \Gamma'^j_{ir} \tau^{ir} + \Gamma'^r_{ir} \tau^{ij} + \varrho F'^j + \varrho' F^j = \varrho f'^j + \varrho' f^j, \qquad (1.3.11)$$

where the density of B' is $\varrho + \varepsilon \varrho'$ with

$$\varrho' = - \varrho G' (2G)^{-1} \qquad (1.3.12)$$

and the double line denotes covariant differentiation with respect to the coordinates ϑ_i and the metric tensor of the first deformed body B.

The surface force components for the strained body B' will be

$$P^j + \varepsilon P'^j = (\tau^{ij} + \varepsilon \tau'^{ij}) (n_i + \varepsilon n'_i), \qquad (1.3.13)$$

$n_i + \varepsilon n'_i$ being the covariant components of the unit normal to the surface of B' referred to the base vectors $\mathbf{G}^i + \varepsilon \mathbf{G}'^i$ and $P^j + \varepsilon P'^j$ the components of the surface force vector referred to $\mathbf{G}_j + \varepsilon \mathbf{G}'_j$.

1. 4. Small deformation superposed on finite uniform extensions

A complete account of this subject can be found in [3]. We give here a brief summary of the necessary formulae for easy reference.

We assume that the undeformed body B_0 is deformed into the body B by uniform finite extensions along the three orthogonal axes to which both B_0 and B are referred, and denote by λ_i the constant extension ratios. The points P_0 of B_0 are defined by the coordinates x_i and for convenience the coordinates of the points P of B will be denoted by x, y, z instead of y_i. The deformed body B' is obtained by superposing on B a small deformation of the type described in Section 1. 3. We take the moving coordinates ϑ_i to coincide with the fixed cartesian coordinates x, y, z.

The strain invariants for uniform extensions are

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{1.4.1}$$

From these formulae and (1. 2. 7) we see that Φ, Ψ and p are constants.

The components of the stress tensor for the body B are

$$\tau^{11} = \Phi\lambda_1^2 + \Psi\lambda_1^2(\lambda_2^2 + \lambda_3^2) + p, \quad \tau^{22} = \Phi\lambda_2^2 + \Psi\lambda_2^2(\lambda_3^2 + \lambda_1^2) + p$$
$$\tau^{33} = \Phi\lambda_3^2 + \Psi\lambda_3^2(\lambda_1^2 + \lambda_2^2) + p, \quad \tau^{12} = \tau^{23} = \tau^{31} = 0. \tag{1.4.2}$$

Since τ^{ij} is constant the equations of equilibrium for the body B are satisfied when the body forces are zero.

The strain invariants for B' are $I_i + \varepsilon I'_i$ with I_i given by (1. 4. 1) and

$$I_1' = 2\left(\lambda_1^2\frac{\partial u}{\partial x} + \lambda_2^2\frac{\partial v}{\partial y} + \lambda_3^2\frac{\partial \omega}{\partial z}\right),$$

$$I_2' = 2\lambda_1^2(\lambda_2^2 + \lambda_3^2)\frac{\partial u}{\partial x} + 2\lambda_2^2(\lambda_3^2 + \lambda_1^2)\frac{\partial v}{\partial y} + 2\lambda_3^2(\lambda_1^2 + \lambda_2^2)\frac{\partial \omega}{\partial z},$$

$$I_3' = 2\lambda_1^2\lambda_2^2\lambda_3^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z}\right), \tag{1.4.3}$$

where we have written u, v, ω instead of $\omega_1, \omega_2, \omega_3$.

If we assume $F^j = 0$, $f^j = 0$ the equations of motion (1. 3. 11) are reduced to

$$\frac{\partial}{\partial \vartheta_i}\left(\tau'^{ij} + \tau^{ir}\frac{\partial \omega_j}{\partial \vartheta_r} + \tau^{jr}\frac{\partial \omega_i}{\partial \vartheta_r}\right) = \varrho f'^j, \tag{1.4.4}$$

if we also take $F'^j = 0$, that is assume B' free of body forces.

In (1. 4. 4) the stress components are referred to the coordinates ϑ_i but for some purposes it is more convenient to have the stress components in the strained body B' referred to rectangular cartesian coordinates y_i with axes coinciding with the axes (x, y, z). Thus $y_i = \vartheta_i + \varepsilon\omega_i$ and if we denote by $t^{ij} + \varepsilon t'^{ij}$ the new stress components we have

$$t^{rs} = \tau^{rs}, \qquad t'^{rs} = \tau'^{rs} + \tau^{rm}\frac{\partial\omega_s}{\partial\vartheta_m} + \tau^{ms}\frac{\partial\omega_r}{\partial\vartheta_m}. \tag{1. 4. 5}$$

Hence the equations of motion (1. 4. 4) take the simpler form

$$\partial t'^{rs} / \partial\vartheta_s = \varrho f'^r, \tag{1. 4. 6}$$

where $\varepsilon f'^r$ are also the acceleration components in the directions of the y_i - axes.

If the boundary surface of B' is given in the parametric form $F(\vartheta_i) = F(x, y, z) = 0$ and $n_r + \varepsilon n'_r$ are the covariant components of the unit normal referred to $\mathbf{G}^r + \varepsilon\mathbf{G}'^r$ then

$$n_r + \varepsilon n'_r = k\,(\partial F / \partial\vartheta_r)$$

where

$$k = \left(\frac{\partial F}{\partial\vartheta_m}\frac{\partial F}{\partial\vartheta_m}\right)^{-\frac{1}{2}}\left[1 + \varepsilon\frac{\partial F}{\partial\vartheta_r}\frac{\partial F}{\partial\vartheta_s}\left(\frac{\partial\omega_r}{\partial\vartheta_s} + \frac{\partial\omega_s}{\partial\vartheta_r}\right)\left(2\frac{\partial F}{\partial\vartheta_n}\frac{\partial F}{\partial\vartheta_n}\right)^{-1}\right], \tag{1. 4. 7}$$

and the surface force components are

$$P^j + \varepsilon P'^j = k\,(\tau^{ij} + \varepsilon\tau'^{ij})\,(\partial F / \partial\vartheta_i), \tag{1. 4. 8}$$

where $P^j + \varepsilon P'^j$ are the components of the surface force referred to $\mathbf{G}_j + \varepsilon\mathbf{G}'_j$.

If $\mathbf{Q}^i + \varepsilon\mathbf{Q}'^i$ are the components of the surface force referred to the y_i - axes we have

$$Q^i = \tau^{ri}\frac{\partial F}{\partial\vartheta_r}\left(\frac{\partial F}{\partial\vartheta_n}\frac{\partial F}{\partial\vartheta_n}\right)^{-\frac{1}{2}},$$

$$Q'^i = \tau'^{ri}\frac{\partial F}{\partial\vartheta_r}\left(\frac{\partial F}{\partial\vartheta_m}\frac{\partial F}{\partial\vartheta_m}\right)^{-\frac{1}{2}} + Q^j\frac{\partial\omega_i}{\partial\vartheta_j} +$$

$$+ Q^i\frac{\partial F}{\partial\vartheta_r}\frac{\partial F}{\partial\vartheta_s}\left(\frac{\partial\omega_r}{\partial\vartheta_s} + \frac{\partial\omega_s}{\partial\vartheta_r}\right)\left(2\frac{\partial F}{\partial\vartheta_n}\frac{\partial F}{\partial\vartheta_n}\right)^{-1} \tag{1. 4. 9}$$

THE FINITE DEFORMATION OF AN ALMOST HOMOGENEOUS ELASTIC SOLID

2. 1. Statement of the problem

The general theory of Finite Elastic Deformations is given by Green and Zerna [1], for an elastic, homogeneous and isotropic body with strain-energy function $W(I_i)$, where I_i are the strain invariants. The result on the deformation of a perturbation of the strain-energy function into $W(I_i) + \varepsilon W'(I_i)$, where ε is a constant number small compared with unity has been described by Spencer [6]. It is of interest to investigate the more general case of an almost homogeneous body in which $W' = W'(I_i, \vartheta_i)$, where ϑ_i are the convected curvilinear coordinates.

We consider an elastic, homogeneous and isotropic body B with strain-energy function $W = W(I_i)$. If B_o is deformed into the first deformed body B we denote this deformation by $D = D(B_o, B, W)$. P_o and P denote corresponding points of B_o and B respectively, and $\mathbf{f}$, $\mathbf{F}$, $\mathbf{P}$ the acceleration, body and surface force applied at the point P in the deformation D.

We also consider an elastic, almost homogeneous and isotropic body with the same initial configuration B_o and strain-energy function

$$W^* = W(I_i) + \varepsilon W'(I_i, \vartheta_i) \quad , \tag{2.1.1}$$

where ε is a number small enough compared with unity for its second and higher powers to be neglected. If this body is now deformed into the second deformed body B', $D^* = D(B_o, B', W^*)$ denotes this deformation, P' is the point of B' which corresponds to the point P_o of B_o and $\mathbf{f}^*$, $\mathbf{F}^*$, $\mathbf{P}^*$ are the acceleration, body and surface force respectively at P' in D^*, the following problem is considered: Assuming that the displacements, acceleration, body and surface forces of D^* differ at most to the order of ε from those of D, that is

$$\mathbf{P}_o\mathbf{P}' = \mathbf{P}_o\mathbf{P} + \varepsilon\boldsymbol{\omega}, \quad \mathbf{F}^* = \mathbf{F} + \varepsilon\mathbf{F}', \quad \mathbf{f}^* = \mathbf{f} + \varepsilon\mathbf{f}', \quad \mathbf{P}^* = \mathbf{P} + \varepsilon\mathbf{P}' \tag{2.1.2}$$

and $D(B_o, B, W)$ is completely specified, we wish to determine the deformation $D^* = D(B_o, B', W^*,)$ in terms of the specified deformation, the perturbation of the strain-energy function and the constant ε.

2. 2. **General Theory**

Because of the first of (2. 1. 2) we see that the result of the perturbation of the strain-energy function is a small deformation superposed on the original one and that the geometry of D^* is the same as in the case of a small deformation superposed on a finite deformation. We also note that the formulae (1. 2. 7) are valid for the perturbed strain-energy function. If we use the superscript * for the quantities relative to D^*, the formulae (1. 2. 7) take the form

$$\tau^{*ij} = \Phi^* g^{ij} + \Psi^* B^{*ij} + p^* G^{*ij} ,$$

$$\Phi^* = 2\, I_3^{*\,-\frac{1}{2}} (\partial W^*/\partial I_1^*), \quad \Psi^* = 2\, I_3^{*\,-\frac{1}{2}} (\partial W^*/\partial I_2^*),$$

$$p^* = 2\, I_3^{*\,\frac{1}{2}} (\partial W^*/\partial I_3^*). \qquad (2.2.1)$$

By expanding these expressions in series and neglecting the second and higher powers of ε we obtain

$$\Phi^* = \Phi + \varepsilon\Phi', \quad \Psi^* = \Psi + \varepsilon\Psi', \quad p^* = p + \varepsilon p' \qquad (2.2.2)$$

where

$$\Phi' = AI'_1 + FI'_2 + EI'_3 - (2\, I_3)^{-1}\Phi I'_3 + 2\, I_3^{-\frac{1}{2}} (\partial W'/\partial I_1)$$

$$\Psi' = FI'_1 + BI'_2 + DI'_3 - (2\, I_3)^{-1}\Psi I'_3 + 2\, I_3^{-\frac{1}{2}} (\partial W'/\partial I_2)$$

$$p' = I_3(EI'_1 + DI'_2 + CI'_3) + (2\, I_3)^{-1} p I'_3 + 2\, I_3^{\frac{1}{2}} (\partial W'/\partial I_3) \qquad (2.2.3)$$

and $A, B, \ldots, F$ are again given by (1. 3. 10).

It can now be easily proved that the stress tensor τ^{*ij} corresponding to the deformation D^* is

$$\tau^{*ij} = \tau^{ij} + \varepsilon\tau'^{ij}, \qquad (2.2.4)$$

where τ'^{ij} is given by (1. 3. 9), Φ', Ψ', p' by (2. 2. 3) and τ^{ij} is the stress tensor corresponding to the specified deformation $D(B_0, B, W)$. It can also be readily seen that the equation of motion is (1. 3. 11) and that the surface force components are given by (1. 3. 13).

If the body is incompressible we have $I_3 = I_3^* = 1$, $I_3' = 0$. Thus

$$\Phi' = AI'_1 + FI'_2 + 2\,(\partial W'/\partial I_1), \quad \Psi' = FI'_1 + BI'_2 + 2\,(\partial W'/\partial I_2), \quad (2.2.5)$$

where the A, B and F are given by (1. 3. 10). In the case of an incompressible body, p' cannot be found from the elastic potential function

and is a scalar invariant function of the coordinates for each value of the time t.

Before we close this Section we note that all the formulae of Section 1. 3 have been proved to apply in our problem with the exception of the formulae for Φ', Ψ' and p'. These were replaced by (2. 2. 3) or, in the case of an incompressible body, by (2. 2. 5).

2. 3. **Uniform extensions**

As an application of the theory described in Section 2. 2 we consider now the case where the original deformation $D(B_0, B, W)$ is a uniform extension.

By using the same coordinate systems and notation as in Section 1. 4 we obtain, after a certain amount of algebra,

$$\Phi' = d_{11}\frac{\partial u}{\partial x} + d_{12}\frac{\partial v}{\partial y} + d_{13}\frac{\partial \omega}{\partial z} + \frac{2}{\lambda_1\lambda_2\lambda_3}\frac{\partial W'}{\partial I_1},$$

$$\Psi' = d_{21}\frac{\partial u}{\partial x} + d_{22}\frac{\partial v}{\partial y} + d_{23}\frac{\partial \omega}{\partial z} + \frac{2}{\lambda_1\lambda_2\lambda_3}\frac{\partial W'}{\partial I_2},$$

$$p' = d_{31}\frac{\partial u}{\partial x} + d_{32}\frac{\partial v}{\partial y} + d_{33}\frac{\partial \omega}{\partial z} + 2\lambda_1\lambda_2\lambda_3\frac{\partial W'}{\partial I_3}, \qquad (2.3.1)$$

where the constants d_{ij} are given by

$$d_{1j} = 2A\lambda^2_j + 2F\lambda^2_j(\lambda^2_k + \lambda^2_l) + 2E\lambda_1^2\lambda_2^2\lambda_3^2 - \Phi,$$

$$d_{2j} = 2F\lambda^2_j + 2B\lambda^2_j(\lambda^2_k + \lambda^2_l) + 2D\lambda_1^2\lambda_2^2\lambda_3^2 - \Psi,$$

$$d_{3j} = \lambda_1^2\lambda_2^2\lambda_3^2[2E\lambda^2_j + 2D\lambda^2_j(\lambda^2_k + \lambda^2_l) + 2C\lambda_1^2\lambda_2^2\lambda_3^2] + p,$$

$$j, k, l = 1, 2, 3 \qquad j \neq k \neq l \neq j \qquad (2.3.2)$$

In the above formulae Φ, Ψ and p are given by (1. 2. 7) and $A, \ldots, F$ by (1. 3. 10).

The stress tensor τ'^{ij} is given by

$$\tau'^{ii} = c_{i1}\frac{\partial u}{\partial x} + c_{i2}\frac{\partial v}{\partial y} + c_{i3}\frac{\partial \omega}{\partial z} + \frac{2\lambda^2_i}{\lambda_1\lambda_2\lambda_3}\left[\frac{\partial W'}{\partial I_1} + (\lambda^2_k + \lambda^2_l)\frac{\partial W'}{\partial I_2}\right] + 2\lambda_1\lambda_2\lambda_3\frac{\partial W'}{\partial I_3},$$

$$\tau'^{23} = c_{44}\left(\frac{\partial v}{\partial z} + \frac{\partial \omega}{\partial y}\right), \quad \tau'^{31} = c_{55}\left(\frac{\partial \omega}{\partial x} + \frac{\partial u}{\partial z}\right), \quad \tau'^{12} = c_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \qquad (2.3.3)$$

where the constant coefficients c_{ij} are given by

$$c_{ii} = -\tau^{ii} + 2A\lambda^4_i + 2B\lambda^4_i(\lambda^2_k + \lambda^2_l)^2 + 2C\lambda_1^4\lambda_2^4\lambda_3^4 + \\ + 4D\lambda^4_i\lambda^2_k\lambda^2_l(\lambda^2_k + \lambda^2_l) + 4E\lambda^4_i\lambda^2_k\lambda^2_l + \\ + 4F\lambda^4_i(\lambda^2_k + \lambda^2_l), \tag{2.3.4}$$

$$c_{ij} = -\Phi\lambda^2_i + \Psi\lambda^2_i(\lambda^2_j - \lambda^2_k) + p + 2A\lambda^2_i\lambda^2_j + \\ + 2B\lambda^2_i\lambda^2_j(\lambda^2_i + \lambda^2_k)(\lambda^2_j + \lambda^2_k) + 2C\lambda_1^4\lambda_2^4\lambda_3^4 + \\ + 2D\lambda_1^2\lambda_2^2\lambda_3^2(2\lambda^2_i\lambda^2_j + \lambda^2_i\lambda^2_k + \lambda^2_j\lambda^2_k) + \\ + 2E\lambda_1^2\lambda_2^2\lambda_3^2(\lambda^2_i + \lambda^2_j) + 2F\lambda^2_i\lambda^2_j(\lambda^2_i + \lambda^2_j + 2\lambda^2_k), \tag{2.3.5}$$

$$c_{44} = -\Psi\lambda_2^2\lambda_3^2 - p, \quad c_{55} = -\Psi\lambda_3^2\lambda_1^2 - p, \quad c_{66} = -\Psi\lambda_1^2\lambda_2^2 - p. \tag{2.3.6}$$

In (2.3.3) to (2.3.6) double indices are not to be summed, $i, k, l = 1, 2, 3$ and $i \neq j \neq k \neq i$. In the formulae (2.3.4) to (2.3.6) Φ, Ψ, p are given by (1.2.7), $A, \ldots, F$ by (1.3.10) and τ^{ij} by (1.4.2).

If both D and D^* are deformations of incompressible bodies $I'_3 = 0$ and

$$\lambda_1^2\lambda_2^2\lambda_3^2 = 1, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z} = 0. \tag{2.3.7}$$

In this case the components of the stress tensor τ'^{ij} reduce to

$$\tau'^{ii} = p' + \alpha_{i1}\frac{\partial u}{\partial x} + \alpha_{i2}\frac{\partial v}{\partial y} + \alpha_{i3}\frac{\partial \omega}{\partial z} + 2\lambda^2_i\left[\frac{\partial W'}{\partial I_1} + (\lambda^2_j + \lambda^2_k)\frac{\partial W'}{\partial I_2}\right], \tag{2.3.8}$$

where

$$\alpha_{ii} = -2p + 2\lambda^4_i[A + B(\lambda^2_j + \lambda^2_k)^2 + 2F(\lambda^2_j + \lambda^2_k)], \\ \alpha_{ij} = \alpha_{ji} = 2\lambda^2_i\lambda^2_j[\Psi + A + B(\lambda^2_i + \lambda^2_k)(\lambda^2_j + \lambda^2_k) + \\ + F(\lambda^2_i + \lambda^2_j + 2\lambda^2_k)]. \tag{2.3.9}$$

In (2.3.8) and (2.3.9) repeated indices are not to summed and $i \neq j \neq k \neq i$. The remaining stress components are given by the last three of (2.3.3) and (2.3.6).

2.4. **Extension of an almost homogeneous circular cylinder**

We shall now apply the results of the Section 2.1 to 2.3 to an incompressible right circular cylinder with axis Ox_3. Though in this case the equations of equilibrium are reduced to a simpler form, their solution

does not seem to be feasible in general. In order to obtain a solution we assume a simpler form for the perturbation of the strain-energy function and consider the case in which this perturbation depends on I_1, I_2 and one of the curvilinear coordinates, ϑ_3.

As in the preceding Section we refer the undeformed body B_0 and the first deformed body B to rectangular cartesian coordinates (x_1, x_2, x_3) and (x, y, z) respectively, both systems having the same axes, and take the moving coordinates ϑ_i to coincide with (x, y, z). The undeformed body B_0 is an incompressible right circular cylinder with axis x_3. The deformation $D(B_0, B, W)$ is a uniform extension of the type described in Section 1.4 and (2.1.2) are valid. We also assume that the deformed bodies B and B' are in equilibrium and free of body forces. Finally, that the perturbation of the strain-energy function and the extension ratios are

$$W' = W'(I_1, I_2, \vartheta_3), \qquad \lambda_1 = \lambda_2 = \lambda, \qquad \lambda_3 = \mu. \tag{2.4.1}$$

As boundary conditions we assume that B' is in equilibrium under the action of surface forces applied on its end surfaces only.

With the above assumptions the incompressibility conditions (2.3.7) take the form

$$\lambda^2\mu = 1, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z} = 0. \tag{2.4.2}$$

The stress tensor τ^{ij} for the deformation $D(B_0, B, W)$ is given by (1.4.2) and from the equations of equilibrium we see that p is constant. Hence we can specify the value of p and write

$$\tau^{11} = \tau^{22} = 0, \quad \tau^{33} = (\Phi + \Psi\lambda^2)(\mu^2 - \lambda^2), \quad \tau^{12} = \tau^{23} = \tau^{31} = 0,$$

$$p = -\Phi\lambda^2 - \Psi\lambda^2(\lambda^2 + \mu^2). \tag{2.4.3}$$

These relations and (1.2.6) give the following components of the surface force for the original deformation

curved surface of B: $\quad P^1 = P^2 = P^3 = 0$

$$\vartheta_3 = \text{const.}: \; P^1 = P^2 = 0, \; P^3 = \tau^{33} = (\mu^2 - \lambda^2)(\Phi + \Psi\lambda^2). \tag{2.4.4}$$

The formulae (2. 3. 6), (2. 3. 9) and (2. 4. 1) give the coefficients

$$\alpha_{11} = \alpha_{22} = -2p + 2\lambda^4 [A + B(\lambda^2 + \mu^2)^2 + 2F(\lambda^2 + \mu^2)],$$

$$\alpha_{33} = -2p + 2\mu^4 (A + 4B\lambda^4 + 4F\lambda^2),$$

$$\alpha_{12} = \alpha_{21} = 2\lambda^4 [\Psi + A + B(\lambda^2 + \mu^2)^2 + 2F(\lambda^2 + \mu^2)],$$

$$\alpha_{23} = \alpha_{32} = \alpha_{31} = \alpha_{13} = 2\lambda^2\mu^2 [\Psi + A + 2B\lambda^2(\lambda^2 + \mu^2) + F(3\lambda^2 + \mu^2)],$$

$$c_{44} = c_{55} = -\Psi\lambda^2\mu^2 - p, \qquad c_{66} = -\Psi\lambda^4 - p, \tag{2. 4. 5}$$

and if we write

$$2\lambda^2 \left[\frac{\partial W'}{\partial I_1} + (\lambda^2 + \mu^2)\frac{\partial W'}{\partial I_2}\right] = f(\vartheta_3), \quad 2\mu^2 \left[\frac{\partial W'}{\partial I_1} + 2\lambda^2 \frac{\partial W'}{\partial I_2}\right] = h(\vartheta_3), \tag{2. 4. 6}$$

the components of the stress tensor τ'^{ij} are

$$\tau'^{11} = p' + \alpha_{11}\frac{\partial u}{\partial x} + \alpha_{12}\frac{\partial v}{\partial y} + \alpha_{13}\frac{\partial \omega}{\partial z} + f(z),$$

$$\tau'^{22} = p' + \alpha_{12}\frac{\partial u}{\partial x} + \alpha_{11}\frac{\partial v}{\partial y} + \alpha_{13}\frac{\partial \omega}{\partial z} + f(z),$$

$$\tau'^{33} = p' + \alpha_{13}\frac{\partial u}{\partial x} + \alpha_{13}\frac{\partial v}{\partial y} + \alpha_{33}\frac{\partial \omega}{\partial z} + h(z),$$

$$\tau'^{23} = c_{44}\left(\frac{\partial v}{\partial z} + \frac{\partial \omega}{\partial y}\right), \qquad \tau'^{31} = c_{44}\left(\frac{\partial \omega}{\partial x} + \frac{\partial u}{\partial z}\right),$$

$$\tau'^{12} = c_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right). \tag{2. 4. 7}$$

In the above expressions the stress tensor is referred to the curvilinear coordinates ϑ_i. By use of (1. 4. 5) we now refer τ'^{ij} to the rectangular coordinates y_i the axes of which coincide with the axes (x, y, z) and we denote its new components by t'^{ij} where

$$t'^{11} = \tau'^{11}, \qquad t'^{22} = \tau'^{22}, \qquad t'^{33} = \tau'^{33} + 2\tau^{33}\frac{\partial \omega}{\partial z},$$

$$t'^{12} = \tau'^{12}, \qquad t'^{23} = \tau'^{23} + \tau^{33}\frac{\partial v}{\partial z}, \qquad t'^{31} = \tau'^{31} + \tau^{33}\frac{\partial u}{\partial z}. \tag{2. 4. 8}$$

The equations of equilibrium can now easily be calculated from

(1. 4. 6) and (2. 4. 8). After simplification, using the second incompressibility condition (2. 4. 2) and (2. 4 5), they reduce to

$$\frac{\partial p'}{\partial x} + c_{66}\frac{\partial^2 u}{\partial x^2} + c_{66}\frac{\partial^2 u}{\partial y^2} + (c_{44} + \tau^{33})\frac{\partial^2 u}{\partial z^2} + \\ + (c_{44} + \alpha_{13} - \alpha_{12} - c_{66})\frac{\partial^2 \omega}{\partial z\,\partial x} = 0,$$

$$\frac{\partial p'}{\partial y} + c_{66}\frac{\partial^2 v}{\partial x^2} + c_{66}\frac{\partial^2 v}{\partial y^2} + (c_{44} + \tau^{33})\frac{\partial^2 v}{\partial z^2} + \\ + (c_{44} + \alpha_{13} - \alpha_{12} - c_{66})\frac{\partial^2 \omega}{\partial y\,\partial z} = 0,$$

$$\frac{\partial p'}{\partial z} + c_{44}\frac{\partial^2 \omega}{\partial x^2} + c_{44}\frac{\partial^2 \omega}{\partial y^2} + (\alpha_{33} + \tau^{33} - c_{44} - \alpha_{13})\frac{\partial^2 \omega}{\partial z^2} + \frac{dh}{dz} = 0. \qquad (2.4.9)$$

Since the bodies B_0 and B are symmetric with respect to the Ox_3 axis, p is a constant and W' is independent of ϑ_1, ϑ_2 it is reasonable to expect that, if B' is in equilibrium under the action of external forces symmetrical with respect to the same axis, the additional displacements due to the perturbation of the strain-energy function will be axially symmetric and, also, that p' and ϱ will be independent of ϑ. Hence we may tentatively assume

$$p' = p'(r, z), \qquad \varrho = \varrho(r, z),$$
$$u = \varrho\cos\vartheta, \qquad v = \varrho\sin\vartheta, \qquad \omega = \omega(r, z), \qquad (2.4.10)$$

where ϱ is the radial projection of $\boldsymbol{\omega}$. Thus, if we change to cylindrical coordinates (r, ϑ, z) the equations of equilibrium are

$$\frac{\partial p'}{\partial r} + k\left(\frac{\partial^2 \varrho}{\partial r^2} + \frac{1}{r}\frac{\partial \varrho}{\partial r} - \frac{\varrho}{r^2}\right) + l\frac{\partial^2 \varrho}{\partial z^2} = 0,$$

$$\frac{\partial p'}{\partial z} + m\left(\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r}\right) + n\frac{\partial^2 \omega}{\partial z^2} + \frac{dh}{dz} = 0,$$

$$\frac{\partial \varrho}{\partial r} + \frac{\varrho}{r} + \frac{\partial \omega}{\partial z} = 0, \qquad (2.4.11)$$

where for convenience we have written

$$k = \alpha_{12} - \alpha_{13} + 2c_{66} - c_{44}, \qquad l = c_{44} + \tau^{33},$$
$$m = c_{44}, \qquad n = \alpha_{33} + \tau^{33} - c_{44} - \alpha_{13} \qquad (2.4.12)$$

One particular integral of (2. 4. 11) is

$$p' = -h(z), \qquad \varrho = \omega = 0, \tag{2. 4. 13}$$

hence (2. 4. 13) plus any solution of

$$\frac{\partial p'}{\partial r} + k\left(\frac{\partial^2 \varrho}{\partial r^2} + \frac{1}{r}\frac{\partial \varrho}{\partial r} - \frac{1}{r^2}\varrho\right) + l\frac{\partial^2 \varrho}{\partial z^2} = 0,$$

$$\frac{\partial p'}{\partial z} + m\left(\frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r}\right) + n\frac{\partial^2 \omega}{\partial z^2} = 0,$$

$$\frac{\partial \varrho}{\partial r} + \frac{\varrho}{r} + \frac{\partial \omega}{\partial z} = 0, \tag{2. 4. 14}$$

is a solution of (2. 4. 11).

The system (2. 4. 14) is a particular case of a more general system the solution of which is given by Wilkes [7]. It can be verified that (2. 4. 14) has solutions of the form

$$\varrho = f_1^{(\varkappa)}(r)\cos \nu_\varkappa z, \qquad \omega = f_2^{(\varkappa)}(r)\sin \nu_\varkappa z, \qquad p' = f_3^{(\varkappa)}(r)\cos \nu_\varkappa z \tag{2. 4. 15}$$

where $\varkappa$ is not to be summed and the functions $f_i^{(\varkappa)}(r)$ are given by

$$f_1^{(\varkappa)}(r) = A_1^{(\varkappa)} I_1(k_1 \nu_\varkappa r) + A_2^{(\varkappa)} I_1(k_2 \nu_\varkappa r),$$

$$f_2^{(\varkappa)}(r) = -k_1 A_1^{(\varkappa)} I_0(k_1 \nu_\varkappa r) - k_2 A_2^{(\varkappa)} I_0(k_2 \nu_\varkappa r),$$

$$f_3^{(\varkappa)}(r) = k_1 \nu_\varkappa A_1^{(\varkappa)}(n - mk_1^2) I_0(k_1 \nu_\varkappa r) + k_2 \nu_\varkappa A_2^{(\varkappa)}(n - mk_2^2) I_0(k_2 \nu_\varkappa r), \tag{2. 4. 16}$$

where $A_1^{(\varkappa)}$, $A_2^{(\varkappa)}$, $\nu_\varkappa$ are arbitrary constants, k_1^2 and k_2^2 the roots of

$$m\zeta^2 - (n + k)\zeta + 1 = 0 \tag{2. 4. 17}$$

and $I_0(\tau)$, $I_1(\tau)$ the modified Bessel functions of the first kind given by

$$I_0(\tau) = \sum_{t=0}^{\infty} \frac{(\tau/2)^{2t}}{t!\,t!}, \qquad I_1(\tau) = \sum_{t=0}^{\infty} \frac{(\tau/2)^{2t+1}}{t!\,(t+1)!} \tag{2. 4. 18}$$

After a considerable amount of algebra it can be proved that the solution of (2. 4. 11) which satisfies the boundary conditions is

$$\varrho = \sum_{\varkappa=1}^{\infty}\left[A_1^{(\varkappa)} I_1\left(k_1 \frac{\varkappa\pi}{l} r\right) + A_2^{(\varkappa)} I_1\left(k_2 \frac{\varkappa\pi}{l} r\right)\right]\cos\frac{\varkappa\pi}{l} z,$$

$$\omega = \sum_{\varkappa=1}^{\infty}\left[-k_1 A_1^{(\varkappa)} I_0\left(k_1 \frac{\varkappa\pi}{l} r\right) - k_2 A_2^{(\varkappa)} I_0\left(k_2 \frac{\varkappa\pi}{l} r\right)\right]\sin\frac{\varkappa\pi}{l} z,$$

$$p' = -h(z) - \frac{1}{2}\alpha_0 + \sum_{\varkappa=1}^{\infty}\frac{\varkappa\pi}{l}\left[k_1 A_1^{(\varkappa)}(n - mk_1^2) I_0\left(k_1 \frac{\varkappa\pi}{l} r\right) + \right.$$

$$\left. + k_2 A_2^{(\varkappa)}(n - mk_2^2) I_0\left(k_2 \frac{\varkappa\pi}{l} r\right)\right]\cos\frac{\varkappa\pi}{l} z \tag{2. 4. 19}$$

where

$$\alpha_n = \frac{2}{l}\int_0^l \varphi(z)\cos\left(\frac{n\pi}{l}z\right)dz\,, \qquad n = 0, 1, 2, \ldots$$

$$\varphi(z) = f(z) - h(z) = 2(\lambda^2 - \mu^2)\left(\frac{\partial W'}{\partial I_1} + \lambda^2\frac{\partial W'}{\partial I_2}\right) =$$

$$= \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty}\alpha_n\cos\frac{n\pi}{l}z\,, \tag{2.4.20}$$

the constants $A_i^{(\varkappa)}$ are given by

$$A_1^{(n)} = -\alpha_n(k_2^2+1)I'_0(nr_2)\Delta_n^{-1}\,,$$

$$A_2^{(n)} = \alpha_n(k_1^2+1)I'_0(nr_1)\Delta_n^{-1}\,, \qquad r_\alpha = k_\alpha\pi r_0/l$$

$$\Delta_n = \frac{n\pi}{l}[\beta(k_2^2+1)I_0(nr_1)I'_0(nr_2) - \gamma(k_1^2+1)I'_0(nr_1)I_0(nr_2)] +$$

$$+ \frac{2c_{66}}{r_0}(k_1^2 - k_2^2)I'_0(nr_1)I'_0(nr_2)\,,$$

$$\beta = k_1(\alpha_{11} + \alpha_{33} + \tau^{33} - c_{44} - 2\alpha_{13} - c_{44}k_1^2)\,,$$

$$\gamma = k_2(\alpha_{11} + \alpha_{33} + \tau^{33} - c_{44} - 2\alpha_{13} - c_{44}k_2^2)\,, \tag{2.4.21}$$

and l, r_0 is the length and radius respectively of the cylinder B.

It can also be proved that the resultant force (Y^i) acting on the end surface of the second deformed body B' and referred to the y_i - axes is given by

$$(Y^i) = \left[0,\ 0,\ \pi r_0^2\tau^{33} + \varepsilon 2\pi\int_0^{r_0}F(r)\,r\,dr\right] \tag{2.4.22}$$

where

$$F(r) = -\frac{1}{2}\alpha_0 + \sum_{\varkappa=1}^{\infty}(-1)^\varkappa\left[f_3^{(\varkappa)}(r) + (\alpha_{33} - \alpha_{13} + \tau^{33})\frac{\varkappa\pi}{l}f_2^{(\varkappa)}(r)\right],$$

$$\nu_\varkappa = \frac{\varkappa\pi}{l}\,. \tag{2.4.23}$$

Thus, the resultant force equals the resultant force acting on the end surface of the first deformed body B plus a small force parallel to the axis of the cylinder and the magnitude of which is of the order of ε.

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REFERENCES

1. A. E. GREEN and W. ZERNA: Theoretical Elasticity, (Oxford, 1954).
2. A. E. GREEN and J. E. ADKINS: Large Elastic Deformations and Non-Linear Continuum Mechanics, (Oxford, 1960).
3. A. E. GREEN, R. S. RIVLIN and R. T. SHIELD: Proc. Roy. Soc. A 211 (1952), 128.
4. J. E. ADKINS: Progress in Solid Mechanics 2 (1961), 1.
5. A. D. KYDONIEFS and A. J. M. SPENCER: Int. J. Engng Sci. 3, 173 (1965).
6. A. J. M. SPENCER: Quart. Journ. Mech. and Applied Math., 12 (1959), 129.
7. E. W. WILKES: Quart. Journ. Mech. and Appl. Math. 8 (1955).

ΠΕΡΙΛΗΨΙΣ

Λόγῳ τῆς μὴ γραμμικότητος τῶν σχετικῶν διαφορικῶν ἐξισώσεων, ὁ ἀριθμὸς τῶν προβλημάτων μεγάλων ἐλαστικῶν παραμορφώσεων εἰς τὰ ὁποῖα ἔχει δοθῆ ἀκριβὴς λύσις εἶναι πολὺ περιωρισμένος. Αἱ πλεῖσται τῶν λύσεων αὐτῶν περιγράφονται ὑπὸ τῶν GREEN καὶ ZERNA [1] καὶ GREEN καὶ ADKINS [2] οἱ ὁποῖοι ἀναφέρουν ἐπίσης καὶ τὰς ἀρχικὰς πηγάς. Ἕτεραι λύσεις εἶναι δυνατὸν νὰ εὑρεθοῦν, ἐὰν δεχθῶμεν εἰδικὰς ἁπλουστέρας μορφὰς διὰ τὴν συνάρτησιν ἐλαστικοῦ δυναμικοῦ. Ἀλλὰ πολὺ συχνά, καὶ παρὰ τὴν ἁπλούστευσιν ταύτην, αἱ ἐξισώσεις παραμένουν ἀνεπίλυτοι. Διὰ τὸν λόγον αὐτὸν σημαντικὸν ἐνδιαφέρον ἔχει γεννηθῆ περὶ τὴν ἀνάπτυξιν προσεγγιστικῶν μεθόδων καθὼς καὶ μεθόδων παραλλάξεως. Οὕτω, ἐπὶ παραδείγματι, διάφοροι ἐρευνηταί, πολλῶν τῶν ὁποίων αἱ ἐργασίαι περιγράφονται εἰς τὸ [2], ἔχουν χρησιμοποιήσει μεθόδους διαδοχικῶν προσεγγίσεων· οἱ GREEN, RIVLIN καὶ SHIELD [3] ἀνέπτυξαν τὴν μέθοδον τῶν μικρῶν παραμορφώσεων ἐπιπροστιθεμένων εἰς μεγάλας τοιαύτας· ὁ ADKINS [4] παρετήρησε τὴν δυνατότητα παραλλάξεως τοῦ σχήματος εἴτε τοῦ παραμορφωμένου εἴτε τοῦ πρὸ τῆς παραμορφώσεως σώματος· οἱ SPENCER καὶ ΚΥΔΩΝΙΕΥΣ [5] ἐχρησιμοποίησαν μέθοδον ἀναπτύξεως τῶν λύσεων εἰς σειρὰς τῇ βοηθείᾳ γεωμετρικῆς παραμέτρου τοῦ προβλήματος δυναμένης νὰ θεωρηθῇ μικρᾶς.

Ἑτέρα δυνατότης πορουσιάζεται διὰ τῆς παραλλάξεως τῆς συναρτήσεως ἐλαστικοῦ δυναμικοῦ $W(I_i)$, ὅπου I_i εἶναι αἱ ἀναλλοίωτοι τῆς παραμορφώσεως. Τὸ πρῶτον τῶν προβλημάτων αὐτοῦ τοῦ εἴδους περιεγράφη ὑπὸ τοῦ SPENCER [6]

ὁ ὁποῖος ἐξήτασε τὴν περίπτωσιν ὁμογενοῦς σώματος. Εἰς τὴν παροῦσαν ἐργασίαν ἐξετάζεται ἡ ἐπίδρασις ἐπὶ τῆς παραμορφώσεως τοῦ μὴ ὁμογενοῦς παραλλακτικοῦ ὅρου $\varepsilon W'(I_i, \vartheta_i)$ ὅπου I_i καὶ ϑ_i εἶναι αἱ ἀναλλοίωτοι τῆς παραμορφώσεως καὶ αἱ συμπαραμορφούμεναι καμπυλόγραμμοι συντεταγμέναι ἀντιστοίχως καὶ ε εἷς σταθερὸς ἀριθμός, μικρὸς ἐν συγκρίσει πρὸς τὴν μονάδα. Δίδεται ἡ γενικὴ θεωρία καὶ ἡ ἐφαρμογὴ αὐτῆς εἰς τὴν περίπτωσιν ὅπου ἡ ἀρχικὴ παραμόρφωσις εἶναι ὁμοιόμορφος διόγκωσις. Τέλος, ἐν εἴδει παραδείγματος, ἐπιλύεται τὸ πρόβλημα ἑνὸς ἀσυμπιέστου ὀρθοῦ κυκλικοῦ κυλίνδρου ἔχοντος ἄξονα Ox_3 καὶ παραλλακτικὸν ὅρον $W' = W'(I_1, I_2, x_3)$.

Τὸ ἀποτέλεσμα τῆς παραλλάξεως τῆς συναρτήσεως ἐλαστικοῦ δυναμικοῦ εἶναι ἐν γένει μία μικρὰ παραμόρφωσις ἐπιπροστιθεμένη εἰς τὴν ἀρχικὴν μεγάλην παραμόρφωσιν. Ἂν ἡ ἀρχικὴ παραμόρφωσις εἶναι γνωστή, καὶ ἔχουν δοθῆ κατάλληλοι συνοριακαὶ συνθῆκαι, ἡ πρόσθετος παραμόρφωσις δύναται νὰ προσδιορισθῇ ἐκ τῶν ἐξισώσεων ἰσορροπίας καὶ τῶν σχέσεων τάσεων - παραμορφώσεων αἱ ὁποῖαι δίδονται εἰς τὴν γενικὴν θεωρίαν.

Αἱ ἀναλλοίωτοι τῆς παραμορφώσεως, αἱ σχέσεις τάσεων - παραμορφώσεων καὶ αἱ ἐξισώσεις ἰσορροπίας ἔχουν τὴν αὐτὴν μορφὴν μὲ ἐκείνην τῆς θεωρίας μικρῶν παραμορφώσεων ἐπιπροστιθεμένων εἰς μεγάλας τοιαύτας. Οἱ συντελεσταὶ τῶν σχέσεων τάσεων - παραμορφώσεων ἔχουν τὴν αὐτὴν μορφὴν ἥτις δίδεται εἰς [6], ἀλλὰ οἱ πρόσθετοι ὅροι ἐξαρτῶνται καὶ ἐκ τῶν συμπαραμορφουμένων καμπυλογράμμων συντεταγμένων.

Παρ' ὅλον ὅτι αἱ ἐξισώσεις ἰσορροπίας ἁπλοποιοῦνται σημαντικῶς ἐὰν ἡ ἀρχικὴ παραμόρφωσις εἶναι μία ὁμοιόμορφος διόγκωσις καὶ τὸ ἀπαραμόρφωτον σῶμα εἷς ὀρθὸς κυκλικὸς κύλινδρος, ἐν τούτοις αὗται δὲν εἶναι δυνατὸν νὰ ἐπιλυθοῦν εἰς τὴν γενικὴν περίπτωσιν. Διὰ τὴν εὕρεσιν λύσεως τινὸς δεχόμεθα τὴν ἁπλουστέραν μορφὴν $W' = W'(I_1, I_2, x_3)$ τοῦ παραλλακτικοῦ ὅρου τῆς συναρτήσεως ἐλαστικοῦ δυναμικοῦ. Ὑπὸ αὐτὰς τὰς προϋποθέσεις προσδιορίζεται ἡ πρόσθετος παραμόρφωσις.