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ON A GENERAL THEORY OF ANISOTROPY OF MATTER. THE SPECTRAL DECOMPOSITION OF THE COMPLIANCE TENSOR: APPLICATION TO CRYSTALLOGRAPHY

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ΑΘΗΝΑΙ

ΓΡΑΦΕΙΟΝ ΔΗΜΟΣΙΕΥΜΑΤΩΝ ΤΗΣ ΑΚΑΔΗΜΙΑΣ ΑΘΗΝΩΝ

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"... γεννώντάς τε και εκτρέφοντας παίδας, καθάπερ λαμπάδα τον βίον παραδίδοντας άλλοις εξ' άλλων θεραπεύοντας αεί Θεούς κατά νόμους".

"Νόμοι" Πλάτωνος
Βιβλίον VI, 776 B

"... begetting and rearing children and so handing on life, like a torch, from one generation to another and even worshipping the Gods as the laws direct".

Plato, "Laws"
Book VI, 776 B.
The Loeb Classical Library,
Harvard Univ. Press MCMLXI

CONTENTS

Keywords: Spectral decomposition, compliance tensor, crystal systems, eigentensors, eigenangles, anisotropy.

ABSTRACT

A newly developed approach based on the spectral decomposition principle, which is especially useful in crystallography, is applied for the first time in this paper. The compliance fourth-rank tensor of crystalline media belonging to the monoclinic system is, thus, spectrally decomposed, its eigenvalues are evaluated, together with the elementary idempotent fourth-rank tensors, which decompose uniquely the fourth-rank tensor space into orthogonal subspaces. Accordingly, energy orthogonal states of stress and strain are derived from the idempotent tensors, namely eigenstates which do not perform work along the deformations produced by the others. Then, in terms of such eigenstates, the elastic potential of the monoclinic medium is given a definite decomposition in distinct, non interacting parts, indicating the non-existence of a pure dilatational strain energy component. It is subsequently shown that the constitutive parameters, required for an invariant specification of the elastic characteristics of a monoclinic crystal, are the distinct eigenvalues of the compliance tensor, in addition to a group of seven dimensionless parameters, referred to as the eigenangles. Interestingly, these parameters are shown to be responsible for the orientation and alignment of the stress and strain eigentensors in the six-dimensional stress space. Next, the compliance tensor is spectrally decomposed for the case of anisotropic media belonging to the orthorhombic, tetragonal, hexagonal and cubic crystal systems, as special cases of the spectral decomposition of the compliance tensor for the monoclinic medium. Therefore, the characteristic values, idempotent fourth-rank tensors, stress and strain second-rank eigentensors and finally the strain energy parts corresponding to these eigenstates are derived for all the above mentioned symmetries. Moreover, the individual criteria imposed by the classical thermodynamical argument on the components of the compliance tensor, which are necessary and sufficient for the elastic strain energy to be positive definite, are investigated in detail for the class of monoclinic media. Then, the corresponding criteria for the remaining crystal classes are obtained by

reduction of the conditions valid for the monoclinic symmetry. Finally, several examples of representative inorganic crystals of the monoclinic, orthorhombic, tetragonal, hexagonal, and cubic crystal systems illustrate the results of the theoretical analysis.

1. INTRODUCTION

Fourth-rank tensors are indispensable tools for an appropriate description of either the underlying geometrical symmetry of crystal structures, or the mechanical properties of anisotropic materials. Fourth-rank tensors were initially presented by Nye (1957) and Mason (1966) for the characterisation of either elastic stiffness and compliance, elasto-optical and piezo-optical coefficients, or electrostriction and magnetostriction in crystalline media.

In addition, fourth-rank tensors, embodying the elastic or other property of crystalline anisotropic substances, were expanded (Srinivasan and Nigam, 1969) as a linear combination of independent elementary tensors, corresponding to scalar coefficients, which present themselves as universal material constants, being invariant under arbitrary orthogonal co-ordinate transformations. Next, Walpole (1981;1984) reduced for the first time the algebra of fourth-rank tensors to irreducible subalgebras, which were simpler than the initial one, thus facilitating operations between these tensors, and offering insight into the tensor structure. However, the decomposition used by Walpole in his applications did not correspond to the spectral type, except in the trivial cases of the isotropic and cubic system.

It was Rychlewski (1984), who first confirmed the possibility of decomposing the elastic stiffness **C** and compliance **S** fourth-rank tensors by means of the spectral decomposition. This decomposition was favourable due to its ability to split these tensors into idempotent fourth-rank tensors, which successively defined energy orthogonal stress and strain eigentensors. Energy orthogonality specifies stress tensors that are simultaneously mutually orthogonal and collinear with their corresponding strain tensors.

Nevertheless, Rychlewski's spectral analysis was a re-examination of Kelvin's ideas (Thomson, 1856; 1878), who in a time when the tensorial formulation of the mathematical theory of elasticity was not existent, introduced the concepts of elastic eigenvalues, as well as those of stress and strain eigentensors. However, the analysis undertaken by Lord Kelvin was based on non-tensorial terminology, which was not clearly understood, and thus, was criticised with scepticism by Todhunter and Pearson (1886). In fact, Lord Kelvin's formulation was not considered in the numerous comprehensive overviews of the elasticity theory, and was entirely overlooked for over a century, until Rychlewski reconstituted and reexpressed the key notions of his analysis, defining the mathematical structure of an arbitrary linearly elastic anisotropic body.

Despite the fact that Rychlewski substantiated the application and simplicity of the spectral decomposition principle on the class of the symmetric fourth-rank tensors of compliance **S** and stiffness **C** for anisotropic media, he did not proceed to determine the eigenvalues and eigentensors of the corresponding tensors. The spectral decomposition of the compliance **S** and stiffness **C** tensors was performed for transversely isotropic media by Theocaris and Philippidis (1989;1990;1991), and it was proven that the four characteristic values of the corresponding tensors, together with the value of the eigenangle ω constitute the five necessary parameters required for the specification of both the elastic and failure behaviour of such a medium. Moreover, angle ω was confirmed to offer an effective monoparametric indication of the degree of anisotropy of transversely isotropic materials. Hence, while the angle ω is always equal to 125.26° for isotropic media, its value tends to either 90° or 180° when the anisotropy of the material is increased.

Furthermore, Mehrabadi and Cowin (1990) also dealt with the spectral decomposition, which they applied for the analysis of only the stress and strain second-rank tensors. Moreover, they presented results of their decomposition for only the cubic, hexagonal, tetragonal and trigonal crystalline systems. Nevertheless, the authors did not attempt to show the "full" decomposition of the fourth-rank tensors.

Subsequently, the three-dimensional spectral decomposition of the fourth-rank tensors was extended in order to incorporate the equally important two-dimensional equivalent, under plane-stress conditions (Theocaris and Sokolis, 1998). On the basis of the two-dimensional spectral decomposition, the elastic properties of thin plates of a transversely isotropic material were efficiently described via three eigenvalues, associated with three orthogonal stress and strain eigenstates, together with another parameter ω_p, designated as the plane eigenangle. Once again, it was proven that the value of the plane eigenangle ω_p may be employed for a monoparametric indication of the degree of anisotropy of thin transversely isotropic plates. In conclusion, it was exhibited that, for isotropic media, ω_p equals 135, whereas, when the anisotropy of the material is increased, its value tends to either 90° or 180°.

Until now, the special cases of anisotropic crystalline media were confronted with no unique solution for the different elastic symmetries, which were hence dealt with separately. In this paper, a newly developed approach, based on the spectral decomposition principle, is applied for the general case of the monoclinic medium, which allows all symmetries compatible with the monoclinic one to be tackled with a single solution. In accordance, the compliance fourth-rank tensor **S** of crystalline media belonging to the monoclinic system is orthogonally expanded by means of the spectral decomposition, its eigenvalues are evaluated and established, together with the elementary idempotent fourth-rank tensors, which serve to analyse uniquely the fourth-rank tensor space into orthogonal subspaces. Based on the specific properties of the spectral decomposition, energy orthogonal states of stress and strain are derived from the idempotent tensors, namely eigenstates which do not perform work along the deformations produced by the others. It is subsequently affirmed that the constitutive parameters, required for an invariant specification of the elastic characteristics of a monoclinic crystal, are the six distinct characteristic values of the compliance tensor **S**, in addition to a group of seven dimensionless parameters, referred to as the eigenangles ψ, ρ, ν, μ, θ, ω and φ. Interestingly, these parameters are ascertained to be responsible for the orientation and alignment of the stress and strain eigentensors in the six-dimensional stress space. Furthermore, the different stress and strain eigenstates divide the total elastic strain energy density of monoclinic media into distinct, autonomous constituents, pointing out, however, the absence of a pure dilatational strain energy part. Next, the compliance tensor **S** of anisotropic media belonging to the orthorhombic, tetragonal, hexagonal and cubic crystal systems is decomposed, by reduction of the results obtained from the spectral decomposition of the compliance tensor **S** for the monoclinic medium, since the compliance tensors **S** for these symmetries share the same form with the corresponding tensor of the monoclinic medium. Yet, the trigonal medium compliance tensor **S** is not related to the one of the monoclinic medium, thus, it is excluded from the current investigation. Hence, the eigenvalues, idempotent fourth-rank tensors, stress and strain second-rank eigentensors, and finally the strain energy parts associated to these eigenstates are acquired for the orthotropic, tetragonal and hexagonal media, as well as for the less interesting cases of cubic and isotropic media. Moreover, the restrictive bounds dictated by the classical thermodynamical argument on the values of the components of the compliance fourth-rank tensor **S** for the monoclinic medium are obtained, based on the positiveness of the characteristic values of tensor **S**. Then, the individual criteria in terms of the elements of the compliance tensor **S**, which are necessary and sufficient for the elastic strain energy to be positive definite, are examined subsequently for the orthorhombic, tetragonal, hexagonal and cubic symmetries, by reduction of the results derived for the monoclinic crystal system. Finally, several examples with experimentally measured values of the compliance tensor components of representative inorganic crystals belonging to the monoclinic, orthorhombic, tetragonal, hexagonal and cubic crystal systems illustrate the results of the theoretical analysis. This paper is closed with a discussion of the advantages offered by the spectral representation in the theory of anisotropic elasticity.

2. THE DESCRIPTION OF ANISOTROPIC ELASTIC BEHAVIOUR

Given that **L** denotes the space of symmetric tensors of the second rank over $\mathbf{R}^3$, then **L**, together with the ordinary definition of the scalar product, constitutes a 6-dimensional Euclidean space. Additionally, assuming that the symmetric tensor square of **L**, that is the space of symmetric fourth-rank tensors, is designated by **M**, then, the definitions of tensor spaces **L** and **M** are expressed in symbolic notation by:

$$\mathbf{L} \equiv \mathrm{sym}(\mathbf{R}^3 \otimes \mathbf{R}^3), \tag{1a}$$

$$\mathbf{M} \equiv \mathrm{sym}(\mathbf{L} \otimes \mathbf{L}), \tag{1b}$$

representing a pertinent field for the mathematical characterisation of the hyperelastic solid. Such a solid corresponds to the class of material behaviour, whose elements are characterised by stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\varepsilon}$ second-rank tensors correspondingly, as well as by a potential function T, referred to as the elastic potential, for which it is true that:

$$\boldsymbol{\sigma}=\frac{\partial T}{\partial \boldsymbol{\varepsilon}} \quad \text{or} \quad \sigma_{ij}=\frac{\partial T}{\partial \varepsilon_{ij}}. \tag{2}$$

For sufficiently small strains $\boldsymbol{\varepsilon}$ and isothermal or adiabatic conditions, Eq. (2) is equivalent to the postulation of Hooke's law (Hooke, 1678), stating that stress $\boldsymbol{\sigma}$ is proportional to strain $\boldsymbol{\varepsilon}$; strictly speaking, the stress components are linearly related to the strain components. In symbolic indicial notation, this relation, in its generalised statement for an anisotropic medium, may be taken as (Wooster, 1938; Sokolnikoff, 1946; Nye, 1957):

$$\boldsymbol{\sigma}=\mathbf{C}\cdot\boldsymbol{\varepsilon} \quad \text{or} \quad \sigma_{ij}=C_{ijkl}\,\varepsilon_{kl}, \tag{3}$$

with summation over repeated indices being implied. The eighty-one coefficients C_{ijkl} are termed the elastic stiffness constants, and are evidently the components of a fourth-rank tensor. The relation given in Eq. (3) may be inverted to express the strain components as linear functions of the stress components in the form:

$$\boldsymbol{\varepsilon}=\mathbf{S}\cdot\boldsymbol{\sigma} \quad \text{or} \quad \varepsilon_{ij}=S_{ijkl}\,\sigma_{kl}, \tag{4}$$

where the constants of proportionality S_{ijkl} introduced hereby are called the elastic compliance constants. Substitution of Eq. (3) into Eq. (4) shows that tensors **C** and **S** are related by:

$$\mathbf{C}\cdot\mathbf{S}=\mathbf{S}\cdot\mathbf{C}=\mathbf{I}, \tag{5a}$$

$$\text{or} \quad C_{ijmn}\cdot S_{mnkl}=S_{ijmn}\cdot C_{mnkl}=I_{ijkl}, \tag{5b}$$

where the fourth-rank tensor **I** is the unit element of the fourth-rank tensor space **M**:

$$I_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{6}$$

Furthermore, the elastic potential T is defined by the scalar product on the second-rank tensor space **L**:

$$2T = \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ij}. \tag{7}$$

Then, inserting Eqs. (3) and (4) into the defining relation (7) of the elastic potential T yields:

$$2T = \boldsymbol{\sigma}\cdot\mathbf{S}\cdot\boldsymbol{\sigma} = \boldsymbol{\varepsilon}\cdot\mathbf{C}\cdot\boldsymbol{\varepsilon}, \tag{8a}$$

or

$$2T = S_{ijkl}\,\sigma_{ij}\,\sigma_{kl} = C_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl}. \tag{8b}$$

It turns out that the stress **σ** and strain **ε** tensors are symmetric (Sokolnikoff, 1946), so that:

$$\sigma_{ij} = \sigma_{ji}, \qquad \varepsilon_{ij} = \varepsilon_{ji}, \tag{9}$$

reducing the number of independent components of either tensor to six. It would appear by Eqs. (3) and (4) that there are eighty-one components of the stiffness **C** and compliance **S** tensors in the most general case. However, owing to the symmetry of the stress **σ** and strain **ε** tensors, this number is reduced to thirty-six, as a consequence of the requirement that tensors **C** and **S** are symmetrical with respect to the interchange of index i by j, and index k by l:

$$C_{ijkl} = C_{jikl}, \qquad C_{ijkl} = C_{ijlk}, \tag{10a}$$

$$S_{ijkl} = S_{jikl}, \qquad S_{ijkl} = S_{ijlk}, \tag{10b}$$

Next, based on the thermodynamical argument that no work is produced by an elastic medium in a closed loading cycle, a supplementary symmetry constraint is imposed on the stiffness **C** and compliance **S** tensors, necessitating that both tensors are symmetric on interchange of the first two, ij, and the last two, kl, indices:

$$C_{ijkl} = C_{klij}, \qquad S_{ijkl} = S_{klij}, \tag{11}$$

thus, bringing down the number of independent stiffness and compliance components for the generally anisotropic elastic medium to twenty-one. In addition, it should be pointed out that reciprocal relations (11) are of thermodynamic origin, hence they are not dependent upon the actual mechanism of elastic behaviour.

A note should be addressed to the numerous different notations, which have been dispensed for the stress and strain components (Todhunter and Pearson, 1886-93; Voigt, 1910; Love, 1927; Wooster, 1938; Southwell, 1941; Cady, 1946; Brodeau, 1946; Mason, 1950; Timoshenko and Goodier, 1951; Nye, 1957; Hearmon, 1961; Lekhnitskii, 1963). In this paper, Nye's notation has been employed all through.

In experimental practice, an abbreviated notation is often adopted for Hooke's law, referred to as the Voigt notation (Voigt, 1910), represented in the form:

$$\boldsymbol{\sigma} = \mathbf{c} \cdot \boldsymbol{\varepsilon} \quad \text{or} \quad \sigma_p = c_{pq}\,\varepsilon_q, \tag{12a}$$

$$\boldsymbol{\varepsilon} = \mathbf{s} \cdot \boldsymbol{\sigma} \quad \text{or} \quad \varepsilon_p = s_{pq}\,\sigma_q, \tag{12b}$$

where in the stiffness c_{pq} and compliance s_{pq} matrices, indices p and q acquire values 1,…,6.

In addition, the full tensor suffixes of stresses $\boldsymbol{\sigma}$ and strains $\boldsymbol{\varepsilon}$ are contracted according to the scheme:

$$\sigma_{11}=\sigma_1, \quad \sigma_{22}=\sigma_2, \quad \sigma_{33}=\sigma_3, \quad \sigma_{23}=\sigma_4, \quad \sigma_{13}=\sigma_5, \quad \sigma_{12}=\sigma_6, \tag{13a}$$

$$\varepsilon_{11}=\varepsilon_1, \quad \varepsilon_{22}=\varepsilon_2, \quad \varepsilon_{33}=\varepsilon_3, \quad 2\varepsilon_{23}=\varepsilon_4, \quad 2\varepsilon_{13}=\varepsilon_5, \quad 2\varepsilon_{12}=\varepsilon_6. \tag{13b}$$

However, the appearance of the 2-factor for the shear strains ε_{ij}, i,j=1,2 or 3, $i \neq j$ in Eqs. (13b) should be particularly noted. Further, the shear strains ε_{ij}, which form the components of a second-rank tensor in three dimensions, should be cautiously identified from the contracted shear strain components e_p, p=4,5 or 6, which do not form a tensor.

Finally, the equivalence between the components of the stiffness fourth-rank tensor **C** and the components of the 6×6 matrix **c** of the Voigt notation is readily shown to be:

$$C_{ijkl}=c_{pq}, \tag{14}$$

whereas, the ensuing relationship between the components of the compliance fourth-rank tensor **S** and the components of the 6×6 matrix **s** of the Voigt notation is proven to be more complicated than the analogous Eq. (14):

$$S_{ijkl}=s_{pq}, \quad \text{if p and q are both 3 or less,} \tag{15a}$$

$$S_{ijkl}=\frac{1}{2}s_{pq}, \quad \text{if either p or q is greater than 3,} \tag{15b}$$

$$S_{ijkl}=\frac{1}{4}s_{pq}, \quad \text{if both p and q are greater than 3,} \tag{15c}$$

in which a contraction rule is applied to abbreviate a double subscript by a single contracted index, running from 1 to 6, by means of the following pattern:

$$11 \rightarrow 1, 22 \rightarrow 2, 33 \rightarrow 3, 23 \rightarrow 4, 13 \rightarrow 5 \text{ and } 12 \rightarrow 6. \tag{16}$$

Nevertheless, it is substantial to recall, whenever employing the abbreviated notation named after Voigt, that this is a non-tensorial notation, specifically the 6×6 stiffness **c** and compliance **s** matrices do not represent tensors, such as the **C** and **S** tensors, which constitute Cartesian fourth-rank tensors in three dimensions. Nevertheless, this contracted notation is significant, mainly because it is almost invariably used in experimental work of elasticity, and secondly, since it has come over to be a basic, standard tool in anisotropic elasticity (Voigt, 1910; Nye, 1957; Hearmon, 1961).

3. SPECTRAL DECOMPOSITION OF THE MONOCLINIC COMPLIANCE TENSOR

The internal structure of crystalline media is responsible for their elastic characteristics. Whenever the internal configuration of an elastic body maintains geometric symmetry, symmetry is likewise encountered in its elastic characteristics, these being alike along the directions of symmetry, namely the equivalent directions. In fact, a principle was introduced for crystalline bodies (Neumann, 1885), associating the internal symmetry and the elastic symmetry, thus postulating that a material, with respect to its physical properties, exhibits identical sort of symmetry as its crystallographic form.

Further, depending on its structure, the crystal exhibits one or another geometric symmetry. Such symmetries were investigated at the end of last century, and it was proven that there are thirty-two forms of internal symmetry, which can be partitioned into seven crystal systems (syngonies) as follows: (1) triclinic, (2) monoclinic, (3) orthorhombic, (4) trigonal (rhombohedral), (5) tetragonal, (6) hexagonal and (7) cubic (Fig. 1). These crystal systems are identified by a distinct set of symmetry elements, comprising of (a) planes of reflection symmetry, denoted by P, (b) n-fold rotation axes, denoted by L^n, (c) n-fold axes of rotatory inversion, denoted by L_n, and (d) centres of symmetry or centres of inversion, denoted by C.

However, if an anisotropic crystalline medium possesses elastic symmetry, the constitutive equations of the generalised Hooke's law are simplified. Furthermore, the number of independent components of the compliance **S** and stiffness **C** fourth-rank tensors is further reduced by the symmetry operations of the respective crystal classes. For instance, there are only nine independent components for the orthorhombic classes, five for the hexagonal classes, and three for the cubic classes. Moreover, in all but the triclinic classes the effect of crystal symmetry is revealed by the presence of zeros and repeated elements among the tensor components. Accordingly, the form of tensors **S** and **C** was established at the beginning of the twentieth century for all crystal classes by Voigt (1910). Thereafter, it was presented in the books of Lekhnitskii (1963), Mason (1966), Gurtin (1972) and Thurston (1974).

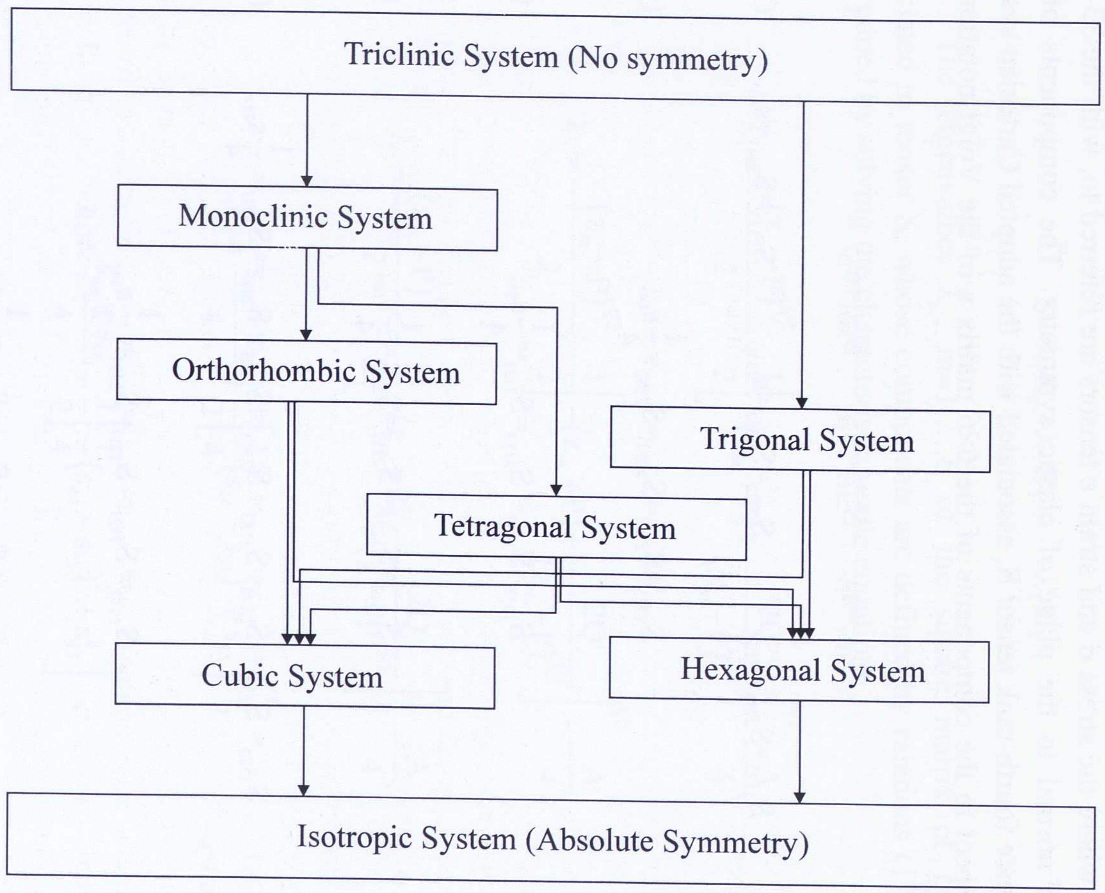

Fig. 1. The relationship between the traditional, distinct crystal systems.

In this paper, our attention is restricted to the monoclinic crystal system, which is characterised by a plane of elastic symmetry. In the following, the compliance fourth-rank tensor **S** of the monoclinic linear elastic solid is decomposed spectrally for the first time. We assume the Cartesian co-ordinate system, where the stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\varepsilon}$ tensors are referred to, with the 33-axis oriented normal to the plane of elastic symmetry. The components of the compliance fourth-rank tensor **S**, associated with the adopted Cartesian system, with respect to the components of the 6×6 matrix **s** of the Voigt notation are given by:

$$S_{1111}=s_{11}, \quad S_{2222}=s_{22}, \quad S_{3333}=s_{33}, \tag{17a}$$

$$S_{1122}=S_{2211}=s_{12}, \quad S_{2233}=S_{3322}=s_{23}, \quad S_{1133}=S_{3311}=s_{13}, \tag{17b}$$

$$S_{2323}=S_{2332}=S_{3223}=S_{3232}=\frac{1}{4}s_{44}, \tag{17c}$$

$$S_{1313}=S_{1331}=S_{3113}=S_{3131}=\frac{1}{4}s_{55}, \tag{17d}$$

$$S_{1212}=S_{1221}=S_{2112}=S_{2121}=\frac{1}{4}s_{66}, \tag{17e}$$

$$S_{1323}=S_{1332}=S_{3123}=S_{3132}=S_{2313}=S_{3213}=S_{2331}=S_{3231}=\frac{1}{4}s_{45}, \tag{17f}$$

$$S_{1112}=S_{1121}=S_{1211}=S_{2111}=\frac{1}{2}s_{16}, \tag{17g}$$

$$S_{2212}=S_{2221}=S_{1222}=S_{2122}=\frac{1}{2}s_{26}, \tag{17h}$$

$$S_{3312}=S_{3321}=S_{1233}=S_{2133}=\frac{1}{2}s_{36}, \tag{17i}$$

and all the remaining S_{ijkl} components are zero.

The compliance fourth-rank tensor **S** of the monoclinic medium is spectrally decomposed in terms of the eigenvalues λ_m, m=1,…,6 and the corresponding idempotent fourth-rank tensors $\mathbf{E_m}$, m=1,...,6, according to the orthogonal expansion:

$$\mathbf{S} = \lambda_1 \mathbf{E_1} + \lambda_2 \mathbf{E_2} + \cdots + \lambda_6 \mathbf{E_6}. \tag{18}$$

The eigenvalues λ_m, m=1,…,6 of the square matrix of rank six associated to tensor **S**, whose components are defined by relations (17), were determined by solving the characteristic sextic equation:

$$\lambda_1 = -\frac{(z_m - P)^{1/2}}{2} + \frac{1}{2}\left[-(z_m + P) + \frac{2Q}{(z_m - P)^{1/2}}\right]^{1/2} - \frac{A}{4}, \tag{19a}$$

$$\lambda_2 = -\frac{(z_m - P)^{1/2}}{2} - \frac{1}{2}\left[-(z_m + P) + \frac{2Q}{(z_m - P)^{1/2}}\right]^{1/2} - \frac{A}{4}, \tag{19b}$$

$$\lambda_3 = \frac{(z_m - P)^{1/2}}{2} + \frac{1}{2}\left[-(z_m + P) - \frac{2Q}{(z_m - P)^{1/2}}\right]^{1/2} - \frac{A}{4}, \tag{19c}$$

$$\lambda_4 = \frac{s_{44} + s_{55}}{4} + \frac{1}{2}\left[\frac{1}{4}(s_{44} - s_{55})^2 + s_{45}^2\right]^{1/2}, \tag{19d}$$

$$\lambda_5 = \frac{s_{44} + s_{55}}{4} - \frac{1}{2}\left[\frac{1}{4}(s_{44} - s_{55})^2 + s_{45}^2\right]^{1/2}, \tag{19e}$$

$$\lambda_6 = \frac{(z_m - P)^{1/2}}{2} - \frac{1}{2}\left[-(z_m + P) - \frac{2Q}{(z_m - P)^{1/2}}\right]^{1/2} - \frac{A}{4}. \tag{19f}$$

The characteristic values λ_m, m=1,…,6, defined by relations (19), constitute the roots of the minimum polynomial of the compliance tensor **S**, which in factored form may be written as:

$$(\mathbf{S}-\lambda_1\mathbf{I})(\mathbf{S}-\lambda_2\mathbf{I})\cdots(\mathbf{S}-\lambda_6\mathbf{I})=0. \tag{20}$$

Quantities z_m, m=1,2,3, emerging in the expressions (19) for the eigenvalues, are defined by:

$$z_1=-\left[-\frac{Q'}{2}+\left(\frac{Q'^2}{4}-k^3\right)^{1/2}\right]^{1/3}\frac{(1+i\sqrt{3})}{2}+\frac{k(-1+i\sqrt{3})/2}{\left[-\frac{Q'}{2}+\left(\frac{Q'^2}{4}-k^3\right)^{1/2}\right]^{1/3}}+\frac{P}{3}, \tag{21a}$$

$$z_2=\left[-\frac{Q'}{2}+\left(\frac{Q'^2}{4}-k^3\right)^{1/2}\right]^{1/3}\frac{(-1+i\sqrt{3})}{2}+\frac{k(-1-i\sqrt{3})/2}{\left[-\frac{Q'}{2}+\left(\frac{Q'^2}{4}-k^3\right)^{1/2}\right]^{1/3}}+\frac{P}{3}, \tag{21b}$$

$$z_3=\left[-\frac{Q'}{2}+\left(\frac{Q'^2}{4}-k^3\right)^{1/2}\right]^{1/3}+\frac{k}{\left[-\frac{Q'}{2}+\left(\frac{Q'^2}{4}-k^3\right)^{1/2}\right]^{1/3}}+\frac{P}{3}, \tag{21c}$$

in which quantities k, P′ and Q′ are expressed by:

$$k=-\frac{P'}{3}, \tag{22a}$$

$$P'=-4R-\frac{P^2}{3}, \tag{22b}$$

$$Q'=-\frac{2P^3}{27}+\frac{8PR}{3}-Q^2, \tag{22c}$$

with quantities P, Q and R defined as:

$$P = B - \frac{3A^2}{8}, \tag{23a}$$

$$Q = \frac{A^3}{8} - \frac{AB}{2} + C, \tag{23b}$$

$$R = -3\left(\frac{A}{4}\right)^4 + \frac{A^2 B}{16} - \frac{A\,C}{4} + D, \tag{23c}$$

where quantities A, B and C are complicated functions of the components of the compliance tensor **S**:

$$A = -s_{11} - s_{22} - s_{33} - \frac{s_{66}}{2}, \tag{24a}$$

$$B = \left(s_{11}s_{22} + s_{11}s_{33} + s_{22}s_{33}\right) + \left(s_{11} + s_{22} + s_{33}\right)\frac{s_{66}}{2} - \left(s_{12}^2 + s_{13}^2 + s_{23}^2 + \frac{s_{16}^2}{2} + \frac{s_{26}^2}{2} + \frac{s_{36}^2}{2}\right), \tag{24b}$$

$$\begin{aligned} C = {} & s_{11}\left(s_{23}^2 + \frac{s_{26}^2}{2} + \frac{s_{36}^2}{2}\right) + s_{22}\left(s_{13}^2 + \frac{s_{16}^2}{2} + \frac{s_{36}^2}{2}\right) + s_{33}\left(s_{12}^2 + \frac{s_{16}^2}{2} + \frac{s_{26}^2}{2}\right) \\ & + \frac{s_{66}}{2}\left(s_{12}^2 + s_{13}^2 + s_{23}^2 - s_{11}s_{22} - s_{11}s_{33} - s_{22}s_{33}\right) \\ & - \left(s_{11}s_{22}s_{33} + s_{23}s_{26}s_{36} + s_{13}s_{16}s_{36} + s_{12}s_{16}s_{26} + 2s_{12}s_{13}s_{23}\right), \end{aligned} \tag{24c}$$

$$\begin{aligned} D = {} & s_{16}s_{26}\left(s_{33}s_{12} - s_{13}s_{23}\right) + s_{26}s_{36}\left(s_{11}s_{23} - s_{12}s_{13}\right) + s_{16}s_{36}\left(s_{22}s_{13} - s_{12}s_{23}\right) \\ & + \frac{s_{66}}{2}\left(s_{11}s_{22}s_{33} + 2s_{12}s_{13}s_{23} - s_{11}s_{23}^2 - s_{22}s_{13}^2 - s_{33}s_{12}^2\right) \\ & + \frac{s_{16}^2}{2}\left(s_{23}^2 - s_{22}s_{33}\right) + \frac{s_{26}^2}{2}\left(s_{13}^2 - s_{11}s_{33}\right) + \frac{s_{36}^2}{2}\left(s_{12}^2 - s_{11}s_{22}\right). \end{aligned} \tag{24d}$$

Furthermore, the elementary idempotent tensors $\mathbf{E_m}$, m=1,...,6 decompose the unit element $\mathbf{I}$ of the fourth-rank symmetric tensor space $\mathbf{M}$ and satisfy the following set of equations:

$$\mathbf{I} = \mathbf{E_1} + \mathbf{E_2} + \cdots + \mathbf{E_6}, \tag{25a}$$

$$\mathbf{E_m} \cdot \mathbf{E_n} = 0, \quad m \neq n \tag{25b}$$

$$\mathbf{E_m} \cdot \mathbf{E_m} = \mathbf{E_m}. \tag{25c}$$

The idempotent fourth-rank tensors $\mathbf{E_m}$, m=1,…,6 offer an orthogonal expansion of the space $\mathbf{M}$ of symmetric fourth-rank tensors, into orthogonal subspaces $\mathbf{M_m}$ as follows:

$$\mathbf{M}=\mathbf{M_1} \oplus \mathbf{M_2} \oplus \cdots \oplus \mathbf{M_6}, \qquad \mathbf{M_m} \perp \mathbf{M_n} \quad \text{for } m \neq n, \tag{26}$$

where $\mathbf{E_m}$ is the idempotent tensor on $\mathbf{M_m}$, for m=1,…,6.

The corresponding six idempotent fourth-rank tensors $\mathbf{E_m}$, m=1,…,6 of the spectral decomposition of $\mathbf{S}$ were obtained as:

$$\mathbf{E_m} = \frac{(\mathbf{S}-\lambda_1\mathbf{I})\cdots(\mathbf{S}-\lambda_{m-1}\mathbf{I})(\mathbf{S}-\lambda_{m+1}\mathbf{I})\cdots(\mathbf{S}-\lambda_6\mathbf{I})}{(\lambda_m-\lambda_1)\cdots(\lambda_m-\lambda_{m-1})(\lambda_m-\lambda_{m+1})\cdots(\lambda_m-\lambda_6)}. \tag{27}$$

Accordingly, tensors $\mathbf{E_m}$ were evaluated to be:

$$\mathbf{E_1} = E^1_{ijkl} = \mathbf{g} \otimes \mathbf{g} = g_{ij} g_{kl}, \tag{28a}$$

$$\mathbf{E_2} = E^2_{ijkl} = \mathbf{r} \otimes \mathbf{r} = r_{ij} r_{kl}, \tag{28b}$$

$$\mathbf{E_3} = E^3_{ijkl} = \mathbf{h} \otimes \mathbf{h} = h_{ij} h_{kl} \quad \text{with } \mathbf{g, r, h, s, t, q} \in \mathbf{L}, \tag{28c}$$

$$\mathbf{E_4} = E^4_{ijkl} = \mathbf{t} \otimes \mathbf{t} = t_{ij} t_{kl}, \tag{28d}$$

$$\mathbf{E}_5 = E^5_{ijkl} = \mathbf{q} \otimes \mathbf{q} = q_{ij} q_{kl}, \tag{28e}$$

$$\mathbf{E}_6 = E^6_{ijkl} = \mathbf{s} \otimes \mathbf{s} = s_{ij} s_{kl}. \tag{28f}$$

The second-rank symmetric tensors **g**, **r**, **h**, **t**, **q** and **s**, appearing in relations (28) for the expressions of the idempotent tensors $\mathbf{E}_m$, m=1,…,6, are defined as follows:

$$\mathbf{g} = g_3 \mathbf{a} + g_2 \mathbf{b} + g_1 \mathbf{c} + g_6 \mathbf{d}, \tag{29a}$$

$$\mathbf{r} = r_3 \mathbf{a} + r_2 \mathbf{b} + r_1 \mathbf{c} + r_6 \mathbf{d}, \tag{29b}$$

$$\mathbf{h} = h_3 \mathbf{a} + h_2 \mathbf{b} + h_1 \mathbf{c} + h_6 \mathbf{d}, \tag{29c}$$

$$\mathbf{t} = \cos\psi \mathbf{f} + \sin\psi \mathbf{e}, \tag{29d}$$

$$\mathbf{q} = -\sin\psi \mathbf{f} + \cos\psi \mathbf{e}, \tag{29e}$$

$$\mathbf{s} = -\sin\rho \mathbf{a} - \sin\nu\cos\rho \mathbf{b} - \sin\mu\cos\rho\cos\nu \mathbf{c} + \cos\rho\cos\nu\cos\mu \mathbf{d}, \tag{29f}$$

in which

$$g_1 = \sin\theta\sin\varphi\cos\mu - \sin\mu(-\sin\theta\cos\varphi\sin\nu + \cos\theta\sin\rho\cos\nu), \tag{30a}$$

$$g_2 = -\sin\theta\cos\varphi\cos\nu - \cos\theta\sin\rho\sin\nu, \tag{30b}$$

$$g_3 = \cos\theta\cos\rho, \tag{30c}$$

$$g_6 = \sin\theta\sin\varphi\sin\mu + \cos\mu(-\sin\theta\cos\varphi\sin\nu + \cos\theta\sin\rho\cos\nu), \tag{30d}$$

$$\begin{aligned} r_1 = {} & (-\sin\omega\cos\varphi - \cos\theta\sin\varphi\cos\omega)\cos\mu \\ & - \sin\mu\left[(-\sin\omega\sin\varphi + \cos\theta\cos\varphi\cos\omega)\sin\nu + \cos\omega\sin\theta\sin\rho\cos\nu\right], \end{aligned} \tag{30e}$$

$$\mathrm{r}_2 = (-\sin\omega\sin\varphi + \cos\theta\cos\varphi\cos\omega)\cos\nu - \cos\omega\sin\theta\sin\rho\sin\nu, \tag{30f}$$

$$\mathrm{r}_3 = \cos\omega\sin\theta\cos\rho, \tag{30g}$$

$$\begin{aligned}\mathrm{r}_6 &= (-\sin\omega\cos\varphi - \cos\theta\sin\varphi\cos\omega)\sin\mu \\ &+ \cos\mu\left[(-\sin\omega\sin\varphi + \cos\theta\cos\varphi\cos\omega)\sin\nu + \cos\omega\sin\theta\sin\rho\cos\nu\right],\end{aligned} \tag{30h}$$

$$\begin{aligned}\mathrm{h}_1 &= (\cos\omega\cos\varphi - \cos\theta\sin\varphi\sin\omega)\cos\mu \\ &- \sin\mu\left[(\cos\omega\sin\varphi + \cos\theta\cos\varphi\sin\omega)\sin\nu + \sin\omega\sin\theta\sin\rho\cos\nu\right],\end{aligned} \tag{30i}$$

$$\mathrm{h}_2 = (\cos\omega\sin\varphi - \cos\theta\sin\varphi\sin\omega)\cos\nu - \sin\omega\sin\theta\sin\rho\sin\nu, \tag{30j}$$

$$\mathrm{h}_3 = \sin\omega\sin\theta\cos\rho, \tag{30k}$$

$$\begin{aligned}\mathrm{h}_6 &= (\cos\omega\cos\varphi - \cos\theta\sin\varphi\sin\omega)\sin\mu \\ &+ \cos\mu\left[(\cos\omega\sin\varphi + \cos\theta\cos\varphi\sin\omega)\sin\nu + \sin\omega\sin\theta\sin\rho\cos\nu\right].\end{aligned} \tag{30l}$$

In addition, the second-rank symmetric tensors **a**, **b**, **c**, **d**, **e** and **f** emerging in relations (29), in the expressions for the second-rank symmetric tensors **g**, **r**, **h**, **t**, **q** and **s**, are defined as:

$$\mathbf{a} = \mathbf{k}\otimes\mathbf{k}, \quad \mathbf{b} = \mathbf{l}\otimes\mathbf{l}, \quad \mathbf{c} = \mathbf{m}\otimes\mathbf{m}, \tag{31a}$$

$$\mathbf{d} = \frac{1}{\sqrt{2}}(\mathbf{l}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{l}), \tag{31b}$$

$$\mathbf{e} = \frac{1}{\sqrt{2}}(\mathbf{k}\otimes\mathbf{l} + \mathbf{l}\otimes\mathbf{k}), \tag{31c}$$

$$\mathbf{f} = \frac{1}{\sqrt{2}}(\mathbf{k}\otimes\mathbf{m} + \mathbf{m}\otimes\mathbf{k}), \tag{31d}$$

with **k**, **l** and **m** being the unit vectors of $\mathbf{R}^3$, associated with the 33, 22 and 11-directions of the Cartesian co-ordinate system.

Further, according to Eq. (17), it is easily noted that the components of tensor **S** are both symmetrical and real, thus, it follows that tensor **S** is self-adjoint or hermitean. Hence, the proof that all the eigenvalues λ_m and idempotent fourth-rank tensors $\mathbf{E_m}$, m=1,…,6 of the spectral decomposition of **S** are real, is obtained at once based on the hermitean nature of the compliance fourth-rank tensor **S**.

Furthermore, the seven angles ψ, θ, φ, ω, ρ, ν and μ, figuring in relations (30), are referred to as eigenangles and are defined as follows:

$$\cos 2\psi = \frac{(s_{44}-s_{55})/2}{\left[\left(\frac{s_{44}-s_{55}}{2}\right)^2 + s_{45}^2\right]^{1/2}}, \tag{32a}$$

$$\tan\theta = \frac{\left(\frac{Z_4^2+W_4^2+1}{Z_4^2+W_4^2+Q_4^2+1} - \frac{Q_1^2}{Z_1^2+W_1^2+Q_1^2+1}\right)^{1/2}}{\frac{Q_1}{(Z_1^2+W_1^2+Q_1^2+1)^{1/2}}}, \tag{32b}$$

$$\tan\varphi = \frac{\left[\frac{\left(\frac{Z_4^2+W_4^2+1}{Z_4^2+W_4^2+Q_4^2+1} - \frac{Q_1^2}{Z_1^2+W_1^2+Q_1^2+1}\right)(Z_4^2+1)}{\frac{(Z_4^2+W_4^2+1)^2}{(Z_4^2+W_4^2+Q_4^2+1)^2}} - \frac{Q_1^2 Q_4^2 W_4^2}{(Z_1^2+W_1^2+Q_1^2+1)(Z_4^2+W_4^2+1)^2}\right]^{1/2}}{\left[\frac{W_2}{(Z_2^2+W_2^2+Q_2^2+1)^{1/2}} + \frac{Q_1 Q_4 W_4}{(Z_1^2+W_1^2+Q_1^2+1)^{1/2}(Z_4^2+W_4^2+1)}\right]}, \tag{32c}$$

$$\tan\omega = \frac{\left(\dfrac{Z_4^2+W_4^2+1}{Z_4^2+W_4^2+Q_4^2+1} - \dfrac{Q_1^2}{Z_1^2+W_1^2+Q_1^2+1} - \dfrac{Q_2^2}{Z_2^2+W_2^2+Q_2^2+1}\right)^{1/2}}{\dfrac{Q_2}{\left(Z_2^2+W_2^2+Q_2^2+1\right)^{1/2}}}, \quad (32d)$$

$$\tan\rho = \frac{Q_4}{\left(Z_4^2+W_4^2+1\right)^{1/2}}, \qquad \tan\nu = \frac{W_4}{\left(Z_4^2+1\right)^{1/2}}, \qquad \tan\mu = Z_4, \quad (32e)$$

in which

$$Q_i = \left[\frac{F_i - \dfrac{C_i^2}{2A_i}}{\dfrac{1}{\sqrt{2}}\left(E_i - \dfrac{B_i C_i}{A_i}\right)}\right], \qquad W_i = \frac{B_i}{A_i}\left[\frac{F_i - \dfrac{C_i^2}{2A_i}}{\dfrac{1}{\sqrt{2}}\left(E_i - \dfrac{B_i C_i}{A_i}\right)}\right] - \frac{C_i}{\sqrt{2}\,A_i}, \quad (33a)$$

$$Z_i = -\frac{s_{12}}{(s_{11}-\lambda_i)}\left\{\frac{B_i}{A_i}\left[\frac{F_i - \dfrac{C_i^2}{2A_i}}{\dfrac{1}{\sqrt{2}}\left(E_i - \dfrac{B_i C_i}{A_i}\right)}\right] - \frac{C_i}{\sqrt{2}\,A_i}\right\}$$
$$+ \frac{s_{13}}{(s_{11}-\lambda_i)}\left[\frac{F_i - \dfrac{C_i^2}{2A_i}}{\dfrac{1}{\sqrt{2}}\left(E_i - \dfrac{B_i C_i}{A_i}\right)}\right] - \frac{s_{16}}{\sqrt{2}(s_{11}-\lambda_i)}, \quad (33b)$$

where

$$A_i = \left[(s_{22}-\lambda_i) - \frac{s_{12}^2}{(s_{11}-\lambda_i)}\right], \qquad B_i = \left[s_{23} - \frac{s_{12}s_{13}}{(s_{11}-\lambda_i)}\right], \quad (34a)$$

$$C_i = \left[s_{26} - \frac{s_{12}s_{16}}{(s_{11}-\lambda_i)}\right], \qquad D_i = \left[(s_{33}-\lambda_i) - \frac{s_{13}^2}{(s_{11}-\lambda_i)}\right], \quad (34b)$$

$$E_i = \left[s_{36} - \frac{s_{13} s_{16}}{(s_{11} - \lambda_i)} \right], \quad F_i = \left[\left(\frac{s_{66}}{2} - \lambda_i \right) - \frac{s_{16}^2}{2(s_{11} - \lambda_i)} \right], \qquad (34c)$$

and subscript i taking values 1, 2, 3 or 4.

In conclusion, it is postulated that the six eigenvalues λ_m, m=1,...,6, together with the set of eigenangles ψ, ρ, ν, μ, θ, ω and φ constitute the thirteen co-ordinate-invariant parameters necessary for the description of the elastic behaviour of crystalline media exhibiting the monoclinic symmetry.

The experimental values of the elastic compliance tensor components of some selected, representative inorganic crystals belonging to the monoclinic system, obtained from Huntington's book ″The elastic constants of crystals″ (1958), are assembled in Table 1. The multiple entries appearing in Table 1 for Di-Potassium Tartrate (DKT) and for Ethylene Diamine Tartrate (EDT) are due to substantial disagreement between different investigators, using usually reliable techniques. For references, Huntington's book together with the books by Nye (1957), Hearmon (1961), Simmons (1965) and Fedorov (1968) may be used as a source of experimental data of the determination of the elastic stiffnesses, and the elastic compliances of several anisotropic crystalline media, belonging to all crystal classes.

Before closing this part of the paper, the characteristic values and the eigenangles of the compliance tensor **S** are evaluated, using the numerical values corresponding to the representative inorganic crystals exhibiting the monoclinic symmetry. For instance, the experimental values of the compliance components for Ethylene Diamine Tartrate (EDT1), in units of $10^{-2}\times GPa^{-1}$, are as follows:

$$s_{11}=3.34, \quad s_{22}=3.65, \quad s_{33}=10.0, \quad s_{44}=19.2, \qquad (35a)$$

$$s_{55}=11.7, \quad s_{66}=19.1, \quad s_{12}=-0.3, \quad s_{13}=-3.0, \qquad (35b)$$

$$s_{23}=-1.8, \quad s_{16}=-1.7, \quad s_{26}=1.5, \quad s_{36}=-2.65, \quad s_{45}=0.38. \qquad (35c)$$

Crystals of the Monoclinic System														
Symbol	Material	Elastic compliances ($10^{-2} \times GPa^{-1}$)												
		s_{11}	s_{22}	s_{33}	s_{44}	s_{55}	s_{66}	s_{12}	s_{13}	s_{23}	s_{16}	s_{26}	s_{36}	s_{45}
$K_2(C_4H_4O_6)\cdot\frac{1}{2}H_2O$	Di-potassium tartrate (DKT1)	4.75	3.53	2.40	11.4	10.2	12.3	-1.74	-0.80	-0.62	-0.75	0.80	-1.40	-0.68
$K_2(C_4H_4O_6)\cdot\frac{1}{2}H_2O$	Di-potassium tartrate (DKT2)	3.87	3.37	2.26	10.4	8.2	11.9	-1.06	-1.64	-0.07	0.85	-0.54	-0.65	0.55
$C_2H_6N_2\cdot H_2C_4H_4O_6$	Ethylene diamine tartrate (EDT1)	3.34	3.65	10.0	19.2	11.7	19.1	-0.3	-3.0	-1.8	-1.7	1.5	-2.65	0.38
$C_2H_6N_2\cdot H_2C_4H_4O_6$	Ethylene diamine tartrate (EDT2)	3.9	3.6	9.8	18.7	17.2	17.4	0.2	-5.2	-1.8	-0.5	0.2	-2.5	-0.2
$Na_2S_2O_3$	Sodium Thiosulfate	5.02	15.6	6.74	22.3	32.7	21.2	-3.23	-0.62	-7.19	1.52	-18.2	11.0	10.0

Table 1. The values of the elastic compliances (in units of $10^{-2} \times GPa^{-1}$) for a series of crystalline media belonging to the monoclinic system.

Then, the set of quantities A, B, C, D, P, Q, R, P', Q', k and z_3, in units of 10^{-2} GPa^{-1}, appearing in expressions (19) for the eigenvalues, are equal to:

$$A=-0.265, \qquad B=0.023, \tag{36a}$$

$$C=-6.755\times10^{-4}, \qquad D=5.868\times10^{-6}, \tag{36b}$$

$$P=-3.821\times10^{-4}, \quad Q=-1.405\times10^{-5}, \qquad R=2.374\times10^{-6}, \tag{36c}$$

$$P'=-1.436\times10^{-5}, \quad Q'=-2.025\times10^{-8}, \qquad k=4.787\times10^{-6}, \tag{36d}$$

$$z_3=3.086\times10^{-3}. \tag{36e}$$

The eigenvalues λ_m, m=1,...,6 of the compliance tensor **S**, in units of GPa^{-1}, for the representative inorganic crystals of the monoclinic system are granted in Table 2. For Ethylene Diamine Tartrate (EDT1), the eigenvalues are evaluated to be:

$$\lambda_1=34.748\times10^{-3}, \qquad \lambda_2=14.846\times10^{-3}, \tag{37a}$$

$$\lambda_3=124.770\times10^{-3}, \qquad \lambda_4=91.299\times10^{-3}, \tag{37b}$$

$$\lambda_5=96.096\times10^{-3}, \qquad \lambda_6=58.404\times10^{-3}. \tag{37c}$$

Parameters A_i, B_i, C_i, D_i, E_i, Q_i, i=1,...,4 defining through relations (34) the eigenangles of the monoclinic medium compliance tensor **S**, in the case of Ethylene Diamine Tartrate (EDT1), are equal to the following values:

$$A_1=0.008, \qquad A_2=0.021, \qquad A_3=-0.088, \qquad A_4=-0.055, \tag{38a}$$

$$B_1=0.049, \qquad B_2=-0.023, \qquad B_3=-0.017, \qquad B_4=-0.017, \tag{38b}$$

$$C_1=0.053, \qquad C_2=0.012, \qquad C_3=0.016, \qquad C_4=0.016, \tag{38c}$$

Crystals of the Monoclinic System							
Symbol	Material	Eigenvalues (TPa^{-1})					
		λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
$K_2(C_4H_4O_6)\cdot\frac{1}{2}H_2O$	Di-potassium tartrate (DKT1)	30.575	12.857	69.183	55.685	58.534	49.466
$K_2(C_4H_4O_6)\cdot\frac{1}{2}H_2O$	Di-potassium tartrate (DKT1)	31.454	10.532	65.249	47.265	52.649	40.351
$C_2H_6N_2\cdot H_2C_4H_4O_6$	Ethylene diamine tartrate (EDT1)	34.748	14.846	124.77	91.299	96.096	58.404
$C_2H_6N_2\cdot H_2C_4H_4O_6$	Ethylene diamine tartrate (EDT2)	35.004	50.395	135.15	84.809	93.631	85.869
$Na_2S_2O_3$	Sodium Thiosulfate	13.716	-6.678	310.69	61.864	193.86	81.144

Table 2. The values of the six eigenvalues (in units of TPa^{-1}) of the compliance fourth-rank tensor for a series of crystalline media belonging to the monoclinic system.

$$D_1=0.723, \quad D_2=0.037, \quad D_3=-0.014, \quad D_4=0.024, \tag{38d}$$

$$E_1=0.168, \quad E_2=0.073, \quad E_3=-0.027, \quad E_4=0.007, \tag{38e}$$

$$F_1=0.352, \quad F_2=-0.054, \quad F_3=-0.021, \quad F_4=-0.018, \tag{38f}$$

$$Q_1=0.072, \quad Q_2=-2.40, \quad Q_3=1.526, \quad Q_4=-0.550, \tag{38g}$$

Thus, the eigenangles ψ, ρ, ν, μ, θ, ω and φ of the compliance tensor **S**, defined by relations (32), are evaluated to be:

$$\psi=2.89°, \quad \rho=26.26°, \quad \nu=-2.03°, \quad \mu=-153.78°, \tag{39a}$$

$$\theta=88.96°, \quad \omega=63.24°, \quad \varphi=154.61°, \tag{39b}$$

and the values of the eigenangles for the remaining, representative, monoclinic crystals are tabulated in Table 3.

4. REDUCTION OF THE SPECTRAL DECOMPOSITION FOR THE ORTHORHOMBIC, TETRAGONAL, HEXAGONAL AND CUBIC SYMMETRIES

In this section of the paper, the eigenvalues and idempotent fourth-rank tensors, which characterise the spectral decomposition of the compliance tensor **S**, are determined for the different anisotropic crystal systems. This is achieved by noting that relations (19) and (28) may be considered as general formulae, expressing the spectrum of tensor **S** for the monoclinic medium, and as such, they are also valid for crystalline or anisotropic media belonging to other crystal systems, whose compliance tensors **S**, however, share the same form (Fig. 1). Hence, the eigenvalues and idempotent tensors for the orthorhombic, tetragonal, hexagonal, cubic and isotropic symmetries are acquired, as special cases of the monoclinic symmetry. Nonetheless, this is not possible for the

Crystals of the Monoclinic System								
Symbol	Material	Eigenangles (deg)						
		ρ	ν	μ	θ	ω	φ	ψ
$K_2(C_4H_4O_6)\cdot\frac{1}{2}H_2O$	Di-potassium tartrate (DKT1)	15.89	22.64	-52.72	129.79	168.46	148.32	24.29
$K_2(C_4H_4O_6)\cdot\frac{1}{2}H_2O$	Di-potassium tartrate (DKT2)	-19.47	-21.58	49.73	-60.57	-17.70	170.90	166.72
$C_2H_6N_2\cdot H_2C_4H_4O_6$	Ethylene diamine tartrate (EDT1)	26.26	-2.03	-153.78	88.96	63.24	154.61	2.89
$C_2H_6N_2\cdot H_2C_4H_4O_6$	Ethylene diamine tartrate (EDT2)	-9.61	2.61	15.58	89.02	122.10	17.16	172.5
$Na_2S_2O_3$	Sodium Thiosulfate	27.67	16.99	260.49	36.30	131.51	174.41	58.73

Table 3. The values of the set of eigenangles (deg) of the compliance fourth-rank tensor for a series of crystalline media belonging to the monoclinic system.

trigonal crystal system, since its compliance tensor **S** is not of the form given by Eq. (17).

4.1. Orthorhombic Symmetry

As far as the orthorhombic symmetry is concerned, the components of the compliance fourth-rank tensor **S** for the orthotropic medium are expressed by relations (17) in terms of nine matrix components of the Voigt notation, with the remaining elements s_{16}, s_{26}, s_{36}, s_{45} vanishing, due to the symmetry operations within the orthorhombic system. On that account, the eigenvalues λ_m, m=1,…,6 of tensor **S** were acquired from relations (19), upon substitution of zero values for s_{16}, s_{26}, s_{36} and s_{45}:

$$\lambda_1 = -\left[-\frac{Q}{2}+\left(\frac{Q^2}{4}-k^3\right)^{1/2}\right]^{1/3}\frac{\left(1+i\sqrt{3}\right)}{2}+\frac{k\left(-1+i\sqrt{3}\right)}{2\left[-\frac{Q}{2}+\left(\frac{Q^2}{4}-k^3\right)^{1/2}\right]^{1/3}}-\frac{A}{3}, \quad (40a)$$

$$\lambda_2 = \left[-\frac{Q}{2}+\left(\frac{Q^2}{4}-k^3\right)^{1/2}\right]^{1/3}\frac{\left(-1+i\sqrt{3}\right)}{2}+\frac{k\left(-1-i\sqrt{3}\right)}{2\left[-\frac{Q}{2}+\left(\frac{Q^2}{4}-k^3\right)^{1/2}\right]^{1/3}}-\frac{A}{3}, \quad (40b)$$

$$\lambda_3 = \left[-\frac{Q}{2}+\left(\frac{Q^2}{4}-k^3\right)^{1/2}\right]^{1/3}+\frac{k}{\left[-\frac{Q}{2}+\left(\frac{Q^2}{4}-k^3\right)^{1/2}\right]^{1/3}}-\frac{A}{3}, \quad (40c)$$

$$\lambda_4 = \frac{s_{44}}{2}, \quad (40d)$$

$$\lambda_5 = \frac{s_{55}}{2}, \quad (40e)$$

$$\lambda_6 = \frac{s_{66}}{2}. \quad (40f)$$

Then, all the characteristic values of tensor **S** are of multiplicity one, and quantities Q and k appearing in expressions (40) are given by:

$$Q = \frac{2A^3}{27} - \frac{AB}{3} + C, \tag{41a}$$

$$k = \frac{A^2}{9} - \frac{B}{3}, \tag{41b}$$

in terms of quantities A, B and C:

$$A = -(s_{11} + s_{22} + s_{33}), \tag{42a}$$

$$B = (s_{11}s_{22} + s_{11}s_{33} + s_{22}s_{33}) - (s_{12}^2 + s_{13}^2 + s_{23}^2), \tag{42b}$$

$$C = s_{11}s_{23}^2 + s_{22}s_{13}^2 + s_{33}s_{12}^2 - (s_{11}s_{22}s_{33} + 2s_{12}s_{13}s_{23}). \tag{42c}$$

The corresponding six idempotent fourth-rank tensors $\mathbf{E_m}$, m=1,...,6 of the spectral decomposition of the compliance tensor **S** for the orthotropic medium were also established, by reduction of the defining relations (28) of the idempotent tensors for the monoclinic body, to be given by the following relations:

$$\mathbf{E_1} = E^1_{ijkl} = \mathbf{g} \otimes \mathbf{g} = g_{ij}g_{kl}, \tag{43a}$$

$$\mathbf{E_2} = E^2_{ijkl} = \mathbf{r} \otimes \mathbf{r} = r_{ij}r_{kl}, \tag{43b}$$

$$\mathbf{E_3} = E^3_{ijkl} = \mathbf{h} \otimes \mathbf{h} = h_{ij}h_{kl}, \tag{43c}$$

$$\mathbf{E_4} = E^4_{ijkl} = \frac{1}{2}\left(a_{ik}b_{jl} + a_{il}b_{jk} + a_{jk}b_{il} + a_{jl}b_{ik}\right), \tag{43d}$$

$$\mathbf{E_5} = E^5_{ijkl} = \frac{1}{2}\left(c_{ik}a_{jl} + c_{il}a_{jk} + c_{jk}a_{il} + c_{jl}a_{ik}\right), \tag{43e}$$

$$\mathbf{E_6} = E^6_{ijkl} = \frac{1}{2}\left(b_{ik}c_{jl} + b_{il}c_{jk} + b_{jk}c_{il} + b_{jl}c_{ik}\right). \tag{43f}$$

The definitions of tensors **g**, **r** and **h**, given by Eqs. (29), reduce to the following relations for the orthorhombic symmetry:

$$\mathbf{g} = \sin\theta\sin\varphi\mathbf{c} - \sin\theta\cos\varphi\mathbf{b} + \cos\theta\mathbf{a}, \tag{44a}$$

$$\begin{aligned}\mathbf{r} &= (-\sin\omega\cos\varphi - \cos\omega\sin\varphi\cos\theta)\mathbf{c} \\ &\quad + (-\sin\omega\sin\varphi + \cos\omega\cos\varphi\cos\theta)\mathbf{b} + \cos\omega\sin\theta\mathbf{a},\end{aligned} \tag{44b}$$

$$\begin{aligned}\mathbf{h} &= (\cos\omega\cos\varphi - \sin\omega\sin\varphi\cos\theta)\mathbf{c} \\ &\quad + (\cos\omega\sin\varphi + \sin\omega\cos\varphi\cos\theta)\mathbf{b} + \sin\omega\sin\theta\mathbf{a}.\end{aligned} \tag{44c}$$

Furthermore, regarding the set of eigenangles, whose definitions are given in (32), the corresponding values of eigenangles ρ, μ, ν and ψ for the orthotropic body become equal to zero, whereas, the definitions of the last three eigenangles φ, ω and θ are given by:

$$\tan\varphi = \frac{\left[s_{12}s_{23} + s_{13}(s_{22} - \lambda_1)\right]}{\left[s_{12}s_{13} + s_{23}(s_{11} - \lambda_1)\right]}, \tag{45a}$$

$$\tan\omega = \frac{\left[\dfrac{B_1^2 + \dfrac{C_1^2}{s_{11}^2}(s_{11} - \lambda_1)^2}{B_1^2 + \dfrac{(A_1^2 + C_1^2)}{s_{11}^2}(s_{11} - \lambda_1)^2} - \dfrac{\dfrac{A_2^2}{s_{11}^2}(s_{11} - \lambda_2)^2}{B_2^2 + \dfrac{(A_2^2 + C_2^2)}{s_{11}^2}(s_{11} - \lambda_2)^2}\right]^{1/2}}{\dfrac{\dfrac{A_2}{s_{11}}(s_{11} - \lambda_2)}{\left[B_2^2 + \dfrac{(A_2^2 + C_2^2)}{s_{11}^2}(s_{11} - \lambda_2)^2\right]^{1/2}}}, \tag{45b}$$

$$\tan\theta=\frac{\left[B_1^2+(s_{11}-\lambda_1)^2\frac{C_1^2}{s_{11}^2}\right]^{1/2}}{\left[(s_{11}-\lambda_1)\frac{A_1}{s_{11}}\right]}, \tag{45c}$$

$$\rho=0, \qquad \nu=0, \qquad \mu=0, \qquad \psi=0. \tag{45d}$$

in which quantities A_i, B_i and C_i , i=1,2, are defined by:

$$A_i=(s_{22}-\lambda_i)(s_{11}-\lambda_i)-s_{12}^2, \tag{46a}$$

$$B_i=s_{11}s_{12}s_{23}(s_{11}-\lambda_i)+s_{11}s_{13}(s_{11}-\lambda_i)(s_{22}-\lambda_i), \tag{46b}$$

$$C_i=s_{12}s_{13}+s_{23}(s_{11}-\lambda_i). \tag{46c}$$

Therefore, owing to the fact that the six, distinct eigenvalues λ_m, m=1,…,6, as well as the values of the three eigenangles θ, ω and φ remain invariant under orthogonal co-ordinate transformations, it is concluded that these parameters provide an invariant characterisation of the elastic properties of orthotropic media.

Now, in order to fix ideas, we shall try to evaluate, using the numerical values of the compliance components for Aragonite, the eigenvalues and the eigenangles of its compliance tensor **S**. The experimental values of the elastic compliance tensor components of Aragonite and several other, common, inorganic crystals belonging to the orthorhombic system, taken from Huntington's ″The elastic constants of crystals″ (1958), are listed in Table 4. The experimental values of the compliance components for Aragonite, in units of $10^{-2}\times GPa^{-1}$, are as follows:

$$s_{11}=0.695, \qquad s_{22}=1.32, \qquad s_{33}=1.22, \qquad s_{44}=2.43, \tag{47a}$$

$$s_{55}=3.90, \qquad s_{66}=2.34, \qquad s_{12}=-0.30, \qquad s_{13}=0.04, \qquad s_{23}=-0.24. \tag{47b}$$

Crystals of the Orthorhombic System										
Symbol	Material	Elastic compliances ($10^{-2}\times GPa^{-1}$)								
		s_{11}	s_{22}	s_{33}	s_{44}	s_{55}	s_{66}	s_{12}	s_{13}	s_{23}
$CaCO_3$	Aragonite	0.695	1.32	1.22	2.43	3.90	2.34	-0.30	0.04	-0.24
$BaSO_4$	Baryte	1.72	1.99	1.09	8.55	3.58	3.92	-0.99	-0.17	-0.30
$BaSO_4$	Baryte	1.84	1.74	1.10	8.33	3.48	3.65	-0.95	-0.27	-0.27
$SrSO_4$	Celestite	2.20	2.19	1.14	7.41	3.58	3.76	-1.39	-0.37	-0.40
$KB_5O_8\cdot 4H_2O$	Potassium pentaborate	2.32	7.36	9.83	6.1	21.5	17.5	-1.06	-0.61	-6.0
$NaK(C_4H_4O_6)$	Rochelle salt	5.24	3.54	3.37	7.47	31.1	10.2	-1.54	-1.03	-0.91
$NaK(C_4H_4O_6)$	Rochelle salt	5.18	3.49	3.34	7.98	32.8	10.1	-1.53	-2.11	-1.03
S	Sulfur	7.1	8.3	3.0	23.2	11.5	13.2	-3.6	-1.3	-1.5
$Al_2(F,OH)_2SiO_4$	Topaz	0.443	0.353	0.384	0.923	0.763	0.763	-0.138	-0.086	-0.006

Table 4. The values of the components (in units of $10^{-2}\times GPa^{-1}$) of the compliance fourth-rank tensor for a series of crystalline media belonging to the orthorhombic crystal system.

Parameters A, B, C, Q and k, in units of GPa^{-1}, figuring in expressions (41) and (42) for the eigenvalues, are equal to:

$$A=-0.033, \qquad B=3.230\times10^{-4}, \qquad C=-9.730\times10^{-7}, \tag{48a}$$

$$Q=1.579\times10^{-9}, \qquad k=8.730\times10^{-6}. \tag{48b}$$

Hence, the characteristic values λ_m, m=1,…,6 of the compliance tensor **S** for Aragonite, in units of GPa^{-1}, are evaluated to be:

$$\lambda_1=10.723\times10^{-3}, \qquad \lambda_2=5.696\times10^{-3}, \tag{49a}$$

$$\lambda_3=15.931\times10^{-3}, \qquad \lambda_4=12.150\times10^{-3}, \tag{49b}$$

$$\lambda_5=19.500\times10^{-3}, \qquad \lambda_6=11.700\times10^{-3}, \tag{49c}$$

and the eigenvalues of the remaining representative inorganic crystals of the orthorhombic system are given in Table 5.

Moreover, parameters A_i, B_i, C_i, i=1,2, defining by means of relations (46) the eigenangles of the orthorhombic crystal, are equal to the following values in the case of Aragonite:

$$A_1=-1.835\times10^{-5}, \qquad B_1=5.486\times10^{-12}, \qquad C_1=7.855\times10^{-6}, \tag{50a}$$

$$A_2=4.100\times10^{-7}, \qquad B_2=1.570\times10^{-11}, \qquad C_2=-4.210\times10^{-6}. \tag{50b}$$

Accordingly, the eigenangles θ, ω and φ, defined by relations (45), are found to be:

$$\theta=122.684°, \quad \omega=200°, \quad \varphi=83.965°. \tag{51}$$

At last, Table 5 also displays the values of the eigenangles of the compliance tensor **S** for the whole group of representative crystals of the orthorhombic system.

Crystals of the Orthorhombic System										
Symbol	Material	Eigenvalues (TPa^{-1})						Eigenangles (deg)		
		λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	θ	φ	ω
$CaCO_3$	Aragonite	10.723	5.696	15.931	12.15	19.5	11.7	122.684	200	83.965
$BaSO_4$	Baryte	13.131	6.252	28.618	42.75	17.9	19.6	86.216	40.499	-54.595
$BaSO_4$	Baryte	13.727	5.659	27.414	41.65	17.4	18.25	90.344	46.578	-54.446
$SrSO_4$	Celestite	15.421	4.028	35.852	37.05	17.9	18.8	89.504	45.006	-53.545
$KB_5O_8 \cdot 4H_2O$	Potassium pentaborate	36.023	11.838	147.239	30.5	107.5	87.5	39.285	1.481	-47.942
$NaK(C_4H_4O_6)$	Rochelle salt	42.978	16.275	62.247	37.35	155.5	51.0	79.906	243.161	51.866
$NaK(C_4H_4O_6)$	Rochelle salt	44.229	8.329	67.543	39.9	164.0	50.5	63.244	253.169	45.511
S	Sulfur	55.498	14.887	113.615	116.0	57.5	66.0	87.782	39.672	-37.408
$Al_2(F,OH)_2SiO_4$	Topaz	3.810	2.308	5.682	3.765	4.615	3.815	69.297	237.835	66.801

Table 5. The values of the eigenvalues (in units of TPa^{-1}), and the set of eigenangles (deg) of the compliance fourth-rank tensor for a series of crystalline media belonging to the orthorhombic crystal system.

4.2. Tetragonal Symmetry

Next, concerning the class of tetragonal symmetry, the components of the tetragonal compliance tensor **S** were expressed with respect to the components of the 6×6 matrix **s** of the Voigt notation by relations (17), taking into account the vanishing of s_{36}, s_{45}, and the repeated elements $s_{11}=s_{22}$, $s_{13}=s_{31}=s_{23}=s_{32}$, $s_{44}=s_{55}$, $s_{16}=-s_{26}$, owing to the tetragonal symmetry operations. Accordingly, the eigenvalues $\lambda_{\mathbf{m}}$, m=1,...,6 of tensor **S** were derived from relations (19) to be:

$$\lambda_1=\frac{\left(s_{11}-s_{12}\right)}{2}+\frac{s_{66}}{4}+\left\{\left[\frac{\left(s_{11}-s_{12}\right)}{2}-\frac{s_{66}}{4}\right]^2+s_{16}^2\right\}^{1/2}, \tag{52a}$$

$$\lambda_2=\frac{\left(s_{11}+s_{12}\right)}{2}+\frac{s_{33}}{2}+\left\{\left[\frac{\left(s_{11}+s_{12}\right)}{2}-\frac{s_{33}}{2}\right]^2+2s_{13}^2\right\}^{1/2}, \tag{52b}$$

$$\lambda_3=\frac{\left(s_{11}+s_{12}\right)}{2}+\frac{s_{33}}{2}-\left\{\left[\frac{\left(s_{11}+s_{12}\right)}{2}-\frac{s_{33}}{2}\right]^2+2s_{13}^2\right\}^{1/2}, \tag{52c}$$

$$\lambda_4=\lambda_5=\frac{s_{44}}{2}, \tag{52d}$$

$$\lambda_6=\frac{\left(s_{11}-s_{12}\right)}{2}+\frac{s_{66}}{4}-\left\{\left[\frac{\left(s_{11}-s_{12}\right)}{2}-\frac{s_{66}}{4}\right]^2+s_{16}^2\right\}^{1/2}. \tag{52e}$$

Obviously, there are four eigenvalues of multiplicity one, i.e. λ_1, λ_2, λ_3, λ_6, and one eigenvalue of multiplicity two, namely $\lambda_4=\lambda_5$.

Consequently, the associated six idempotent fourth-rank tensors $\mathbf{E}_{\mathbf{m}}$, m=1,...,6 of the spectral decomposition of the compliance tensor **S** for the tetragonal medium were obtained from Eqs. (28), as follows:

$$\mathbf{E_1} = E^1_{ijkl} = \mathbf{g} \otimes \mathbf{g} = g_{ij} g_{kl}, \tag{53a}$$

$$\mathbf{E_2} = E^2_{ijkl} = \mathbf{r} \otimes \mathbf{r} = r_{ij} r_{kl}, \tag{53b}$$

$$\mathbf{E_3} = E^3_{ijkl} = \mathbf{h} \otimes \mathbf{h} = h_{ij} h_{kl}, \tag{53c}$$

$$\mathbf{E_4} = E^4_{ijkl} = \frac{1}{2}\left(a_{ik} b_{jl} + a_{il} b_{jk} + a_{jk} b_{il} + a_{jl} b_{ik}\right), \tag{53d}$$

$$\mathbf{E_5} = E^5_{ijkl} = \frac{1}{2}\left(c_{ik} a_{jl} + c_{il} a_{jk} + c_{jk} a_{il} + c_{jl} a_{ik}\right), \tag{53e}$$

$$\mathbf{E_6} = E^6_{ijkl} = \mathbf{s} \otimes \mathbf{s} = s_{ij} s_{kl}. \tag{53f}$$

Tensors **g**, **r**, **h** and **s**, originally defined by Eqs. (29), are now given by the ensuing relations:

$$\mathbf{g} = \frac{1}{\sqrt{2}} \cos\rho(-\mathbf{b} + \mathbf{c}) + \sqrt{2}\sin\rho\,\mathbf{d}, \tag{54a}$$

$$\mathbf{r} = -\frac{1}{\sqrt{2}} \sin\omega(\mathbf{b} + \mathbf{c}) + \cos\omega\,\mathbf{a}, \tag{54b}$$

$$\mathbf{h} = \frac{1}{\sqrt{2}} \cos\omega(\mathbf{b} + \mathbf{c}) + \sin\omega\,\mathbf{a}, \tag{54c}$$

$$\mathbf{s} = -\frac{1}{\sqrt{2}} \sin\rho(-\mathbf{b} + \mathbf{c}) + \sqrt{2}\cos\rho\,\mathbf{d}, \tag{54d}$$

and the seven eigenangles ψ, μ, ν, θ, φ, ω and ρ were found to be expressed by:

$$\psi=0, \qquad \mu=0, \qquad \nu=0, \tag{55a}$$

$$\theta=\pi/2, \qquad \varphi=\pi/4, \tag{55b}$$

$$\tan 2\omega=\frac{2\sqrt{2}\,s_{13}}{\left[\left(s_{11}-s_{33}\right)+s_{12}\right]}, \qquad \tan 2\rho=\frac{2s_{16}}{\left[\left(s_{11}-\frac{s_{66}}{2}\right)-s_{12}\right]}. \tag{55c}$$

It is, thus, deduced that an invariant description of the elasticity of crystalline, or other anisotropic media of the tetragonal system is offered by means of the five, different characteristic values λ_m, m=1,…,4 and 6, of the compliance tensor **S**, together with the values of the two eigenangles ω and ρ.

In the following, the eigenvalues and the eigenangles of tensor **S** are computed, using numerical data of the compliance components, in units of $10^{-2}\times GPa^{-1}$, for Barium Titanate, a natural crystal of the tetragonal system. The experimental values are expressed in Table 6 as:

$$s_{11}=0.725, \qquad s_{33}=1.08, \qquad s_{44}=1.24, \tag{56a}$$

$$s_{66}=0.884, \qquad s_{12}=-0.315, \qquad s_{13}=-0.326. \tag{56b}$$

Carrying out the calculations implied by relations (52), the eigenvalues λ_m, m=1,…,6 of the compliance tensor **S**, in units of GPa^{-1}, were computed equal to:

$$\lambda_1=10.400\times10^{-3}, \qquad \lambda_2=13.139\times10^{-3}, \tag{57a}$$

$$\lambda_3=1.751\times10^{-3}, \qquad \lambda_4=6.200\times10^{-3}, \tag{57b}$$

$$\lambda_5=6.200\times10^{-3}, \qquad \lambda_6=4.420\times10^{-3}. \tag{57c}$$

Next, the eigenangles ω and ρ, defined according to relations (55), are computed to be:

$$\omega=26.998°, \qquad \rho=0. \tag{58}$$

Crystals of the Tetragonal System															
Symbol	Material	Elastic compliances ($10^{-2}\times GPa^{-1}$)						Eigenvalues (TPa^{-1})					Eigenangles (deg)	
		s_{11}	s_{33}	s_{44}	s_{66}	s_{12}	s_{13}	λ_1	λ_2	λ_3	λ_4	λ_6	ω	ρ
$(NH_4)H_2PO_4$	Ammonium di-hydrogen phosphate	1.81	4.35	11.53	14.55	0.19	-1.18	72.8	52.2	11.3	57.7	16.2	27.4	0
$(NH_4)H_2PO_4$	Ammonium di-hydrogen phosphate	2.0	4.57	11.7	16.9	0.17	-1.29	84.5	55.5	11.9	58.5	18.3	28.3	0
$(NH_4)H_2PO_4$	Ammonium di-hydrogen phosphate	1.75	4.35	11.4	16.3	0.75	-1.1	81.5	52.3	16.2	57.0	57.0	29.6	0
$(NH_4)D_2PO_4$	Ammonium di-hydrogen phosphate (deuterated)	1.9	4.4	11.0	16.4	0.2	-1.1	82.0	51.9	13.2	55.0	17.0	29.8	0
$BaTiO_3$	Barium titanate	0.725	1.08	1.24	0.884	-0.315	-0.326	10.4	13.1	1.8	6.2	4.4	27.0	0

Table 6. The values of the elastic compliances (in units of $10^{-2}\times GPa^{-1}$), the eigenvalues (in units of TPa^{-1}) and the eigenangles (deg) of the compliance fourth-rank tensor for a series of media belonging to the tetragonal crystal system.

		s_{11}	s_{33}	s_{44}	s_{66}	s_{12}	s_{13}	λ_1	λ_2	λ_3	λ_4	λ_6	ω	ρ
$BaTiO_3$	Barium titanate	0.805	1.57	1.84	0.884	-0.235	-0.524	10.4	19.6	1.8	9.2	4.4	28.0	0
In	Indium	14.94	18.70	15.27	8.20	-5.06	-9.02	200.0	278.0	7.9	76.4	41.0	35.5	0
KH_2PO_4	Potassium di-hydrogen phosphate	1.48	1.95	7.87	15.92	0.170	-0.379	79.6	23.6	12.4	39.4	13.1	37.2	0
KH_2PO_4	Potassium di-hydrogen phosphate	1.78	2.03	8.1	16.4	-0.38	-0.71	82.0	27.7	6.3	405.0	21.6	36.3	0
KH_2PO_4	Potassium di-hydrogen phosphate	1.6	1.9	7.4	15.8	-0.2	-0.55	79.0	24.7	8.3	37.0	18.0	36.1	0
Sn	Tin	1.63	1.41	4.54	4.42	-0.36	-0.41	22.1	19.2	7.6	22.7	19.9	41.6	0
Sn	Tin	1.85	1.18	5.70	13.5	-0.99	-0.25	67.5	14.1	6.3	28.5	28.4	32.8	0
$ZrSiO_4$	Zircon	1.39	2.21	7.2	6.2	-0.16	-0.14	31.0	22.5	11.9	36.0	15.5	11.0	0

Table 6. Cont^n

Furthermore, the compliance tensor components, as well as the characteristic values and the eigenangles of Barium Titanate and other, tetragonal crystalline media (Huntington, 1958) are assembled in Table 6.

4.3. Hexagonal Symmetry

Considering in the following the class of hexagonal media, the components of the compliance fourth-rank tensor **S** are defined by relations (17), with respect to the matrix components of the Voigt notation, with $s_{16}=s_{26}=s_{36}=s_{45}=0$, and $s_{11}=s_{22}$, $s_{13}=s_{31}=s_{23}=s_{32}$, $s_{44}=s_{55}$, s_{11}-$s_{12}=s_{66}/2$. The eigenvalues λ_m, m=1,...,6 of the hexagonal tensor **S** were thus obtained, from the defining relations (19) for the monoclinic medium, as follows:

$$\lambda_1=\lambda_6=s_{11}-s_{12}=\frac{s_{66}}{2}, \tag{59a}$$

$$\lambda_2=\frac{(s_{11}+s_{12})}{2}+\frac{s_{33}}{2}+\left\{\left[\frac{(s_{11}+s_{12})}{2}-\frac{s_{33}}{2}\right]^2+2s_{13}^2\right\}^{1/2}, \tag{59b}$$

$$\lambda_3=\frac{(s_{11}+s_{12})}{2}+\frac{s_{33}}{2}-\left\{\left[\frac{(s_{11}+s_{12})}{2}-\frac{s_{33}}{2}\right]^2+2s_{13}^2\right\}^{1/2}, \tag{59c}$$

$$\lambda_4=\lambda_5=\frac{s_{44}}{2}. \tag{59d}$$

Then, two eigenvalues of tensor **S** are of multiplicity two, namely $\lambda_1=\lambda_6$ and $\lambda_4=\lambda_5$, and two eigenvalues are of multiplicity one, i.e. λ_2 and λ_3.

In addition, the corresponding idempotent fourth-rank tensors $\mathbf{E}_m$, m=1,…,6 were determined from Eqs. (28) to be:

$$\mathbf{E}_1=E^1_{ijkl}=\frac{1}{2}(b_{ik}b_{jl}+b_{il}b_{jk}-b_{ij}b_{kl}+c_{ik}c_{jl} \\ +c_{il}c_{jk}-c_{ij}c_{kl}-b_{ij}c_{kl}-b_{kl}c_{ij}), \tag{60a}$$

$$\mathbf{E}_2 = E^2_{ijkl} = \mathbf{r} \otimes \mathbf{r} = r_{ij} r_{kl}, \tag{60b}$$

$$\mathbf{E}_3 = E^3_{ijkl} = \mathbf{h} \otimes \mathbf{h} = h_{ij} h_{kl}, \tag{60c}$$

$$\mathbf{E}_4 = E^4_{ijkl} = \frac{1}{2}\left(a_{ik} b_{jl} + a_{il} b_{jk} + a_{jk} b_{il} + a_{jl} b_{ik}\right), \tag{60d}$$

$$\mathbf{E}_5 = E^5_{ijkl} = \frac{1}{2}\left(c_{ik} a_{jl} + c_{il} a_{jk} + c_{jk} a_{il} + c_{jl} a_{ik}\right), \tag{60e}$$

$$\mathbf{E}_6 = E^6_{ijkl} = \frac{1}{2}\left(b_{ik} c_{jl} + b_{il} c_{jk} + b_{jk} c_{il} + b_{jl} c_{ik}\right). \tag{60f}$$

Tensors **r** and **h**, defined by Eqs. (29) for the monoclinic system, reduce to the following relations for hexagonal symmetry:

$$\mathbf{r} = -\frac{1}{\sqrt{2}} \sin\omega(\mathbf{c} + \mathbf{b}) + \cos\omega\,\mathbf{a}, \tag{61a}$$

$$\mathbf{h} = \frac{1}{\sqrt{2}} \cos\omega(\mathbf{c} + \mathbf{b}) + \sin\omega\,\mathbf{a}, \tag{61b}$$

where the values of the seven eigenangles ψ, ρ, μ, ν, θ, φ and ω, defined originally for the monoclinic crystal system according to relations (32), are expressed for the hexagonal symmetry by:

$$\psi=0, \qquad \rho=0, \qquad \mu=0, \qquad \nu=0, \tag{62a}$$

$$\theta=\pi/2, \qquad \varphi=\pi/4, \tag{62b}$$

$$\tan 2\omega = \frac{2\sqrt{2}\, s_{13}}{\left[(s_{11} - s_{33}) + s_{12}\right]}. \tag{62c}$$

Therefore, it is inferred that the eigenvalues λ_m, m=1,…,4 of the hexagonal material compliance tensor **S**, and the value of the eigenangle ω are invariant descriptors of the elastic features of hexagonal anisotropic media.

Next, Cobalt is considered as an example of a hexagonal medium, whose compliance components, together with those of other hexagonal crystalline media, are listed in Table 7 (Huntington, 1958). The experimental values for Cobalt, in units of $10^{-2}\times GPa^{-1}$, are as follows:

$$s_{11}=0.472, \qquad s_{33}=0.319, \qquad s_{44}=1.324, \tag{63a}$$

$$s_{12}=-0.231, \qquad s_{13}=-0.069. \tag{63b}$$

Consequently, the characteristic values λ_m, m=1,…,6 of the compliance tensor **S**, in units of GPa^{-1}, were evaluated via Eqs. (59) to be equal to:

$$\lambda_1=7.030\times10^{-3}, \qquad \lambda_2=3.851\times10^{-3}, \tag{64a}$$

$$\lambda_3=1.749\times10^{-3}, \qquad \lambda_4=6.620\times10^{-3}, \tag{64b}$$

$$\lambda_5=6.620\times10^{-3}, \qquad \lambda_6=7.030\times10^{-3}, \tag{64c}$$

and the eigenvalues of the remaining inorganic crystals, displaying the hexagonal symmetry, are tabulated in Table 7. Finally, the value of the eigenangle ω, defined according to relation (62c), is given by:

$$\omega=34.108°, \tag{65}$$

and the values of the eigenangles of the compliance tensor **S** for the remaining representative crystals of the hexagonal system are found in Table 7.

4.4. Cubic-Isotropic Symmetries

Taking into account next the class of cubic media, the components of the compliance fourth-rank tensor **S** are offered in terms of relations (17), with respect to only three components of the matrix notation, due to the fact that cubic symmetry imposes the ensuing restrictions upon the matrix elements:

Crystals of the Hexagonal System											
Symbol	Material	Elastic compliances ($10^{-2}\times GPa^{-1}$)					Eigenvalues (TPa^{-1})				Eigenangle (deg)
		s_{11}	s_{33}	s_{44}	s_{12}	s_{13}	λ_1	λ_2	λ_3	λ_4	ω
$BaTiO_3$	Barium titanate ceramic A	0.818	0.676	1.83	-0.298	-0.195	11.2	8.9	3.1	9.2	37.1
$BaTiO_3$	Barium titanate ceramic A	0.855	0.893	2.33	-0.261	-0.285	11.2	11.2	3.1	11.7	34.8
$BaTiO_3$ +5%$CaTiO_3$	5% calcium barium titanate ceramic	0.805	0.842	2.11	-0.245	-0.265	10.5	11.0	3.0	10.6	34.7
Cd	Cadmium	1.23	3.55	5.40	-0.15	-0.93	13.8	41.1	5.1	27.0	23.4
Cd	Cadmium	1.29	3.69	6.40	-0.15	-0.93	14.4	42.5	5.8	32.0	22.9
Co	Cobalt	0.472	0.319	1.324	-0.231	-0.069	7.0	3.9	1.8	6.6	34.1
H_2O	Ice	10.4	8.5	31.4	-4.3	-2.4	147.0	109.0	37.0	157.0	35.3
Mg	Magnesium	2.20	1.97	6.1	-0.785	-0.50	29.9	24.5	9.3	30.5	34.3
SiO_2	β-quartz	0.941	1.062	2.773	-0.060	-0.262	10.0	13.5	5.9	13.9	38.1

Table 7. The values of the elastic compliances (in units of $10^{-2}\times GPa^{-1}$), as well as the eigenvalues (in units of TPa^{-1}) and the eigenangles (deg) of the compliance fourth-rank tensor for a series of crystalline media belonging to the hexagonal crystal system.

$s_{16}=s_{26}=s_{36}=s_{45}=0$, and $s_{11}=s_{22}=s_{33}$, $s_{12}=s_{21}=s_{13}=s_{31}=s_{23}=s_{32}$, $s_{44}=s_{55}=s_{66}$. As a result, the eigenvalues λ_m, m=1,...,6 of the cubic compliance tensor **S** were obtained from relations (19) as follows:

$$\lambda_1=\lambda_2=s_{11}-s_{12}, \tag{66a}$$

$$\lambda_3=s_{11}+2s_{12}, \tag{66b}$$

$$\lambda_4=\lambda_5=\lambda_6=\frac{s_{44}}{2}. \tag{66c}$$

Obviously, one eigenvalue is of multiplicity two, namely $\lambda_1=\lambda_2$, one eigenvalue is of multiplicity three i.e. $\lambda_4=\lambda_5=\lambda_6$, and one eigenvalue is of multiplicity one.

Furthermore, the corresponding idempotent fourth-rank tensors $\mathbf{E}_m$, m=1,…,6 were derived as follows, from the defining Eqs. (28) for the monoclinic symmetry:

$$\mathbf{E_1}=E^1_{ijkl}=\frac{1}{2}(b_{ik}b_{jl}+b_{il}b_{jk}-b_{ij}b_{kl}+c_{ik}c_{jl}+c_{il}c_{jk}-c_{ij}c_{kl}-b_{ij}c_{kl}-b_{kl}c_{ij}), \tag{67a}$$

$$\mathbf{E_2}=E^2_{ijkl}=\frac{1}{6}(4a_{ij}a_{kl}+b_{ij}b_{kl}+c_{ij}c_{kl}-2a_{ij}b_{kl}-2b_{ij}a_{kl}-2a_{ij}c_{kl}-2c_{ij}a_{kl}+b_{ij}c_{kl}+c_{ij}b_{kl}), \tag{67b}$$

$$\mathbf{E_3}=E^3_{ijkl}=\frac{1}{3}(\mathbf{1}\otimes\mathbf{1})=\frac{1}{3}\delta_{ij}\delta_{kl}, \tag{67c}$$

$$\mathbf{E_4}=E^4_{ijkl}=\frac{1}{2}\left(a_{ik}b_{jl}+a_{il}b_{jk}+a_{jk}b_{il}+a_{jl}b_{ik}\right), \tag{67d}$$

$$\mathbf{E_5}=E^5_{ijkl}=\frac{1}{2}\left(c_{ik}a_{jl}+c_{il}a_{jk}+c_{jk}a_{il}+c_{jl}a_{ik}\right), \tag{67e}$$

$$\mathbf{E}_6 = E^6_{ijkl} = \frac{1}{2}\left(b_{ik}c_{jl} + b_{il}c_{jk} + b_{jk}c_{il} + b_{jl}c_{ik}\right). \quad (67f)$$

The seven eigenangles ψ, ρ, μ, ν, θ, φ and ω of the cubic symmetry were also found, according to their defining relations (32), to be expressed by:

$$\psi=0, \qquad \rho=0, \qquad \mu=0, \qquad \nu=0, \quad (68a)$$

$$\theta=\pi/2, \qquad \varphi=\pi/4, \qquad \omega=35.26°. \quad (68b)$$

In conclusion, the three distinct eigenvalues λ_m, m=1,3,4, of the compliance fourth-rank tensor **S** constitute co-ordinate-invariant parameters, characterising the elastic properties of cubic media.

As a numerical example, the eigenvalues and the eigenangles are computed employing the experimental data of the compliance components for Silver, a characteristic cubic system element, in units of $10^{-2}\times GPa^{-1}$:

$$s_{11}=2.29, \qquad s_{44}=2.17, \qquad s_{12}=-0.983. \quad (69)$$

Hence, the eigenvalues λ_m, m=1,…,6 of the compliance tensor **S**, in units of GPa^{-1}, were calculated according to relation (66):

$$\lambda_1=10.850\times10^{-3}, \qquad \lambda_2=3.240\times10^{-3}, \quad (70a)$$

$$\lambda_3=3.273\times10^{-3}, \qquad \lambda_4=10.850\times10^{-3}, \quad (70b)$$

$$\lambda_5=10.850\times10^{-3}, \qquad \lambda_6=10.850\times10^{-3}. \quad (70c)$$

In addition, the compliance tensor components (Huntington, 1958), and the eigenvalues of several other cubic crystalline elements and compounds are tabulated in Tables 8 and 9.

Finally, in the trivial case of isotropic elastic media, there are only two distinct matrix components in the Voigt notation, since it is valid that: $s_{11}=s_{22}=s_{33}$, $s_{44}=s_{55}=s_{66}=2(s_{11}-s_{12})$ and $s_{12}=s_{21}=s_{13}=s_{31}=s_{32}=s_{23}$. Hence, it was found

Cubic System Elements

Symbol	Material	Elastic compliances ($10^{-2}\times GPa^{-1}$)			Eigenvalues (TPa^{-1})			Symbol	Material	Elastic compliances ($10^{-2}\times GPa^{-1}$)			Eigenvalues (TPa^{-1})		
		s_{11}	s_{44}	s_{12}	λ_1	λ_2	λ_3			s_{11}	s_{44}	s_{12}	λ_1	λ_2	λ_3
Ag	Silver	2.29	2.17	-0.98	10.9	3.2	3.27	K	Potassium	82.3	38.0	-37.0	190.0	83.0	1193.0
Ag	Silver	2.32	2.29	-0.99	11.5	3.4	33.1	Li	Lithium	29.5	9.26	-13.5	46.3	25.0	430.0
Al	Aluminum	1.57	3.51	-0.57	17.6	4.3	21.4	Mo	Molybdenum	0.28	0.91	-0.078	4.6	1.2	3.6
Al	Aluminum	1.74	3.51	-0.66	17.6	4.2	24.0	Na	Sodium	48.61	17.1	-21.0	85.5	66.1	696.1
Au	Gold	2.33	2.38	-1.07	11.9	2.0	34.0	Na	Sodium	42.0	16.2	-19.0	81.0	40.0	610.0
C	Diamond	0.10	0.17	-0.01	0.9	0.8	1.1	Ni	Nickel	0.73	0.80	-0.274	4.0	1.9	10.1
C	Diamond	0.14	0.23	-0.04	1.2	0.6	1.8	Pb	Lead	9.28	6.94	-4.24	34.7	8.0	135.2
C	Diamond	0.11	0.23	-0.02	1.2	0.6	1.3	Si	Silicon	0.77	1.26	-0.214	6.3	3.4	9.8
Cu	Copper	1.50	1.33	-0.63	6.6	2.4	21.3	Th	Thorium	2.72	2.09	-1.07	10.5	5.8	37.9
Ge	Germanium	0.98	1.49	-0.27	7.5	4.5	12.4	W	Tungsten	0.26	0.66	-0.073	3.3	1.1	3.3

Table 8. The values of the elastic compliances (in units of $10^{-2}\times GPa^{-1}$), the eigenvalues (in units of TPa^{-1}) and the eigenangles (deg) of the compliance fourth-rank tensor for a series of cubic system elements.

Cubic System Compounds																
Symbol	Material	Elastic compliances ($10^{-2}\times GPa^{-1}$)			Eigenvalues (TPa^{-1})			Symbol	Material	Elastic compliances ($10^{-2}\times GPa^{-1}$)			Eigenvalues (TPa^{-1})		
		s_{11}	s_{44}	s_{12}	λ_1	λ_2	λ_3			s_{11}	s_{44}	s_{12}	λ_1	λ_2	λ_3
AgBr	Silver bromide	3.13	13.9	-1.17	69.5	7.9	43.0	LiF	Lithium fluoride	1.14	1.59	-0.31	8.0	-608	321.4
AgCl	Silver chloride	3.04	16.0	-1.14	80.0	7.6	41.8	LiF	Lithium fluoride	1.06	1.59	-0.29	8.0	-559	295.6
$Ba(NO_3)_2$	Barium nitrate	1.99	8.26	-0.48	41.3	10.3	24.7	MgO	Magnesium oxide	0.41	0.676	-0.1	3.4	-186	99.1
CaF_2	Fluorspar	0.71	2.88	-0.17	14.4	3.8	8.8	NaBr	Sodium bromide	2.87	10.3	-0.58	51.5	-1131	608.7
Cr_2FeO_4	Chromite	0.43	0.86	-0.13	4.3	1.7	5.6	$NaBrO_3$	Sodium bromate	2.04	6.57	-0.48	32.9	-940	500.4
CuZn	β-brass	3.53	1.22	-1.62	6.1	2.9	51.5	$NaBrO_3$	Sodium bromate	2.24	6.67	-0.58	33.4	-1138	602.4
CuZn	β-brass	4.11	1.34	-1.90	6.7	3.1	60.1	NaCl	Rocksalt	2.29	7.94	-0.465	39.7	-987	487.9

Table 9. The values of the elastic compliances (in units of $10^{-2}\times GPa^{-1}$), as well as the eigenvalues (in units of TPa^{1}) and the eigenangles (deg) of the compliance fourth-rank tensor for a series of cubic system compounds.

		s_{11}	s_{44}	s_{12}	λ_1	λ_2	λ_3			s_{11}	s_{44}	s_{12}	λ_1	λ_2	λ_3
Cu_3Au	Copper-gold	1.34	1.51	-0.565	7.6	2.1	19.1	$NaClO_3$	Sodium chlorate	2.29	8.54	-0.505	42.7	-987	527.9
Fe_3O_4	Magnetite	0.47	1.03	-0.131	5.2	2.1	6.0	$NH_4Al\ SO_4\ 12H_2O$	Ammonium alum	5.35	12.5	-1.59	62.5	-3127	1643.5
FeS_2	Pyrites	0.29	0.96	0.039	4.8	3.6	2.5	NH_4Br	Ammonium bromide	3.62	18.9	-0.60	94.5	24.2	42.2
GaAs	Gallium arsenide	12.6	18.6	-4.234	93.0	41.7	168	NH_4Cl	Ammonium chloride	2.72	14.7	-0.42	73.5	18.8	31.4
GaSb	Gallium antimonide	1.58	2.31	-0.496	11.6	5.9	20.8	PbS	Galena	0.864	4.03	-0.164	20.2	5.4	10.3
InSb	Indium antimonide	2.42	3.31	-0.855	16.6	7.1	32.8	PbS	Galena	1.23	4.0	-0.33	20.0	5.7	15.6
$Kal(SO_4)_2.12HO$	Potassium alum	5.25	11.9	-1.56	59.5	21.3	68.1	TlBr	Thallium bromide	33.9	13.2	-0.95	66.0	320	348.5
KBr	Potassium bromide	3.04	19.8	-0.44	99.0	21.7	34.8	TlCl	Thallium chloride	3.16	13.2	-0.87	66.0	14.2	40.3
KCl	Sylvine	2.62	16.0	-0.35	80.0	19.2	29.7	ZnS	Zinc sulfide	1.94	2.29	-0.73	11.5	4.8	26.7
KI	Potassium iodide	3.92	23.8	-0.54	119.0	28.4	44.6	ZnS	Zinc sulfide	2.0	2.43	-0.802	12.2	4.0	28.0

Table 9. Cont^{n}.

that one eigenvalue of the isotropic compliance tensor **S** is of multiplicity five, i.e. $\lambda_1=\lambda_2=\lambda_4=\lambda_5=\lambda_6$, and one eigenvalue λ_3 is of multiplicity one:

$$\lambda_1=\lambda_2=\lambda_4=\lambda_5=\lambda_6= s_{44}/2, \tag{71a}$$

$$\lambda_3= s_{11} + 2s_{12}, \tag{71b}$$

and the corresponding idempotent fourth-rank tensors $\mathbf{E_m}$, m=1,…,6, as well as the values of the eigenangles are exactly the same as those of cubic symmetry.

5. STATES OF STRESS AND STRAIN FOR THE MONOCLINIC MEDIUM

The action of the idempotent fourth-rank tensors $\mathbf{E_m}$, m=1,…,6 on the symmetric second-rank tensor space **L**, leads to an orthogonal decomposition of the **L**-space into subspaces $\mathbf{L_m}$:

$$\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2 \oplus \cdots \oplus \mathbf{L}_6 , \qquad \mathbf{L}_m \perp \mathbf{L}_n \ \text{ for } m \neq n. \tag{72}$$

If the second-rank stress eigentensors $\overline{\boldsymbol{\sigma}_\mathbf{m}}$ constitute eigenstates of tensor **S**, they satisfy the eigenvalue equation:

$$\mathbf{S} \cdot \overline{\boldsymbol{\sigma}_\mathbf{m}} = (\lambda_1 \mathbf{E}_1 + \lambda_2 \mathbf{E}_2 + \cdots + \lambda_6 \mathbf{E}_6) \cdot \overline{\boldsymbol{\sigma}_\mathbf{m}} = \lambda_m \overline{\boldsymbol{\sigma}_\mathbf{m}}, \tag{73}$$

with index m varying between 1 and 6, and the λ_m-values being given by relations (19).

Therefore, the stress second-rank eigentensors $\overline{\boldsymbol{\sigma}_\mathbf{m}}$ of the compliance fourth-rank tensor **S** for the monoclinic symmetry are derived by the orthogonal projection of a second-rank symmetric tensor $\boldsymbol{\sigma}$ on subspaces $\mathbf{L_m}$, produced by the idempotent fourth-rank tensors $\mathbf{E_m}$, as follows:

$$\overline{\boldsymbol{\sigma}_\mathbf{m}} = \mathbf{E_m} \cdot \boldsymbol{\sigma} , \qquad m = 1,\ldots,6, \tag{74}$$

where $\boldsymbol{\sigma}$ denotes the contracted stress tensor, which is expressed in the form of a 6-D vector by:

$$\boldsymbol{\sigma}=\left[\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6\right]^T. \tag{75}$$

Relations (74) imply a series of computations, which finally yield the stress eigentensors in contracted notation:

$$\overline{\boldsymbol{\sigma}_1}=\left(g_1(\sigma_1)+g_2(\sigma_2)+g_3(\sigma_3)+g_6(\sigma_6)\right)\times\left[g_1,g_2,g_3,0,0,g_6\right]^T, \tag{76a}$$

$$\overline{\boldsymbol{\sigma}_2}=\left(r_1(\sigma_1)+r_2(\sigma_2)+r_3(\sigma_3)+r_6(\sigma_6)\right)\times\left[r_1,r_2,r_3,0,0,r_6\right]^T, \tag{76b}$$

$$\overline{\boldsymbol{\sigma}_3}=\left(h_1(\sigma_1)+h_2(\sigma_2)+h_3(\sigma_3)+h_6(\sigma_6)\right)\times\left[h_1,h_2,h_3,0,0,h_6\right]^T, \tag{76c}$$

$$\overline{\boldsymbol{\sigma}_4}=\left(\cos\psi(\sigma_4)+\sin\psi(\sigma_5)\right)\left[0,0,0,\cos\psi,\sin\psi,0\right]^T, \tag{76d}$$

$$\overline{\boldsymbol{\sigma}_5}=\left(-\sin\psi(\sigma_4)+\cos\psi(\sigma_5)\right)\left[0,0,0,-\sin\psi,\cos\psi,0\right]^T, \tag{76e}$$

$$\begin{aligned}\overline{\boldsymbol{\sigma}_6}=&\left((-\sin\mu\cos\rho\cos\nu)\sigma_1+(-\sin\nu\cos\rho)\sigma_2+(-\sin\rho)\sigma_3+(\cos\rho\cos\nu\cos\mu)\sigma_6\right)\\&\times\left[-\sin\mu\cos\rho\cos\nu,-\sin\nu\cos\rho,-\sin\rho,0,0,\cos\rho\cos\nu\cos\mu\right]^T,\end{aligned} \tag{76f}$$

where g_i, r_i and h_i, i=1,2,3,6, are defined by relations (30).

Relations (76) indicate that the stress eigentensors, corresponding to the spectral decomposition of the compliance tensor **S** for the monoclinic medium, analyse the generic stress tensor $\boldsymbol{\sigma}$ into six distinct parts:

$$\boldsymbol{\sigma}=\overline{\boldsymbol{\sigma}_1}+\overline{\boldsymbol{\sigma}_2}+\cdots+\overline{\boldsymbol{\sigma}_6}. \tag{77}$$

It is observed immediately from relations (76) that the stress states $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$, $\overline{\boldsymbol{\sigma}_3}$ and $\overline{\boldsymbol{\sigma}_6}$ constitute superposition of pure shear together with stressing along the 11, 22 and 33-directions of the adopted Cartesian co-ordinate system, dependent on the value of the set of eigenangles ρ, ν, μ, θ, ω and φ, expressed by relations (32) as complex functions of the components of the monoclinic material compliance tensor **S**. Contrariwise, the remaining two contracted $\overline{\boldsymbol{\sigma}_4}$ and $\overline{\boldsymbol{\sigma}_5}$-stress states represent pure shear states, related to the value of eigenangle ψ, whose definition is given by relation (32a).

Moreover, it is of interest to note that, upon substitution of the spectral representation (18) of the compliance tensor **S**, the generalised anisotropic form of Hooke's law, represented by Eq. (4), is expressed by:

$$\boldsymbol{\varepsilon}=\mathbf{S}\cdot\boldsymbol{\sigma}=\left(\lambda_1\mathbf{E}_1+\lambda_2\mathbf{E}_2+\cdots+\lambda_6\mathbf{E}_6\right)\cdot\boldsymbol{\sigma}=\lambda_1\overline{\boldsymbol{\sigma}_1}+\lambda_2\overline{\boldsymbol{\sigma}_2}+\cdots+\lambda_6\overline{\boldsymbol{\sigma}_6}. \tag{78}$$

Hence, the strain second-rank tensor **ε** is readily decomposed into six eigentensors $\overline{\boldsymbol{\varepsilon}_m}$:

$$\boldsymbol{\varepsilon}=\overline{\boldsymbol{\varepsilon}_1}+\overline{\boldsymbol{\varepsilon}_2}+\cdots+\overline{\boldsymbol{\varepsilon}_6}, \tag{79}$$

and the expression of Hooke's law, valid for crystalline or other anisotropic materials displaying the monoclinic symmetry, may be decomposed into six independent laws of proportionality of the stress and strain eigentensors, in a well-defined manner:

$$\overline{\boldsymbol{\varepsilon}_m}=\lambda_m\overline{\boldsymbol{\sigma}_m}\ ,\qquad \text{for } m=1,\ldots,6. \tag{80}$$

Then, substituting relations (19) into relations (80), the following expressions are acquired for the contracted strain second-rank eigentensors of a monoclinic medium:

$$\overline{\boldsymbol{\varepsilon}_1}=\left\{-\frac{1}{2}+\frac{1}{2}\left[1-2\,z_m+\frac{2Q}{(z_m-P)}\right]^{1/2}-\frac{A}{4}\right\}\times$$
$$\left(g_1(\sigma_1)+g_2(\sigma_2)+g_3(\sigma_3)+g_6(\sigma_6)\right)\left[g_1,g_2,g_3,0,0,g_6\right]^T, \tag{81a}$$

$$\overline{\boldsymbol{\varepsilon}_2} = \left\{ -\frac{1}{2} - \frac{1}{2}\left[1 - 2\,z_m + \frac{2Q}{(z_m - P)} \right]^{1/2} - \frac{A}{4} \right\} \times$$

$$\left(r_1(\sigma_1) + r_2(\sigma_2) + r_3(\sigma_3) + r_6(\sigma_6)\right)\left[r_1, r_2, r_3, 0, 0, r_6\right]^T, \qquad (81b)$$

$$\overline{\boldsymbol{\varepsilon}_3} = \left\{ \frac{1}{2} + \frac{1}{2}\left[1 - 2\,z_m - \frac{2Q}{(z_m - P)} \right]^{1/2} - \frac{A}{4} \right\} \times$$

$$\left(h_1(\sigma_1) + h_2(\sigma_2) + h_3(\sigma_3) + h_6(\sigma_6)\right) \times \left[h_1, h_2, h_3, 0, 0, h_6\right]^T, \qquad (81c)$$

$$\overline{\boldsymbol{\varepsilon}_4} = \left\{ \frac{s_{44} + s_{55}}{4} + \frac{1}{2}\left[\frac{1}{4}(s_{44} - s_{55})^2 + s_{45}^2 \right]^{1/2} \right\}$$

$$\times\left(\cos\psi(\sigma_4) + \sin\psi(\sigma_5)\right)\left[0, 0, 0, \cos\psi, \sin\psi, 0\right]^T,$$

$$\overline{\boldsymbol{\varepsilon}_4} = \left\{ \frac{s_{44} + s_{55}}{4} + \frac{1}{2}\left[\frac{1}{4}(s_{44} - s_{55})^2 + s_{45}^2 \right]^{1/2} \right\}$$

$$\times\left(\cos\psi(\sigma_4) + \sin\psi(\sigma_5)\right)\left[0, 0, 0, \cos\psi, \sin\psi, 0\right]^T, \qquad (81d)$$

$$\overline{\boldsymbol{\varepsilon}_5} = \left\{ \frac{s_{44} + s_{55}}{4} - \frac{1}{2}\left[\frac{1}{4}(s_{44} - s_{55})^2 + s_{45}^2 \right]^{1/2} \right\}$$

$$\times\left(-\sin\psi(\sigma_4) + \cos\psi(\sigma_5)\right)\left[0, 0, 0, -\sin\psi, \cos\psi, 0\right]^T, \qquad (81e)$$

$$\overline{\boldsymbol{\varepsilon}_6} = \left\{ \frac{1}{2} - \frac{1}{2}\left[1 - 2\,z_m - \frac{2Q}{(z_m - P)} \right]^{1/2} - \frac{A}{4} \right\} \times$$

$$\left((-\sin\mu\cos\rho\cos\nu)\sigma_1 + (-\sin\nu\cos\rho)\sigma_2 + (-\sin\rho)\sigma_3 + (\cos\rho\cos\nu\cos\mu)\sigma_6\right)$$

$$\times\left[-\sin\mu\cos\rho\cos\nu, -\sin\nu\cos\rho, -\sin\rho, 0, 0, \cos\rho\cos\nu\cos\mu\right]^T, \qquad (81f)$$

in which g_i, r_i and h_i, i=1,2,3,6 are defined by relations (30).

Finally, it was proven that stress $\overline{\boldsymbol{\sigma}_\mathbf{m}}$ and strain $\overline{\boldsymbol{\varepsilon}_\mathbf{m}}$ second-rank eigentensors correspond to energy orthogonal states of stress and strain, by satisfying the following relations:

$$\begin{aligned} &\overline{\boldsymbol{\sigma}_\mathbf{m}} \cdot \overline{\boldsymbol{\sigma}_\mathbf{n}} = 0 \\ &\overline{\boldsymbol{\sigma}_\mathbf{m}} \cdot \overline{\boldsymbol{\varepsilon}_\mathbf{n}} = 0 \qquad , \qquad \text{for } m \neq n. \\ &\overline{\boldsymbol{\varepsilon}_\mathbf{m}} \cdot \overline{\boldsymbol{\varepsilon}_\mathbf{n}} = 0 \end{aligned} \tag{82}$$

6. GEOMETRIC REPRESENTATION OF STRESS EIGENTENSORS

By projecting the stress eigentensors on the four-dimensional space system $(\sigma_1,\sigma_2,\sigma_3,\sigma_6)$, tensors $\overline{\boldsymbol{\sigma}_4}$ and $\overline{\boldsymbol{\sigma}_5}$ disappear, whereas tensors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$, $\overline{\boldsymbol{\sigma}_3}$ and $\overline{\boldsymbol{\sigma}_6}$ are represented by the following orthonormal vectors $\mathbf{e}_m$, m=1,2,3 and 6:

$$\mathbf{e}_1 = [g_1, g_2, g_3, g_6]^T , \tag{83a}$$

$$\mathbf{e}_2 = [r_1, r_2, r_3, r_6]^T , \tag{83b}$$

$$\mathbf{e}_3 = [h_1, h_2, h_3, h_6]^T , \tag{83c}$$

$$\mathbf{e}_6 = [-\sin\mu\cos\rho\cos\nu, -\sin\nu\cos\rho, -\sin\rho, \cos\rho\cos\nu\cos\mu]^T . \tag{83d}$$

The unit vectors $\mathbf{e}_m$, m=1,2,3 and 6 constitute the base vectors of a co-ordinate system obtained by rotating the stress space $(\sigma_1,\sigma_2,\sigma_3,\sigma_6)$ successively through angles ω, θ, φ, ρ, ν and μ, described by the following transformation matrices $\mathbf{A}_m$, m=1,…,6:

$$\mathbf{A}_1 = \begin{pmatrix} \cos\omega & \sin\omega & 0 & 0 \\ -\sin\omega & \cos\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{A}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (84a)$$

$$\mathbf{A}_3 = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{A}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\rho & \sin\rho \\ 0 & 0 & -\sin\rho & \cos\rho \end{pmatrix}, \quad (84b)$$

$$\mathbf{A}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\nu & 0 & \sin\nu \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\nu & 0 & \cos\nu \end{pmatrix}, \qquad \mathbf{A}_6 = \begin{pmatrix} \cos\mu & 0 & 0 & \sin\mu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\mu & 0 & 0 & \cos\mu \end{pmatrix}. \quad (84c)$$

The complete transformation **A** is therefore received by considering the product of the six transformation matrices, namely:

$$\mathbf{A} = \prod_{m=1}^{6} \mathbf{A}_m \qquad (85)$$

which is an orthogonal matrix. It is, thus, concluded that eigenangles ω, θ, φ, ρ, ν and μ affect the alignment of eigentensors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$, $\overline{\boldsymbol{\sigma}_3}$ and $\overline{\boldsymbol{\sigma}_6}$ in the stress space $(\sigma_1,\sigma_2,\sigma_3,\sigma_6)$. However, this space is four-dimensional and as such, eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_6$ cannot be visualised. In spite of that, it is always feasible to restrict our attention to three-dimensional pictures of the four-dimensional stress space. Then, it is easily observed by projecting the stress eigentensors on an arbitrary stress space $(\sigma_i,\sigma_j,\sigma_k)$, with {i,j,k}={1,2,3 and 6} and $i\neq j\neq k\neq i$, that vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_6$ are nonvanishing. Yet, vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_6$ are linearly dependent, and hence, in order to acquire the three eigenvectors $\mathbf{e}_m$, m=1,2,3 corresponding to the $(\sigma_i,\sigma_j,\sigma_k)$-reference system, one has to consider the transformation of this system, by means of three separate rotations through angles θ_1, θ_2 and θ_3, which can be expressed in matrix form $\mathbf{A}_m$, m=1,2,3, as follows:

$$\mathbf{A}_1 = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{86a}$$

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & \sin\theta_2 \\ 0 & -\sin\theta_2 & \cos\theta_2 \end{pmatrix}, \tag{86b}$$

$$\mathbf{A}_3 = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{86c}$$

and the product matrix **A** is accordingly expressed by an orthogonal matrix:

$$\mathbf{A} = \mathbf{A}_3\,\mathbf{A}_2\,\mathbf{A}_1 \tag{87}$$

$$= \begin{bmatrix} \cos\theta_1\cos\theta_3 - \sin\theta_1\cos\theta_2\sin\theta_3 & \sin\theta_1\cos\theta_3 + \cos\theta_1\cos\theta_2\sin\theta_3 & \sin\theta_2\sin\theta_3 \\ -\cos\theta_1\sin\theta_3 - \sin\theta_1\cos\theta_2\cos\theta_3 & -\sin\theta_1\sin\theta_3 + \cos\theta_1\cos\theta_2\cos\theta_3 & \sin\theta_2\cos\theta_3 \\ \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 & \cos\theta_2 \end{bmatrix}.$$

Moreover, angles θ_1, θ_2 and θ_3 are known as the Euler angles, which are introduced into relations (86) to express in generalised co-ordinates the elements of the orthogonal transformation matrix **A**. Therefore, the projections of stress eigentensors $\overline{\boldsymbol{\sigma}_i}$, $\overline{\boldsymbol{\sigma}_j}$ and $\overline{\boldsymbol{\sigma}_k}$ are represented by a set of three orthogonal vectors with associated unit vectors $\mathbf{e}_i$, $\mathbf{e}_j$ and $\mathbf{e}_k$, having as direction cosines:

$$\mathbf{e}_i = [\,\sin\theta_1\sin\theta_2,\, -\cos\theta_1\sin\theta_2,\, \cos\theta_2\,]^T, \tag{88a}$$

$$\mathbf{e}_j = \begin{bmatrix} -\sin\theta_1\cos\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_3 \\ \cos\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_3 \\ \sin\theta_2\cos\theta_3 \end{bmatrix}, \tag{88b}$$

$$\mathbf{e}_k = \begin{bmatrix} -\sin\theta_1\cos\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 \\ \cos\theta_1\cos\theta_2\sin\theta_3 + \sin\theta_1\cos\theta_3 \\ \sin\theta_2\sin\theta_3 \end{bmatrix}. \tag{88c}$$

Then, the transformation from a generic Cartesian co-ordinate system to another is realised via a succession of three angular displacements θ_1, θ_2, θ_3. Figs. 2(a-c) illustrate the various stages of the sequence. Frame $(\sigma_i,\sigma_j,\sigma_k)$ is initially rotated through an angle θ_1 counterclockwise with respect to the $\sigma_k \equiv \Theta_1''$-axis. The resultant $(\Theta_1,\Theta_1',\Theta_1'')$-system is next rotated counterclockwise by an angle θ_2 about the $\Theta_1 \equiv \Theta_2''$-axis (Fig. 2b), forming the sub-sequent system $(\Theta_2,\Theta_2',\Theta_2'')$, which is finally rotated counterclockwise by an angle θ_3 about the $\Theta_2 \equiv \Theta_3$-axis, hence producing the final frame $(\overline{\boldsymbol{\sigma}_i}, \overline{\boldsymbol{\sigma}_j}, \overline{\boldsymbol{\sigma}_k})$ $\equiv(\Theta_3,\Theta_3',\Theta_3'')$. Therefore, as seen in Fig. 2c, the unit vectors $\mathbf{e}_j$ and $\mathbf{e}_k$ lie on plane (Θ_2',Θ_2''), subtending with plane (σ_k,Θ_2'') an angle equal to $(\pi/2-\theta_2)$. In addition, the Θ_2''-axis is inclined to the σ_j-axis by an angle $(\pi/2-\theta_1)$, and the $\mathbf{e}_j$ and $\mathbf{e}_k$ unit vectors subtend an angle θ_3 with axes Θ_2' and Θ_2''.

Finally, an interesting geometric depiction of the contracted stress eigenstates $\overline{\boldsymbol{\sigma}_4}$ and $\overline{\boldsymbol{\sigma}_5}$ is obtained, if the characteristic stress tensors of the monoclinic medium are projected on the shear stress plane (σ_4,σ_5). In that case, tensors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$, $\overline{\boldsymbol{\sigma}_3}$ and $\overline{\boldsymbol{\sigma}_6}$ vanish, whereas the $\overline{\boldsymbol{\sigma}_4}$ and $\overline{\boldsymbol{\sigma}_5}$-tensors are represented by two orthogonal unit vectors $\mathbf{e}_4$ and $\mathbf{e}_5$, shown in Fig. 3:

$$\mathbf{e}_4 = [\cos\psi, \sin\psi]^T, \tag{89a}$$

$$\mathbf{e}_5 = [-\sin\psi, \cos\psi]^T. \tag{89b}$$

Then, the unit vectors $\mathbf{e}_4$ and $\mathbf{e}_5$ are inclined respectively to the σ_4 and σ_5-

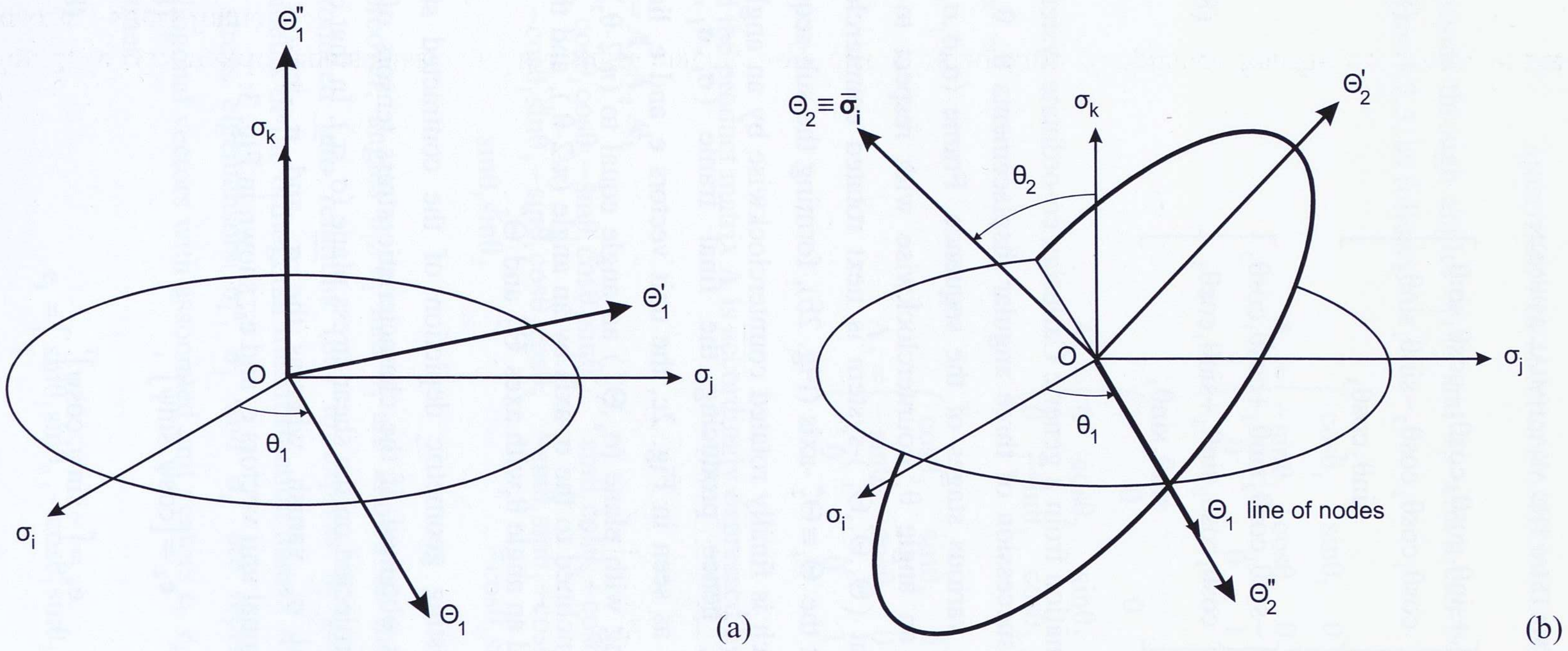

Fig. 2. (a-c) The rotations defining the Euler angles θ_1, θ_2, and θ_3 in the $(\sigma_i,\sigma_j,\sigma_k)$-stress frame.

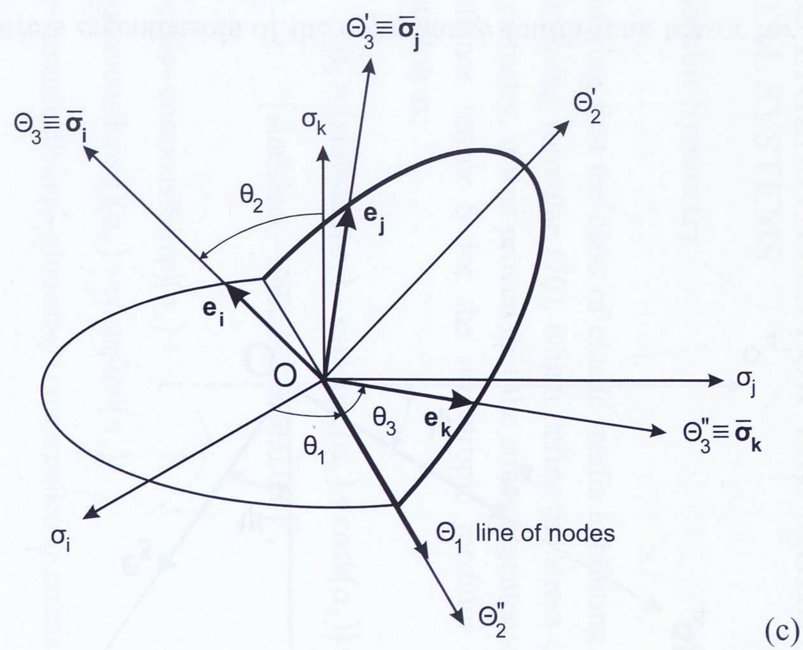

Fig. 2. (c) Geometric representation of the stress eigentensors of the monoclinic compliance fourth-rank tensor in the $(\sigma_i,\sigma_j,\sigma_k)$-stress frame.

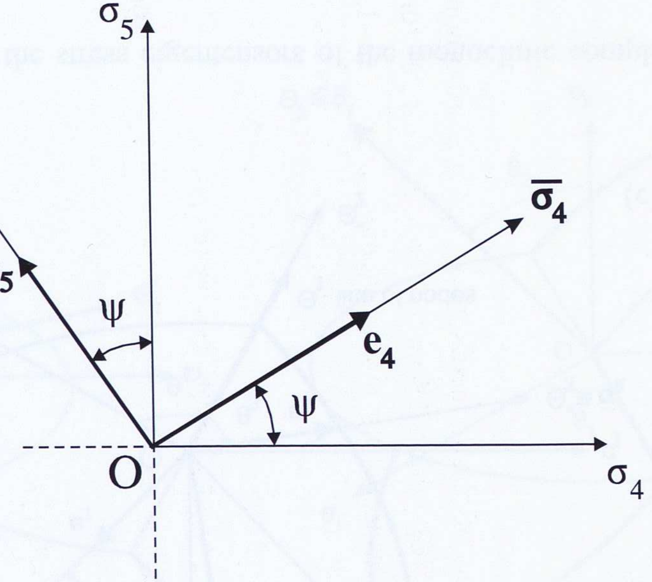

Fig. 3. Representation of the stress eigentensors of the compliance fourth-rank tensor for anisotropic media belonging to the monoclinic system on the (σ_4,σ_5)-shear stress plane.

axes of the (σ_4,σ_5)-plane by an angle ψ. Thus, eigenangle ψ is geometrically interpreted as the angle, which determines the alignment of the characteristic stress states $\overline{\boldsymbol{\sigma}_4}$ and $\overline{\boldsymbol{\sigma}_5}$ of the monoclinic material compliance tensor **S** in the shear stress plane (σ_4,σ_5).

7. STRESS EIGENSTATES FOR THE MORE SYMMETRIC CRYSTAL SYSTEMS

7.1. Orthorhombic Symmetry

Considering first the class of elastic media exhibiting the orthorhombic symmetry, and using formulae (76), which define the stress eigenstates for the monoclinic symmetry, it was proven that the stress eigentensors $\overline{\boldsymbol{\sigma}_m}$, m=1,...,6 of the compliance tensor **S** for the orthotropic medium are expressed in contracted notation as:

$$\overline{\boldsymbol{\sigma}_1} = \left(\sin\theta\sin\varphi(\sigma_1) - \sin\theta\cos\varphi(\sigma_2) + \cos\theta(\sigma_3)\right)\times$$
$$\left[\sin\theta\sin\varphi, -\sin\theta\cos\varphi, \cos\theta, 0,0,0\right]^T, \tag{90a}$$

$$\overline{\boldsymbol{\sigma}_2} = \{(-\sin\omega\cos\varphi - \cos\omega\cos\theta\sin\varphi)(\sigma_1) +$$
$$(-\sin\omega\sin\varphi + \cos\omega\cos\theta\cos\varphi)(\sigma_2) + \cos\omega\sin\theta(\sigma_3)\}$$
$$\times[-\sin\omega\cos\varphi - \cos\omega\cos\theta\sin\varphi, -\sin\omega\sin\varphi + \cos\omega\cos\theta\cos\varphi, \cos\omega\sin\theta, 0,0,0]^T, \tag{90b}$$

$$\overline{\boldsymbol{\sigma}_3} = \{(\cos\omega\cos\varphi - \sin\omega\cos\theta\sin\varphi)(\sigma_1) +$$
$$(\cos\omega\sin\varphi + \sin\omega\cos\theta\cos\varphi)(\sigma_2) + \sin\omega\sin\theta(\sigma_3)\}$$
$$\times[\cos\omega\cos\varphi - \sin\omega\cos\theta\sin\varphi, \cos\omega\sin\varphi + \sin\omega\cos\theta\cos\varphi, \sin\omega\sin\theta, 0,0,0]^T, \tag{90c}$$

$$\overline{\boldsymbol{\sigma}_4} = [0,0,0,\sigma_4,0,0]^T, \tag{90d}$$

$$\overline{\boldsymbol{\sigma}_5} = \left[0,0,0,0,\sigma_5,0\right]^{\mathrm{T}}, \tag{90e}$$

$$\overline{\boldsymbol{\sigma}_6} = \left[0,0,0,0,0,\sigma_6\right]^{\mathrm{T}}. \tag{90f}$$

As can be observed in relations (90), the contracted stress eigentensors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ are dependent on the values of the eigenangles θ, φ and ω, given by relations (45), and the engineering elastic constants of the material. Moreover, the $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$-states constitute superposition of stressing along the 11, 22 and 33-directions of the adopted Cartesian co-ordinate system. On the contrary, the last three characteristic stress tensors $\overline{\boldsymbol{\sigma}_4}$, $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$ represent pure shear states and are independent of the eigenangles θ, φ and ω, as well as of the material properties, hence remaining the same for the whole class of orthotropic media.

When the contracted eigentensors $\overline{\boldsymbol{\sigma}_m}$, m=1,...,6 are projected on the principal stress space $(\sigma_1,\sigma_2,\sigma_3)$, the $\overline{\boldsymbol{\sigma}_4}$, $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$-states vanish, whereas the $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$-states are represented by a set of three orthogonal vectors with associated unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, having as direction cosines:

$$\mathbf{e}_1 = [\ \sin\theta\sin\varphi, -\sin\theta\cos\varphi, \cos\theta\ ]^{\mathrm{T}}, \tag{91a}$$

$$\mathbf{e}_2 = \left[-\cos\theta\sin\varphi\cos\omega - \cos\varphi\sin\omega, \cos\theta\cos\varphi\cos\omega - \sin\varphi\sin\omega, \sin\theta\cos\omega\right]^{\mathrm{T}}, \tag{91b}$$

$$\mathbf{e}_3 = \left[-\cos\theta\sin\varphi\sin\omega + \cos\varphi\cos\omega, \cos\theta\cos\varphi\sin\omega + \sin\varphi\cos\omega, \sin\theta\sin\omega\right]^{\mathrm{T}}. \tag{91c}$$

As seen in Fig. 4, the unit vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ lie on plane (Θ',Θ''), which is inclined to plane (σ_3,Θ'') by an angle (π/2-θ). In addition, the Θ''-axis is inclined to the σ_2-axis by an angle (π/2-φ), and the $\mathbf{e}_2$ and $\mathbf{e}_3$-unit vectors are inclined respectively to axes Θ' and Θ'' by an angle ω.

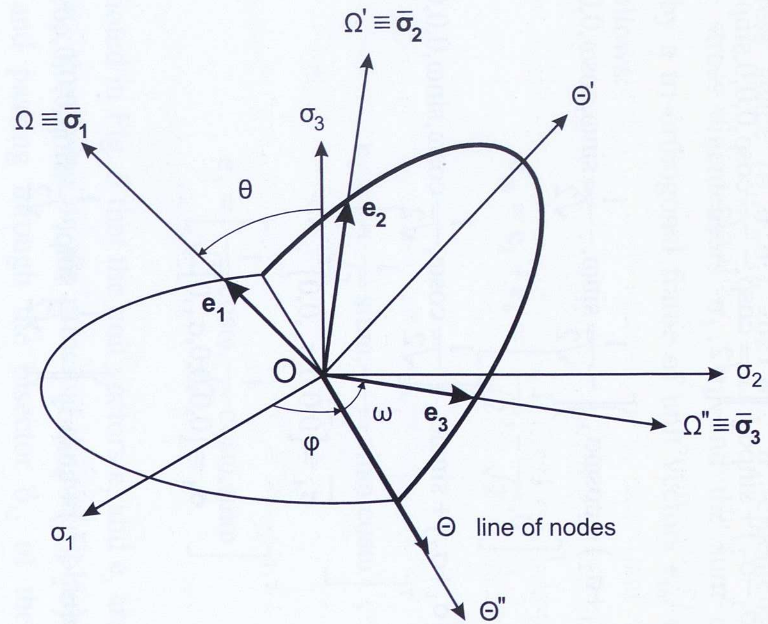

Fig. 4. Stress eigenstates of the orthotropic material compliance fourth-rank tensor in the principal stress space $(\sigma_1,\sigma_2,\sigma_3)$.

7.2. Tetragonal Symmetry

Concerning subsequently the tetragonal crystal system, and employing the generalised formulae (76) expressing the characteristic stress states for the monoclinic medium compliance tensor **S**, it was found that, in the case of tetragonal symmetry, the stress eigentensors $\overline{\boldsymbol{\sigma}_{\mathbf{m}}}$, m=1,...,6 reduce to:

$$\overline{\boldsymbol{\sigma}_1} = \left(\frac{1}{\sqrt{2}} \cos\rho(\sigma_1 - \sigma_2) + \sin\rho\sigma_6 \right) \left[\frac{1}{\sqrt{2}} \cos\rho, -\frac{1}{\sqrt{2}} \cos\rho, 0,0,0, \sin\rho \right]^{\mathrm{T}}, \quad (92a)$$

$$\overline{\boldsymbol{\sigma}_2} = \left(-\frac{1}{\sqrt{2}} \sin\omega(\sigma_1 + \sigma_2) + \cos\omega\sigma_3 \right) \left[-\frac{1}{\sqrt{2}} \sin\omega, -\frac{1}{\sqrt{2}} \sin\omega, \cos\omega, 0,0,0 \right]^{\mathrm{T}}, (92b)$$

$$\overline{\boldsymbol{\sigma}_3} = \left(\frac{1}{\sqrt{2}} \cos\omega(\sigma_1 + \sigma_2) + \sin\omega\sigma_3 \right) \left[\frac{1}{\sqrt{2}} \cos\omega, \frac{1}{\sqrt{2}} \cos\omega, \sin\omega, 0,0,0 \right]^{\mathrm{T}}, \quad (92c)$$

$$\overline{\boldsymbol{\sigma}_4} = \left[0,0,0,\sigma_4,0,0 \right]^{\mathrm{T}}, \quad (92d)$$

$$\overline{\boldsymbol{\sigma}_5} = \left[0,0,0,0,\sigma_5,0 \right]^{\mathrm{T}}, \quad (92e)$$

$$\overline{\boldsymbol{\sigma}_6} = \left(\frac{1}{\sqrt{2}} \sin\rho(\sigma_2 - \sigma_1) + \cos\rho\sigma_6 \right) \left[-\frac{1}{\sqrt{2}} \sin\rho, \frac{1}{\sqrt{2}} \sin\rho, 0,0,0, \cos\rho \right]^{\mathrm{T}}. \quad (92f)$$

Thus, relations (92) imply that four of the stress eigentensors, that is $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_4}$, $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$, correspond to shear states. Indeed, the characteristic tensors $\overline{\boldsymbol{\sigma}_1}$ and $\overline{\boldsymbol{\sigma}_6}$ are superposition of pure and simple shear, with $\overline{\boldsymbol{\sigma}_4}$ and $\overline{\boldsymbol{\sigma}_5}$ being pure shear loadings. Moreover, the characteristic stress $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ -tensors constitute an equilateral stressing in the (σ_1,σ_2)-plane, together with a tension or compression along the 33-axis of the adopted Cartesian co-ordinate system. Similarly, the $\overline{\boldsymbol{\sigma}_1}$ and $\overline{\boldsymbol{\sigma}_6}$ -eigenstates are stressing in the (σ_1,σ_2)-plane, as well as tension along the 12-axis. Secondly, the stress eigentensors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$,

$\overline{\boldsymbol{\sigma}_3}$ and $\overline{\boldsymbol{\sigma}_6}$ are dependent on the elastic constants of the medium, with the $\overline{\boldsymbol{\sigma}_1}$ and $\overline{\boldsymbol{\sigma}_6}$-eigentensors being functions of the eigenangle ρ, and the $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$-eigentensors being functions of the eigenangle ω, whose definition is given by relation (55c).

In addition, if we consider the projections of the contracted stress eigentensors $\overline{\boldsymbol{\sigma}_m}$, m=1,...,6, characterising the tetragonal symmetry, on the principal stress space $(\sigma_1,\sigma_2,\sigma_3)$, the $\overline{\boldsymbol{\sigma}_4}$, $\overline{\boldsymbol{\sigma}_5}$-eigentensors readily vanish, whereas the stress eigentensors $\overline{\boldsymbol{\sigma}_2}$, $\overline{\boldsymbol{\sigma}_3}$ and the sum of $\overline{\boldsymbol{\sigma}_1}$ and $\overline{\boldsymbol{\sigma}_6}$ are represented by a tri-orthogonal frame of unit vectors $\mathbf{e}_{16}$, $\mathbf{e}_2$ and $\mathbf{e}_3$, which are defined as follows:

$$\mathbf{e}_{16} = \mathbf{e}_1 + \mathbf{e}_6 = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right]^T, \tag{93a}$$

$$\mathbf{e}_2 = \left[-\frac{1}{\sqrt{2}}\sin\omega, -\frac{1}{\sqrt{2}}\sin\omega, \cos\omega \right]^T, \tag{93b}$$

$$\mathbf{e}_3 = \left[\frac{1}{\sqrt{2}}\cos\omega, \frac{1}{\sqrt{2}}\cos\omega, \sin\omega \right]^T. \tag{93c}$$

It is noted in Fig. 5 that the unit vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ are equally inclined to axes σ_1 and σ_2, thus lying on the principal diagonal plane (σ_3,δ_{12}), containing the σ_3-axis and passing through the bisector δ_{12} of the angle $\sigma_1 \hat{O} \sigma_2$. In addition, vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ subtend angles equal to ω and (π/2-ω) with respect to the σ_3-axis, whereas vector $\mathbf{e}_1$ is perpendicular to the σ_3-axis, hence lies on the intersection of the deviatoric π-plane and the plane σ_3=0.

Moreover, regarding the projections of the contracted stress eigentensors $\overline{\boldsymbol{\sigma}_m}$, m=1,...,6, given by relations (92), on the stress space $(\sigma_1,\sigma_2,\sigma_6)$, it is readily derived that the projections of the characteristic $\overline{\boldsymbol{\sigma}_4}$, $\overline{\boldsymbol{\sigma}_5}$-tensors, which are expressed as pure shear loadings, become equal to zero. On the other hand, the projections of the stress eigenstates $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_6}$ and the sum of $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ are generally represented by three mutually orthogonal vectors.

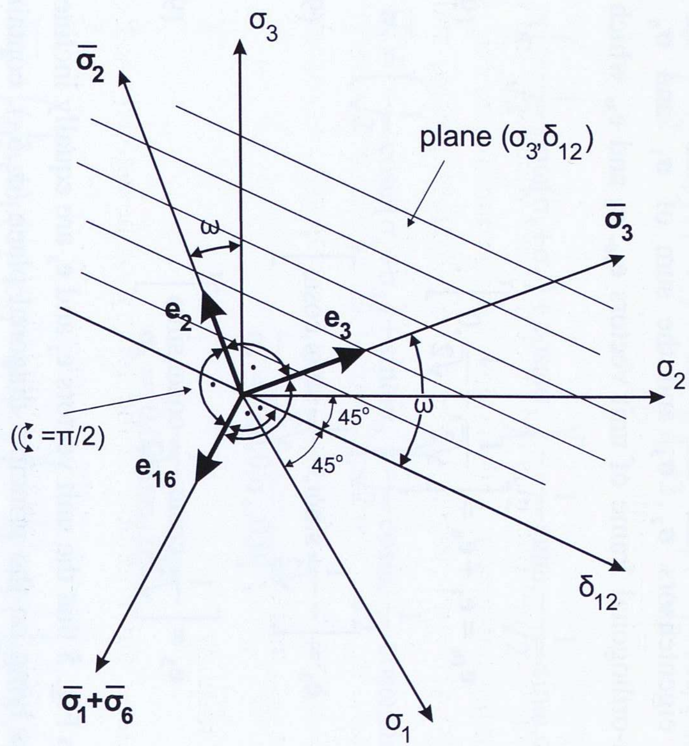

Fig. 5. Projection of the characteristic stress states of the compliance fourth-rank tensor valid for tetragonal media in the principal stress frame $(\sigma_1,\sigma_2,\sigma_3)$.

These are associated with the respective unit vectors $\mathbf{e}_1$, $\mathbf{e}_{23}$ and $\mathbf{e}_6$, which are defined by:

$$\mathbf{e}_1 = \left[\frac{1}{\sqrt{2}}\cos\rho, -\frac{1}{\sqrt{2}}\cos\rho, \sin\rho \right]^T, \tag{94a}$$

$$\mathbf{e}_{23} = \mathbf{e}_2 + \mathbf{e}_3 = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right]^T, \tag{94b}$$

$$\mathbf{e}_6 = \left[-\frac{1}{\sqrt{2}}\sin\rho, \frac{1}{\sqrt{2}}\sin\rho, \cos\rho \right]^T. \tag{94c}$$

It may be derived from relations (94) that the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_6$ lie on the plane containing the σ_6-axis and passing through the line σ_1=-σ_2, since they are equally inclined with respect to the principal stress axes σ_1 and -σ_2. Furthermore, vector $\mathbf{e}_1$ subtends an angle equal to (π/2-ρ) with axis σ_6, and vector $\mathbf{e}_{23}$ is perpendicular to the same axis, thus lies on the intersection of the (σ_1,-σ_2)-plane and the plane σ_6=0. Fig. 6 presents the geometric arrangement of these three vectors $\mathbf{e}_1$, $\mathbf{e}_{23}$ and $\mathbf{e}_6$.

7.3. Hexagonal Symmetry

Next, confining our attention to the family of hexagonal media, the stress eigentensors $\overline{\boldsymbol{\sigma}_m}$, m=1,...,6 of the compliance tensor **S** were derived via relations (76) to be expressed in contracted notation as:

$$\overline{\boldsymbol{\sigma}_1} = \left[\frac{1}{2}(\sigma_1 - \sigma_2), \frac{1}{2}(\sigma_2 - \sigma_1), 0,0,0,0 \right]^T, \tag{95a}$$

$$\overline{\boldsymbol{\sigma}_2} = \left(-\frac{1}{\sqrt{2}}\sin\omega(\sigma_1+\sigma_2) + \cos\omega\sigma_3 \right) \left[-\frac{1}{\sqrt{2}}\sin\omega, -\frac{1}{\sqrt{2}}\sin\omega, \cos\omega, 0,0,0 \right]^T, \tag{95b}$$

$$\overline{\boldsymbol{\sigma}_3} = \left(\frac{1}{\sqrt{2}}\cos\omega(\sigma_1+\sigma_2) + \sin\omega\sigma_3 \right) \left[\frac{1}{\sqrt{2}}\cos\omega, \frac{1}{\sqrt{2}}\cos\omega, \sin\omega, 0,0,0 \right]^T, \tag{95c}$$

$$\overline{\boldsymbol{\sigma}_4} = [0,0,0,\sigma_4,0,0]^T, \tag{95d}$$

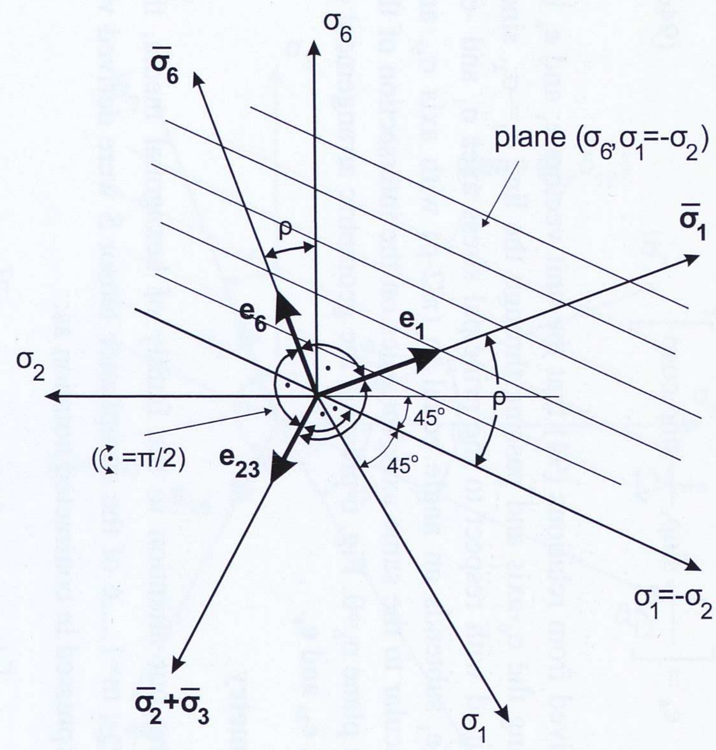

Fig. 6. Projection of the characteristic stress states of the compliance fourth-rank tensor valid for tetragonal media in the stress frame $(\sigma_1,\sigma_2,\sigma_6)$.

$$\overline{\boldsymbol{\sigma}_5} = \left[0,0,0,0,\sigma_5,0\right]^{\mathrm{T}}, \tag{95e}$$

$$\overline{\boldsymbol{\sigma}_6} = \left[0,0,0,0,0,\sigma_6\right]^{\mathrm{T}}. \tag{95f}$$

Then, the characteristic stress tensors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_4}$, $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$ correspond to shear loadings, with $\overline{\boldsymbol{\sigma}_1}$ being a simple shear, and $\overline{\boldsymbol{\sigma}_4}$, $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$ being pure shears. Besides, the $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$-eigenstates constitute an equilateral stressing in the (σ_1,σ_2)-plane, together with a tension or compression along the 33-axis of symmetry. In addition, if we consider the projections of the contracted stress eigentensors $\overline{\boldsymbol{\sigma}_m}$, m=1,...,6 on the principal stress space $(\sigma_1,\sigma_2,\sigma_3)$, the $\overline{\boldsymbol{\sigma}_4}$, $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$-eigentensors readily vanish, whereas the stress eigenstates $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ are represented by three mutually orthogonal unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, with the following direction cosines (Fig. 7):

$$\mathbf{e}_1 = \left[\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right]^{\mathrm{T}}, \tag{96a}$$

$$\mathbf{e}_2 = \left[-\frac{1}{\sqrt{2}}\sin\omega,-\frac{1}{\sqrt{2}}\sin\omega,\cos\omega\right]^{\mathrm{T}}, \tag{96b}$$

$$\mathbf{e}_3 = \left[\frac{1}{\sqrt{2}}\cos\omega,\frac{1}{\sqrt{2}}\cos\omega,\sin\omega\right]^{\mathrm{T}}. \tag{96c}$$

Then, it is easily noted in Fig. 7 that the unit vector $\mathbf{e}_1$ is perpendicular to the diagonal plane $\sigma_1=\sigma_2$ and the σ_3-axis, thus lying on the intersection of the deviatoric π-plane and the plane $\sigma_3=0$. Vectors $\mathbf{e}_2$ and $\mathbf{e}_3$, mutually orthogonal, lie always along the plane $\sigma_1=\sigma_2$, with vector $\mathbf{e}_3$ subtending an angle equal to $(\pi/2-\omega)$ with the σ_3-axis, as shown in Fig. 7.

7.4. Cubic-Isotropic Symmetries

Next, considering the rather trivial cases of the cubic and isotropic

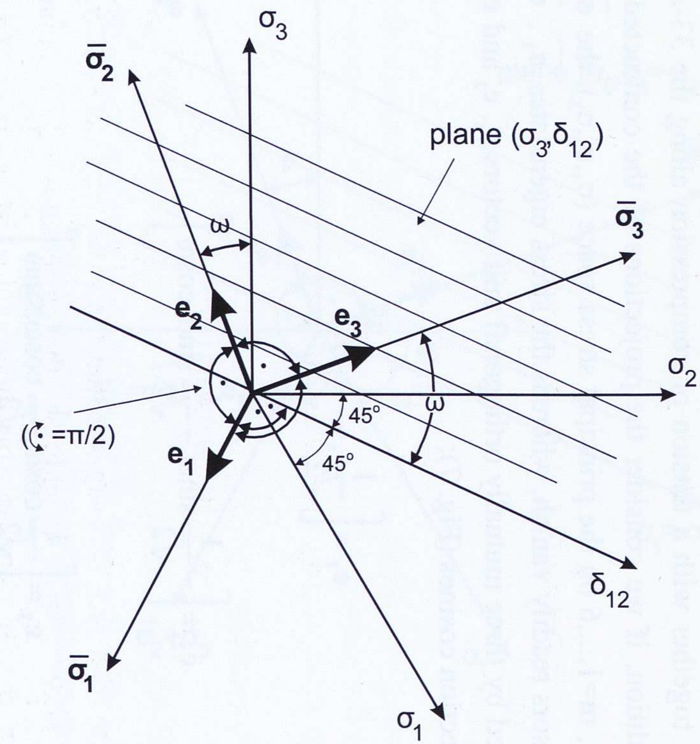

Fig. 7. Geometric representation of the contracted second-rank stress eigentensors of the compliance fourth-rank tensor for hexagonal media in the principal stress space $(\sigma_1,\sigma_2,\sigma_3)$.

crystal systems, it was proven by means of relations (76) that the stress eigentensors $\overline{\boldsymbol{\sigma}_{\mathbf{m}}}$, m=1,...,6 of the compliance tensor **S** are expressed in contracted notation as:

$$\overline{\boldsymbol{\sigma}_1} = \left[\frac{1}{2}(\sigma_1 - \sigma_2), \frac{1}{2}(\sigma_2 - \sigma_1), 0,0,0,0\right]^T, \tag{97a}$$

$$\overline{\boldsymbol{\sigma}_2} = \left(-\frac{1}{\sqrt{6}}\sigma_1 - \frac{1}{\sqrt{6}}\sigma_2 + \frac{2}{\sqrt{6}}\sigma_3\right)\left[-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0,0,0\right]^T, \tag{97b}$$

$$\overline{\boldsymbol{\sigma}_3} = \left(\frac{1}{\sqrt{3}}\sigma_1 + \frac{1}{\sqrt{3}}\sigma_2 + \frac{1}{\sqrt{3}}\sigma_3\right)\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0,0,0\right]^T, \tag{97c}$$

$$\overline{\boldsymbol{\sigma}_4} = \left[0,0,0,\sigma_4,0,0\right]^T, \tag{97d}$$

$$\overline{\boldsymbol{\sigma}_5} = \left[0,0,0,0,\sigma_5,0\right]^T, \tag{97e}$$

$$\overline{\boldsymbol{\sigma}_6} = \left[0,0,0,0,0,\sigma_6\right]^T. \tag{97f}$$

In that case, relations (97) suggest that the stress eigentensors $\overline{\boldsymbol{\sigma}_4}$, $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$ are pure shear states. In addition, the projections of the contracted stress eigentensors $\overline{\boldsymbol{\sigma}_4}$, $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$ on the principal stress space $(\sigma_1,\sigma_2,\sigma_3)$ become zero, whereas the projected stress eigenstates $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ are represented by three mutually orthogonal unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, with the following definitions:

$$\mathbf{e}_1 = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right]^T, \tag{98a}$$

$$\mathbf{e}_2 = \left[-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right]^T, \tag{98b}$$

$$\mathbf{e}_3 = \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]^T. \tag{98c}$$

Then, as noted in Fig. 8 and verified by relations (98), the unit vector $\mathbf{e}_3$ lies on the positive direction of the hydrostatic axis, whereas the $\mathbf{e}_1$ and $\mathbf{e}_2$-vectors lie on the deviatoric π-plane. Finally, both vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ remain on the principal diagonal plane $\sigma_1=\sigma_2$, as seen in Figs. 8b and 8c.

8. SPLITTING OF STRAIN ENERGY IN ANISOTROPIC MEDIA

Considering the definition (7) of the total elastic strain energy density, we have that:

$$2T(\boldsymbol{\sigma}) = \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon} = \boldsymbol{\sigma}\cdot\mathbf{S}\cdot\boldsymbol{\sigma} =$$

$$= \left(\overline{\boldsymbol{\sigma}_1} + \overline{\boldsymbol{\sigma}_2} + \ldots + \overline{\boldsymbol{\sigma}_6}\right)\cdot\left(\lambda_1\mathbf{E}_1 + \lambda_2\mathbf{E}_2 + \ldots + \lambda_6\mathbf{E}_6\right)\cdot\left(\overline{\boldsymbol{\sigma}_1} + \overline{\boldsymbol{\sigma}_2} + \ldots + \overline{\boldsymbol{\sigma}_6}\right)$$
$$= \lambda_1\overline{\boldsymbol{\sigma}_1}\cdot\overline{\boldsymbol{\sigma}_1} + \lambda_2\overline{\boldsymbol{\sigma}_2}\cdot\overline{\boldsymbol{\sigma}_2} + \ldots + \lambda_6\overline{\boldsymbol{\sigma}_6}\cdot\overline{\boldsymbol{\sigma}_6}. \tag{99}$$

Recalling relation (80), the expression for the potential function may be recast as:

$$2T(\boldsymbol{\sigma}) = \overline{\boldsymbol{\sigma}_1}\cdot\overline{\boldsymbol{\varepsilon}_1} + \overline{\boldsymbol{\sigma}_2}\cdot\overline{\boldsymbol{\varepsilon}_2}\cdots + \overline{\boldsymbol{\sigma}_6}\cdot\overline{\boldsymbol{\varepsilon}_6} \tag{100}$$

or

$$2T(\boldsymbol{\sigma}) = T\left(\overline{\boldsymbol{\sigma}_1}\right) + T\left(\overline{\boldsymbol{\sigma}_2}\right) + \cdots + T\left(\overline{\boldsymbol{\sigma}_6}\right), \tag{101}$$

that is the elastic potential is decomposed in distinct energy components, each associated to the same stress eigentensor. Denoting by $T(\overline{\boldsymbol{\sigma}_m})$ the following quantity:

$$T\left(\overline{\boldsymbol{\sigma}_m}\right) = \lambda_m\left(\overline{\boldsymbol{\sigma}_m}\cdot\overline{\boldsymbol{\sigma}_m}\right), \qquad m = 1,\ldots,6, \tag{102}$$

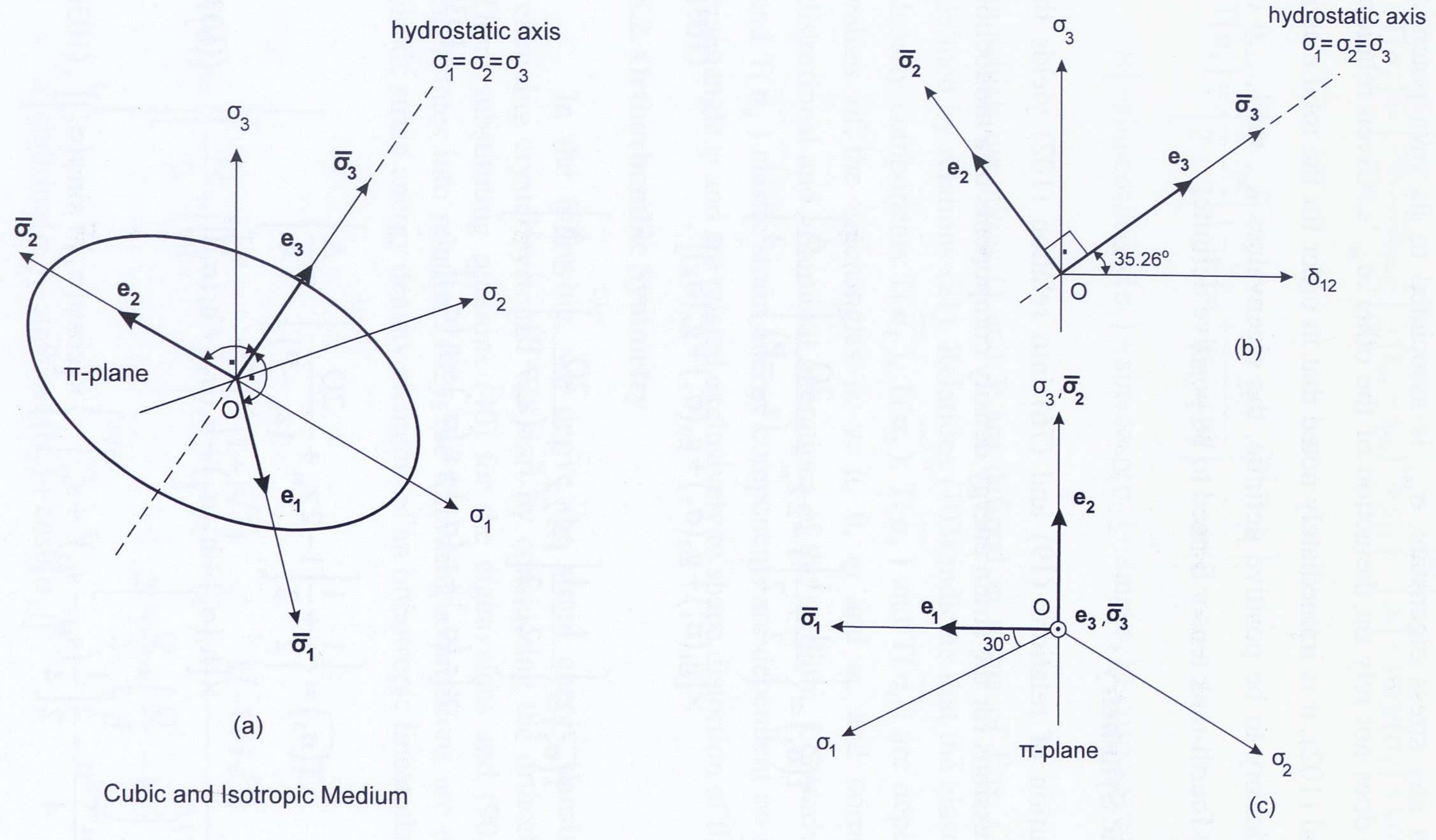

Fig. 8. Representation of the characteristic stress states of the compliance fourth-rank tensor for both cubic and isotropic materials (a) in the principal stress space $(\sigma_1,\sigma_2,\sigma_3)$, (b) their projection on the principal diagonal plane (σ_3,δ_{12}), and (c) their projection on the deviatoric π-plane.

it is noted that any stress eigenstate $\overline{\boldsymbol{\sigma}_m}$ is associated to its own potential T($\overline{\boldsymbol{\sigma}_m}$), which does not rely on the action of the other $\overline{\boldsymbol{\sigma}_m}$'s. Given relations (100), (101) and (102), it is immediately noted that in order for the total elastic strain energy density to be positive definite, the eigenvalues λ_m, m=1,…,6 of the compliance fourth-rank tensor **S** need to be positive definite.

8.1. Monoclinic Symmetry

Substitution of relations (19) and (76) into relation (102) yields the following expressions for the strain energy density components of a monoclinic medium:

$$T(\overline{\boldsymbol{\sigma}_1}) = \left\{-\frac{1}{2}+\frac{1}{2}\left[1-2\,z_m+\frac{2Q}{(z_m-P)}\right]^{1/2}-\frac{A}{4}\right\}$$

$$\times\left[g_1(\sigma_1)+g_2(\sigma_2)+g_3(\sigma_3)+g_6(\sigma_6)\right]^2, \tag{103a}$$

$$T(\overline{\boldsymbol{\sigma}_2}) = \left\{-\frac{1}{2}-\frac{1}{2}\left[1-2\,z_m+\frac{2Q}{(z_m-P)}\right]^{1/2}-\frac{A}{4}\right\}$$

$$\times\left[r_1(\sigma_1)+r_2(\sigma_2)+r_3(\sigma_3)+r_6(\sigma_6)\right]^2, \tag{103b}$$

$$T(\overline{\boldsymbol{\sigma}_3}) = \left\{\frac{1}{2}+\frac{1}{2}\left[1-2\,z_m+\frac{2Q}{(z_m-P)}\right]^{1/2}-\frac{A}{4}\right\}$$

$$\times\left[h_1(\sigma_1)+h_2(\sigma_2)+h_3(\sigma_3)+h_6(\sigma_6)\right]^2, \tag{103c}$$

$$T(\overline{\boldsymbol{\sigma}_4}) = \left\{\frac{s_{44}+s_{55}}{4}+\frac{1}{2}\left[\frac{1}{4}(s_{44}-s_{55})^2+s_{45}^2\right]^{1/2}\right\}\times\left[\cos\psi(\sigma_4)+\sin\psi(\sigma_5)\right]^2, \tag{103d}$$

$$T(\overline{\boldsymbol{\sigma}_5}) = \left\{ \frac{s_{44}+s_{55}}{4} - \frac{1}{2}\left[\frac{1}{4}(s_{44}-s_{55})^2 + s_{45}^2\right]^{1/2} \right\} \times \left[-\sin\psi(\sigma_4)+\cos\psi(\sigma_5)\right]^2, \quad (103e)$$

$$T(\overline{\boldsymbol{\sigma}_6}) = \left\{ \frac{1}{2} - \frac{1}{2}\left[1-2z_m+\frac{2Q}{(z_m-P)}\right]^{1/2} - \frac{A}{4} \right\}$$
$$\times\left[-\sin\mu\cos\rho\cos\nu(\sigma_1)-\sin\nu\cos\rho(\sigma_2)-\sin\rho(\sigma_3)+\cos\rho\cos\nu\cos\mu(\sigma_6)\right]^2, \quad (103f)$$

in which quantities Q and P are defined in terms of Eqs. (23), and z_m are defined by relations (21). Relations (103) indicate that the elastic strain energy density components $T(\overline{\boldsymbol{\sigma}_1})$, $T(\overline{\boldsymbol{\sigma}_2})$, $T(\overline{\boldsymbol{\sigma}_3})$ and $T(\overline{\boldsymbol{\sigma}_6})$ are dependent upon the values of the eigenangles ρ, ν, μ, θ, ω and φ, and correspond to both distortional and voluminal alterations of the medium. Contrariwise, the $T(\overline{\boldsymbol{\sigma}_5})$ and $T(\overline{\boldsymbol{\sigma}_6})$ elastic strain energy components are dependent on the value of the eigenangle ψ and are related exclusively to shape distortion of the medium.

8.2. Orthorhombic Symmetry

In the following, we derive the strain energy density parts of the remaining crystal systems. We start by considering the orthorhombic system. Then, substituting relations (40) for the eigenvalues and (90) for the stress eigenstates into relation (102), the following expressions are obtained for the elastic strain energy density elements of an orthotropic linear elastic body:

$$T(\overline{\boldsymbol{\sigma}_1}) = \left\{ \left[-\frac{Q}{2}+\left(\frac{Q^2}{4}-k^3\right)^{1/2}\right]^{1/3}\frac{(1+i\sqrt{3})}{2} + \frac{k(-1+i\sqrt{3})}{2\left[-\frac{Q}{2}+\left(\frac{Q^2}{4}-k^3\right)^{1/2}\right]^{1/3}} - \frac{A}{3} \right\}$$
$$\times\left[\sin\theta\sin\varphi(\sigma_1)-\sin\theta\cos\varphi(\sigma_2)+\cos\theta(\sigma_3)\right]^2, \quad (104a)$$

$$T(\overline{\boldsymbol{\sigma}_2}) = \left\{ \left[-\frac{Q}{2} + \left(\frac{Q^2}{4} - k^3 \right)^{1/2} \right]^{1/3} \frac{(-1+i\sqrt{3})}{2} + \frac{k(-1-i\sqrt{3})}{2\left[-\frac{Q}{2} + \left(\frac{Q^2}{4} - k^3 \right)^{1/2} \right]^{1/3}} - \frac{A}{3} \right\}$$

$$\times \begin{bmatrix} (-\sin\omega\cos\varphi - \cos\omega\cos\theta\sin\varphi)(\sigma_1) \\ +(-\sin\omega\sin\varphi + \cos\omega\cos\theta\cos\varphi)(\sigma_2) + \cos\omega\sin\theta(\sigma_3) \end{bmatrix}^2, \tag{104b}$$

$$T(\overline{\boldsymbol{\sigma}_3}) = \left\{ \left[-\frac{Q}{2} + \left(\frac{Q^2}{4} - k^3 \right)^{1/2} \right]^{1/3} + \frac{k}{\left[-\frac{Q}{2} + \left(\frac{Q^2}{4} - k^3 \right)^{1/2} \right]^{1/3}} - \frac{A}{3} \right\}$$

$$\times \begin{bmatrix} (\cos\omega\cos\varphi - \sin\omega\cos\theta\sin\varphi)(\sigma_1) + \\ (\cos\omega\sin\varphi + \sin\omega\cos\theta\cos\varphi)(\sigma_2) + \sin\omega\sin\theta(\sigma_3) \end{bmatrix}^2, \tag{104c}$$

$$T(\overline{\boldsymbol{\sigma}_4}) = \frac{s_{44}}{2}(\sigma_4)^2, \tag{104d}$$

$$T(\overline{\boldsymbol{\sigma}_5}) = \frac{s_{55}}{2}(\sigma_5)^2, \tag{104e}$$

$$T(\overline{\boldsymbol{\sigma}_6}) = \frac{s_{66}}{2}(\sigma_6)^2, \tag{104f}$$

in which, quantities Q and k are defined in terms of Eqs. (41).

According to relations (104), the three strain energy density components $T(\overline{\boldsymbol{\sigma}_1})$, $T(\overline{\boldsymbol{\sigma}_2})$ and $T(\overline{\boldsymbol{\sigma}_3})$ are dependent upon the values of the three eigenangles θ, φ and ω, and are associated with both distortional and dilatational forms of energy. On the contrary, the remaining three strain energy density constituents, namely $T(\overline{\boldsymbol{\sigma}_4})$, $T(\overline{\boldsymbol{\sigma}_5})$ and $T(\overline{\boldsymbol{\sigma}_6})$, are independent of the values of the set of eigenangles θ, φ and ω and correspond solely to a distortional type of energy.

8.3. Tetragonal Symmetry

Next, the tetragonal crystalline system is concerned and substituting relations (52) for the eigenvalues and (92) for the stress eigenstates into relation (102), the following expressions are derived for the strain energy parts of tetragonal symmetry:

$$T(\overline{\boldsymbol{\sigma}_1}) = \left\{ \frac{(s_{11}-s_{12})}{2} + \frac{s_{66}}{4} + \left\{ \left[\frac{(s_{11}-s_{12})}{2} - \frac{s_{66}}{4} \right]^2 + s_{16}^2 \right\}^{1/2} \right\} \tag{105a}$$

$$\times \left[\frac{1}{\sqrt{2}} \cos\rho(\sigma_1 - \sigma_2) + \sin\rho(\sigma_6) \right]^2,$$

$$T(\overline{\boldsymbol{\sigma}_2}) = \left\{ \frac{(s_{11}+s_{12})}{2} + \frac{s_{33}}{2} + \left\{ \left[\frac{(s_{11}+s_{12})}{2} - \frac{s_{33}}{2} \right]^2 + 2s_{13}^2 \right\}^{1/2} \right\} \tag{105b}$$

$$\times \left[-\frac{1}{\sqrt{2}} \sin\omega(\sigma_1 + \sigma_2) + \cos\omega(\sigma_3) \right]^2,$$

$$T(\overline{\boldsymbol{\sigma}_3}) = \left\{ \frac{(s_{11}+s_{12})}{2} + \frac{s_{33}}{2} - \left\{ \left[\frac{(s_{11}+s_{12})}{2} - \frac{s_{33}}{2} \right]^2 + 2s_{13}^2 \right\}^{1/2} \right\} \tag{105c}$$

$$\times \left[\frac{1}{\sqrt{2}} \cos\omega(\sigma_1 + \sigma_2) + \sin\omega(\sigma_3) \right]^2,$$

$$T(\overline{\boldsymbol{\sigma}_4}) = \frac{s_{44}}{2}(\sigma_4)^2, \tag{105d}$$

$$T\left(\overline{\boldsymbol{\sigma}_5}\right)=\frac{s_{44}}{2}\left(\sigma_5\right)^2, \tag{105e}$$

$$T\left(\overline{\boldsymbol{\sigma}_6}\right)=\left\{\frac{\left(s_{11}-s_{12}\right)}{2}+\frac{s_{66}}{4}-\left\{\left[\frac{\left(s_{11}-s_{12}\right)}{2}-\frac{s_{66}}{4}\right]^2+s_{16}^2\right\}^{1/2}\right\} \tag{105f}$$

$$\times\left[\frac{1}{\sqrt{2}}\sin\rho\left(\sigma_2-\sigma_1\right)+\cos\rho\left(\sigma_6\right)\right]^2.$$

As seen from relations (105), the $T(\overline{\boldsymbol{\sigma}_4})$ and $T(\overline{\boldsymbol{\sigma}_5})$-strain energy parts do not rely on the value of the eigenangles ρ and ω, and correspond totally to pure shape alterations of the tetragonal solid without any volume change. Nevertheless, the remaining constituents of the strain energy are associated with mixtures of both shape distortional and dilatational components of elastic energy, with the $T(\overline{\boldsymbol{\sigma}_1})$ and $T(\overline{\boldsymbol{\sigma}_6})$-strain energy parts depending on the value of ρ, and the $T(\overline{\boldsymbol{\sigma}_2})$ and $T(\overline{\boldsymbol{\sigma}_3})$-strain energy parts depending on the value of ω, which consequently influence the form of the strain energy stored.

8.4. Hexagonal Symmetry

For hexagonal symmetry, the following expressions are obtained for the strain energy components corresponding to the stress eigentensors given by relations (95):

$$T\left(\overline{\boldsymbol{\sigma}_1}\right)=\left(s_{11}-s_{12}\right)\times\left(\sigma_1-\sigma_2\right)^2, \tag{106a}$$

$$T\left(\overline{\boldsymbol{\sigma}_2}\right)=\left\{\frac{\left(s_{11}+s_{12}\right)}{2}+\frac{s_{33}}{2}+\left\{\left[\frac{\left(s_{11}+s_{12}\right)}{2}-\frac{s_{33}}{2}\right]^2+2s_{13}^2\right\}^{1/2}\right\} \tag{106b}$$

$$\times\left[-\frac{1}{\sqrt{2}}\sin\omega\left(\sigma_1+\sigma_2\right)+\cos\omega\left(\sigma_3\right)\right]^2,$$

$$T(\overline{\boldsymbol{\sigma}_3}) = \left\{ \frac{(s_{11}+s_{12})}{2} + \frac{s_{33}}{2} - \left\{ \left[\frac{(s_{11}+s_{12})}{2} - \frac{s_{33}}{2} \right]^2 + 2s_{13}^2 \right\}^{1/2} \right\} \tag{106c}$$

$$\times \left[\frac{1}{\sqrt{2}} \cos\omega(\sigma_1 + \sigma_2) + \sin\omega(\sigma_3) \right]^2 ,$$

$$T(\overline{\boldsymbol{\sigma}_4}) = \frac{s_{44}}{2}(\sigma_4)^2 , \tag{106d}$$

$$T(\overline{\boldsymbol{\sigma}_5}) = \frac{s_{44}}{2}(\sigma_5)^2 , \tag{106e}$$

$$T(\overline{\boldsymbol{\sigma}_6}) = (s_{11} - s_{12}) \times (\sigma_6)^2 . \tag{106f}$$

Relations (106) suggest that the remaining constituents of the strain energy are associated with both shape distort and volume changes, with the $T(\overline{\boldsymbol{\sigma}_2})$ and $T(\overline{\boldsymbol{\sigma}_3})$-energy parts depending on the value of ω, which thus affect the type of the strain energy stored. On the contrary, the $T(\overline{\boldsymbol{\sigma}_1})$, $T(\overline{\boldsymbol{\sigma}_4})$, $T(\overline{\boldsymbol{\sigma}_5})$ and $T(\overline{\boldsymbol{\sigma}_6})$-strain energy components are independent of the value of eigenangle ω, and correspond totally to shape alterations of the hexagonal solid.

8.5. Cubic-Isotropic Symmetries

Finally, the strain energy components corresponding to the stress eigentensors defined by relations (97) for the cubic symmetry are given by:

$$T(\overline{\boldsymbol{\sigma}_1}) = (s_{11} - s_{12}) \times (\sigma_1 - \sigma_2)^2 , \tag{107a}$$

$$T(\overline{\boldsymbol{\sigma}_2}) = \frac{(s_{11} - s_{12})}{6} \times (2\sigma_3 - \sigma_2 - \sigma_1)^2 , \tag{107b}$$

$$T(\overline{\boldsymbol{\sigma}_3}) = \frac{(s_{11} + 2s_{12})}{3} \times (\sigma_1 + \sigma_2 + \sigma_3)^2 , \tag{107c}$$

$$T\left(\overline{\boldsymbol{\sigma}_4}\right)=\frac{s_{44}}{2}\left(\sigma_4\right)^2, \tag{107d}$$

$$T\left(\overline{\boldsymbol{\sigma}_5}\right)=\frac{s_{44}}{2}\left(\sigma_5\right)^2, \tag{107e}$$

$$T\left(\overline{\boldsymbol{\sigma}_6}\right)=\frac{s_{44}}{2}\left(\sigma_6\right)^2, \tag{107f}$$

where the $T(\overline{\boldsymbol{\sigma}_1})$, $T(\overline{\boldsymbol{\sigma}_2})$, $T(\overline{\boldsymbol{\sigma}_4})$, $T(\overline{\boldsymbol{\sigma}_5})$ and $T(\overline{\boldsymbol{\sigma}_6})$-strain energy components correspond to distortional types of energy, and the $T(\overline{\boldsymbol{\sigma}_3})$-energy part corresponds to the dilatational form of energy. Exactly the same formulae are valid also for the isotropic medium, except for the first term of relations (107a,b), i.e. s_{11}-s_{12}, which is replaced by $s_{44}/2$.

Therefore, the decomposition of the strain energy density for the monoclinic, orthotropic, tetragonal, hexagonal and cubic media given by relations (103-107), which is based on the spectral decomposition of the compliance **S** and stiffness **C** tensors of such materials, constitutes the simplest one for the monoclinic, orthorhombic, tetragonal, hexagonal and cubic symmetries respectively. In addition, it is reduced to the generally accepted, classical decomposition of the elastic strain energy of the isotropic materials. In conclusion, the spectral analysis described above, which results in an efficient decomposition of the elastic strain energy density of the general monoclinic medium, was made possible by introducing the theory of the spectral decomposition of the fourth-rank tensor of compliance **S** and stiffness **C**.

9. CONSTRAINTS IN THE ANISOTROPIC COMPLIANCE TENSOR COMPONENTS

It has long been known that the existence of the thermodynamical constraint of positive-definite elastic potential sets restrictive bounds on the values of the elements of the constitutive tensors within the domain of classical elasticity. These constraints, derived from the compliance tensor **S**, which ensure positive-definite elastic strain energy, are investigated for all symmetries in this section of the paper. We start by noting that one of the very interesting features of the spectral analysis is its simplicity and clarity in proving the positive definite character of the elastic strain energy. The associated mathematical expression is:

$$\lambda_m > 0, \qquad m=1,\ldots,6 \tag{108}$$

requiring that the characteristic values of the compliance tensor **S** are all positive.

9.1. Monoclinic Symmetry

It was proven that this constraint demands:

$$\{s_{11}, s_{22}, s_{33}, s_{44}, s_{55}, s_{66}\} > 0, \tag{109a}$$

$$s_{12}^2 < s_{11}s_{22}, \qquad s_{13}^2 < s_{11}s_{33}, \qquad s_{23}^2 < s_{22}s_{33}, \tag{109b}$$

$$s_{16}^2 < s_{11}s_{66}, \qquad s_{26}^2 < s_{22}s_{66}, \qquad s_{36}^2 < s_{33}s_{66}, \qquad s_{45}^2 < s_{44}s_{55}, \tag{109c}$$

$$s_{11}\left(s_{22}s_{33} - s_{23}^2\right) - s_{12}\left(s_{12}s_{33} - s_{13}s_{23}\right) + s_{13}\left(s_{12}s_{23} - s_{13}s_{22}\right) > 0, \tag{109d}$$

$$s_{11}\left(s_{22}s_{66} - s_{26}^2\right) - s_{12}\left(s_{12}s_{66} - s_{16}s_{26}\right) + s_{16}\left(s_{12}s_{26} - s_{16}s_{22}\right) > 0, \tag{109e}$$

$$s_{11}\left(s_{33}s_{66} - s_{36}^2\right) - s_{13}\left(s_{13}s_{66} - s_{16}s_{36}\right) + s_{16}\left(s_{13}s_{36} - s_{16}s_{33}\right) > 0, \tag{109f}$$

$$s_{22}\left(s_{33}s_{66} - s_{36}^2\right) - s_{23}\left(s_{23}s_{66} - s_{26}s_{36}\right) + s_{26}\left(s_{23}s_{36} - s_{26}s_{33}\right) > 0, \tag{109g}$$

$$\begin{aligned} & s_{16}s_{26}\left(s_{33}s_{12} - s_{13}s_{23}\right) + s_{26}s_{36}\left(s_{11}s_{23} - s_{12}s_{13}\right) + s_{16}s_{36}\left(s_{22}s_{13} - s_{12}s_{23}\right) \\ & + \frac{s_{66}}{2}\left(s_{11}s_{22}s_{33} + 2s_{12}s_{13}s_{23} - s_{11}s_{23}^2 - s_{22}s_{13}^2 - s_{33}s_{12}^2\right) \\ & + \frac{s_{16}^2}{2}\left(s_{23}^2 - s_{22}s_{33}\right) + \frac{s_{26}^2}{2}\left(s_{13}^2 - s_{11}s_{33}\right) + \frac{s_{36}^2}{2}\left(s_{12}^2 - s_{11}s_{22}\right) > 0. \end{aligned} \tag{109h}$$

It is essential that inequalities (109) are all simultaneously satisfied, in order for the elastic strain energy density to be positive definite. Hence, bounds of the elastic constants based on partial fulfilment of these inequalities are considered improper and should be excluded. In the sequel, the bounds of the compliance tensor **S** components for the remaining crystal systems are obtained by reduction of Eqs. (109).

9.2. Orthorhombic Symmetry

For the orthorhombic symmetry, the constraints are expressed by:

$$\{s_{11}, s_{22}, s_{33}, s_{44}, s_{55}, s_{66}\} > 0, \tag{110a}$$

$$s_{12}^2 < s_{11}s_{22}, \qquad s_{13}^2 < s_{11}s_{33}, \qquad s_{23}^2 < s_{22}s_{33}, \tag{110b}$$

$$s_{11}\left(s_{22}s_{33} - s_{23}^2\right) - s_{12}\left(s_{12}s_{33} - s_{13}s_{23}\right) + s_{13}\left(s_{12}s_{23} - s_{13}s_{22}\right) > 0. \tag{110c}$$

9.3. Tetragonal Symmetry

Moreover, for tetragonal symmetry, the constraints are given by:

$$\{s_{11}, s_{33}, s_{44}, s_{66}\} > 0, \tag{111a}$$

$$s_{11} > s_{12}, \qquad s_{13}^2 < s_{11}s_{33}, \qquad s_{16}^2 < s_{11}s_{66}, \tag{111b}$$

$$(s_{11} - s_{12})s_{66} > 2s_{16}^2, \qquad (s_{11} + s_{12})s_{33} > 2s_{13}^2. \tag{111c}$$

9.4. Hexagonal Symmetry

Next, the corresponding constraints for hexagonal symmetry are given by:

$$\{s_{11}, s_{33}, s_{44}, s_{66}\} > 0, \tag{112a}$$

$$s_{13}^2 < s_{11}s_{33}, \qquad (s_{11} + s_{12})s_{33} > 2s_{13}^2. \tag{112b}$$

9.5. Cubic Symmetry

Finally, the bounds of the compliance tensor **S** components for cubic symmetry are expressed by:

$$\{s_{11}, s_{44}\} > 0, \qquad s_{11} > s_{12}, \qquad s_{11} + 2s_{12} > 0. \tag{113}$$

In the field of classical elasticity, the constraints entailed on the elements of the general anisotropic compliance matrix were first stated by Voigt (1910). In more recent times, they have been proclaimed by Born and Huang (1954), as well as by Hearmon (1961). Considering now the conditions imposed on the elastic constants of isotropic media, these are all well known and found in (Love, 1927). Furthermore, the restrictions applicable to media belonging to the cubic or hexagonal crystal systems are explicitly stated by Nye (1957). Relations for the bounds of elastic compliances for transversely isotropic media were determined separately by Eubanks and Sternberg (1954), along with Lempriere (1968) and Christensen (1979), employing different, but mathematically equivalent formulations, which guaranteed positive values for the elastic potential. Lempriere (1968) also examined the restrictions on the components of the compliance tensor **S** valid for orthotropic media.

10. CONCLUSIONS

The compliance fourth-rank tensor **S**, appropriate for crystalline media belonging to the monoclinic system, was decomposed in a spectral manner in this paper for the first time. Therefore, the characteristic values λ_m of the compliance tensor **S** were defined and the elementary idempotent fourth-rank tensors $\mathbf{E}_m$ were established, offering an orthogonal expansion of the space **M** of symmetric fourth-rank tensors into irreducible subspaces $\mathbf{M}_m$. In addition, it was shown that the combination of the six eigenvalues of the compliance fourth-rank tensor **S** and the values of the seven eigenangles formed the necessary parameters, needed for a coordinate invariant characterisation of the elastic properties of anisotropic crystalline media belonging to the monoclinic system.

Moreover, the effect of the idempotent fourth-rank tensors $\mathbf{E}_m$ on the symmetric second-rank tensor space $\mathbf{L}$ leads to a decomposition of the $\mathbf{L}$ space into subspaces $\mathbf{L}_m$, resulting in a decomposition of the stress second-rank tensor $\boldsymbol{\sigma}$ obtained for monoclinic crystals into six energy-orthogonal stress states. Thusly, the stress tensor is effectively pictured both by stress eigentensors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$, $\overline{\boldsymbol{\sigma}_3}$ and $\overline{\boldsymbol{\sigma}_6}$, which are superposition of pure shear with stressing along the directions of the coordinate system, and by stress eigentensors $\overline{\boldsymbol{\sigma}_4}$ and $\overline{\boldsymbol{\sigma}_5}$, which constitute pure shear states. Eigentensors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$, $\overline{\boldsymbol{\sigma}_3}$ and $\overline{\boldsymbol{\sigma}_6}$ are dependent on the value of eigenangles ρ, ν, μ, θ, ω and φ, whereas eigentensors $\overline{\boldsymbol{\sigma}_4}$ and $\overline{\boldsymbol{\sigma}_5}$ depend solely on the value of eigenangle ψ.

Furthermore, noting that the compliance tensors $\mathbf{S}$ suitable for the orthorhombic, tetragonal, hexagonal and cubic symmetries share an identical form with the corresponding tensor of the monoclinic medium, the compliance tensor $\mathbf{S}$ suitable for these anisotropic media was decomposed, by reduction of the results obtained for the monoclinic medium. The compliance tensor $\mathbf{S}$ for the trigonal symmetry was not related to that of the monoclinic medium, and was excluded from the current study. Hence, the eigenvalues, idempotent fourth-rank tensors, stress and strain second-rank eigentensors, and finally the strain energy parts associated to these eigenstates were acquired for the orthotropic, tetragonal and hexagonal media, as well as for the less interesting cases of cubic and isotropic media.

Nonetheless, the most consequential aspect of the spectral analysis is its capability in exposing most naturally the analogy in the elastic characteristics of isotropic and anisotropic media. Concerning for instance the form of the compliance tensor $\mathbf{S}$, this was established for the isotropic body during the first quartet of the last century:

$$\mathbf{S} = \frac{1}{3\mathrm{K}}\left(\frac{1}{3}\mathbf{1}\otimes\mathbf{1}\right) + \frac{1}{2\mathrm{G}}\left(\mathbf{I} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1}\right), \tag{114}$$

where K and G are the elastic bulk and shear moduli of the material. Indeed, the analogy between this expression and the spectral expansion of the compliance tensor $\mathbf{S}$ for anisotropic media belonging to the monoclinic system is easily recognised, whereas instead of K and G one has the eigenvalues of the compliance tensor **S**.

Furthermore, the analogy revealed by the spectral formulation of the theory of anisotropic elasticity is also confirmed by considering the splitting of the stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\varepsilon}$ second-rank tensors into their respective eigentensors. It is well known that the eigentensors of linearly isotropic elastic media consist of the deviatoric second-rank tensor and a tensor proportional to the unit tensor, the so called isotropic, or spherical, or hydrostatic parts of the tensor. Thus, the stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\varepsilon}$ tensors are decomposed as the sum of a spherical and a deviatoric tensor as follows:

$$\boldsymbol{\sigma}=\boldsymbol{\sigma}_P+\boldsymbol{\sigma}_D, \tag{115a}$$

$$\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_P+\boldsymbol{\varepsilon}_D, \tag{115b}$$

where subscript D in $\boldsymbol{\sigma}_D$ and $\boldsymbol{\varepsilon}_D$ denotes the deviatoric parts, the second terms denote the hydrostatic parts of the stress $\boldsymbol{\sigma}_P$ and strain $\boldsymbol{\varepsilon}_P$ tensors correspondingly, defined by:

$$\boldsymbol{\sigma}_P=\frac{1}{3}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{1}, \tag{116a}$$

$$\boldsymbol{\varepsilon}_P=\frac{1}{3}(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{1}, \tag{116b}$$

and $\mathrm{tr}\boldsymbol{\sigma}$ is the first invariant of the stress tensor $\boldsymbol{\sigma}$, whereas **1** is the unit tensor of the second-rank tensor space **L**, which in Cartesian form is represented by the Kronecker delta.

Then, this characteristic of isotropic elasticity encounters its analogy in the spectral decomposition of the stress and strain tensors into six distinct, non-interacting stress states. However, it should be emphasised that in contrast to the corresponding eigentensors of anisotropic elasticity, which are dependent on the elastic compliance components, and, therefore, accept different values for different media, the deviatoric and spherical eigentensors remain constant for all isotropic materials.

Furthermore, it was proven in this paper that the generalised, anisotropic Hooke's law may be formulated alternatively in the equivalent form of a system of six non-interacting, mutually orthogonal laws of direct proportionality, as expressed by relations (80), thus offering to the anisotropic theory of linear elasticity a status comparable to that of isotropic elasticity.

Indeed, the decomposition of Hooke's law for isotropic materials into two equations originates from the decomposition of the stress and strain tensors in isotropic elastic bodies into spherical and deviatoric constituents:

$$\frac{1}{3}(\mathrm{tr}\boldsymbol{\sigma})\mathbf{1} = \frac{1}{3}3\mathrm{K}(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{1}, \tag{117a}$$

$$\boldsymbol{\sigma}_{\mathbf{D}} = 2\mathrm{G}\boldsymbol{\varepsilon}_{\mathbf{D}}, \tag{117b}$$

where the two equations state that the spherical and deviatoric stress and strain eigentensors are proportional respectively with constants of proportionality 3K in the first and 2G in the second equation. Hence, the form of Hooke's law valid for anisotropic media may be thought of as a generalisation of equations (117).

Finally, the energy orthogonal states of stress and strain were proven to partition directly the elastic strain energy density in distinct strain energy constituents, $\mathrm{T}(\overline{\boldsymbol{\sigma}_{\mathbf{m}}})$, pointing out the absence of a pure dilatational strain energy component. Therefore, it appears that the $\mathrm{T}(\overline{\boldsymbol{\sigma}_1})$, $\mathrm{T}(\overline{\boldsymbol{\sigma}_2})$, $\mathrm{T}(\overline{\boldsymbol{\sigma}_3})$ and $\mathrm{T}(\overline{\boldsymbol{\sigma}_6})$ strain energies not only are related to with both volume changes and shape distortions of the medium, but they also depend on the values of the eigenangle ρ, ν, μ, θ, ω and φ, that subsequently affect the type of strain energy stored. Additionally, the value of the eigenangle ψ does influence the strain energy components $\mathrm{T}(\overline{\boldsymbol{\sigma}_4})$ and $\mathrm{T}(\overline{\boldsymbol{\sigma}_5})$, which are also related solely to shape distortion of the medium.

Another analogy is revealed between the splitting of the total elastic strain energy density of the monoclinic medium and the corresponding splitting valid for the isotropic medium, which is given in the form:

$$\mathrm{T}(\boldsymbol{\sigma}) = \mathrm{T}(\boldsymbol{\sigma}_{\mathbf{P}}) + \mathrm{T}(\boldsymbol{\sigma}_{\mathbf{D}}), \tag{118}$$

where $T(\boldsymbol{\sigma}_P)$ is the hydrostatic strain energy, and $T(\boldsymbol{\sigma}_D)$ denotes the deviatoric strain energy, corresponding to the hydrostatic and deviatoric stresses and strains respectively:

$$T(\boldsymbol{\sigma}_P) = \frac{1}{18K}(\mathrm{tr}\boldsymbol{\sigma})^2, \tag{119a}$$

$$T(\boldsymbol{\sigma}_D) = \frac{1}{2G}\left[\mathrm{tr}\boldsymbol{\sigma}^2 - \frac{1}{3}(\mathrm{tr}\boldsymbol{\sigma})^2\right]. \tag{119b}$$

Nevertheless, the decomposition of the elastic potential which is firmly established for the isotropic medium, is not valid for the monoclinic one. It is, therefore, inferred that a generalisation of the decomposition of the elastic strain energy density into components corresponding to sole dilatational and distortional types of energy, valid for the isotropic medium, as well as for cubic crystals cannot be achieved for the monoclinic medium, since the spherical tensor **1** is not part of the second-rank eigentensors of the compliance fourth-rank tensor **S**. Thus, the attempts made by Olszak and his co-workers (1956;1985), which aimed to introduce a generalisation of the Huber-Mises-Hencky criterion to hold for anisotropic media, therefore establishing the distortional component of the elastic strain energy density as the critical failure quantity, did not succeed for the abovementioned reasons.

In addition, the restrictive limits were set on the components of the compliance fourth-rank tensor **S** for monoclinic crystalline media, based on the classical thermodynamic argument, according to which the elastic strain energy density is required to be strictly positive. Such a requirement is fulfilled when the compliance fourth-rank tensor **S** is positive definite, and therefore, according to the algebra of fourth-rank tensors, when the eigenvalues of the compliance tensor **S** are strictly positive. In addition, the constraints dictated to the compliance tensor components were also examined for the orthotropic, tetragonal, hexagonal and cubic media, by reduction of the results derived for the monoclinic crystal system.

In conclusion, the generalisation of well-known characteristics of isotropic linear elastic bodies to anisotropic ones is succeeded by the spectral decomposition of the compliance fourth-rank tensor **S**, bridging in a well-defined manner the gap between isotropic and anisotropic elasticity, and offering to the theory of anisotropic media in the elastic domain a status comparable to that of isotropic elasticity. Furthermore, whereas all the elastic symmetries, exhibited by the anisotropic crystalline media, were formerly considered individually, with no unique decomposition valid simultaneously for the different crystal systems, by means of the spectral analysis, all symmetries are undertaken concurrently with a single spectral type of decomposition.

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Π Ε Ρ Ι Λ Η Ψ Ι Σ

ΓΕΝΙΚΗ ΘΕΩΡΙΑ ΤΗΣ ΑΝΙΣΟΤΡΟΠΙΑΣ ΤΗΣ ΥΛΗΣ. Η ΦΑΣΜΑΤΙΚΗ ΑΝΑΛΥΣΙΣ ΤΟΥ ΤΑΝΥΣΤΟΥ ΕΝΔΟΣΕΩΣ : ΕΦΑΡΜΟΓΑΙ ΕΙΣ ΤΗΝ ΚΡΥΣΤΑΛΛΟΓΡΑΦΙΑΝ

Οι τανυσταί τετάρτης τάξεως κρίνονται απαραίτητοι διά την ρεαλιστικήν και αποδοτικήν ανάλυσιν τόσον της γεωμετρικής συμμετρίας των κρυστάλλων, όσον και των μηχανικών ιδιοτήτων των ανισοτρόπων μέσων. Οι τανυσταί τετάρτης τάξεως εισήχθησαν υπό των Nye (1957) και Mason (1966) διά τον χαρακτηρισμόν της ελαστικής δυσκαμψίας και ενδόσεως, καθώς και διά την περιγραφήν των ελαστο-οπτικών και πιεζο-οπτικών συντελεστών των κρυσταλλικών μέσων.

Μέχρι σήμερον μόνο ειδικαί περιπτώσεις των ανισοτρόπων κρυσταλλικών μέσων αντεμετωπίσθησαν κατά ξεχωριστόν τρόπον, και, κατά συνέπειαν εδίδετο λύσις διάφορος διά τας διαφόρους ελαστικάς συμμετρίας ανεξάρτητος η μία της άλλης. Ήδη νέα προσέγγισης, βασιζομένη εις την θεωρίαν της φασματικής αναλύσεως, η οποία κρίνεται ως ιδιαιτέρως χρήσιμος διά την μελέτην των κρυστάλλων, εφαρμόζεται εις την πραγματείαν αυτήν διά πρώτην φορά. Η προσέγγισις αυτή εφαρμόζεται διά την γενικωτάτην περίπτωσιν του μονοκλινούς μέσου, και επεκτείνεται εις όλας τας λοιπάς συμμετρίας, συμβατάς προς την μονοκλινικήν ώστε να επιλύονται με την αυτήν λύσιν.

Κατά την θεωρίαν αυτήν ο τανυστής τετάρτης τάξεως ενδόσεως κρυσταλλικών μέσων, που ανήκουν εις το μονοκλινές σύστημα, ανεπτύχθη ορθογωνίως, μέσω της φασματικής αναλύσεως. Αι ιδιοτιμαί του υπελογίσθησαν, μαζί με τους στοιχειώδεις τανυστάς τετάρτης τάξεως οι οποίοι αναλύουν κατά μοναδικόν τρόπον τον χώρον των τανυστών τετάρτης τάξεως εις ορθογωνίους υποχώρους.

Κατ' αυτήν, βασιζομένοι εις τας συγκεκριμένας ιδιότητας της φασματικής αναλύσεως, λαμβάνομεν εκ των στοιχειωδών τανυστών τετάρτης τάξεως, ενεργειακώς ορθογωνίους ιδιοτανυστάς δευτέρας τάξεως τάσεων και παραμορφώσεων. Εν συνεχεία, αποδεικνύεται ότι αι απαραίτηται καταστατικαί παράμετροι, αναγκαίαι δια τον αναλλοίωτον καθορισμόν των ελαστικών χαρακτηριστικών του μονοκλινούς κρυστάλλου, ανέρχονται εις έξ ξεχωριστάς ιδιοτιμάς του τανυστού ενδόσεως μετά του συνόλου των επτά αδιαστάτων παραμέτρων, ήτοι των ιδιογωνιών θ, φ, ω, μ, ν, ρ και ψ.

Αξιοπρόσεκτον είναι το γεγονός ότι αι παράμετροι αυταί ευρέθησαν υπεύθυνοι διά τον προσανατολισμόν των ιδιοτανυστών τάσεων και παραμορφώσεων εις τον εξαδιάστατον χώρον των τάσεων. Επιπλέον, απεδείχθη ότι οι ιδιοτανυσταί τάσεων και παραμορφώσεων διαχωρίζουν την συνολικήν πυκνότητα ενεργείας παραμορφώσεων των μονοκλινών μέσων εις ανεξαρτήτους και διαφόρους συνιστώσας, υποδηλούσας, ωστόσο, την απουσίαν ενεργείας μεταβολής του όγκου.

Εν συνεχεία, ευρέθησαν αι ιδιοτιμαί, αι ιδιογωνίαι και οι ιδιοτανυσταί τάσεων, αι οποίαι χαρακτηρίζουν την φασματικήν ανάλυσιν του τανυστού ενδόσεως, διά τα διάφορα κρυσταλλικά συστήματα, δεδομένου ότι αι εκφράσεις των ιδιοτιμών και των ιδιοτανυστών του τανυστού ενδόσεως του μονοκλινούς σώματος είναι γενικαί εκφράσεις, ισχύουσαι και διά την κατηγορίαν των ορθοτρόπων, των τετραγωνικών, των εξαγωνικών, των κυβικών και των ισοτρόπων ελαστικών μέσων.

Ευρέθη μόνον επιπλέον ότι ο τανυστής ενδόσεως διά την τριγωνικήν συμμετρίαν δεν είναι της μορφής του μονοκλινούς μέσου, και επομένως, αι ιδιοτιμαί και ιδιοτανυσταί του τριγωνικού συστήματος δεν ήτο δυνατόν να καθορισθούν κατ' αυτόν τον τρόπον.

Δια το ορθότροπον στερεόν, αι ιδιοτιμαί ανέρχονται εις έξ. Αι τιμαί των ιδιογωνιών θ, φ και ω μεταβάλλονται, ενώ αι τιμαί των ιδιογωνιών ρ, μ, ν και ψ ισούνται με μηδέν. Εν συνεχεία και δια την τετραγωγικήν συμμετρίαν, αι χαρακτηριστικαί τιμαί αριθμούν πέντε εις το σύνολον, μία εκ των οποίων είναι διπλής πολλαπλότητος. Αι τιμαί των χαρακτηριστικών γωνιών ρ και ω αποτελούν συναρτήσεις των συνιστωσών του τανυστού ενδόσεως, ενώ αντιθέτως, αι τιμαί των υπολοίπων ιδιογωνιών παραμένουν σταθεραί.

Θεωρούμε περαιτέρω την κατηγορίαν των εξαγωνικών μέσων. Ευρέθη ότι αι χαρακτηριστικαί τιμαί του είναι τέσσαρες, δύο εκ των οποίων είναι διπλής πολλαπλότητος. Επιπροσθέτως, απεδείχθη ότι μόνον μία ιδιογωνία μεταβάλλεται, ενώ αι τιμαί των υπολοίπων ιδιογωνίων παραμένουν σταθεραί.

Όσον αφορά εις την τάξιν των κυβικών και των ισοτρόπων μέσων, αι ιδιοτιμαί δια το μεν κυβικόν μέσον είναι τρεις, μία εκ των οποίων είναι τετραπλής πολλαπλότητος. Αντιθέτως, αι ιδιοτιμαί διά το ισότροπον σώμα είναι δύο, μία εκ των οποίων είναι πενταπλούς πολλαπλότητος. Αι τιμαί των επτά ιδιογωνιών παραμένον σταθεραί δια όλα τα κυβικά και ισότροπα υλικά.

Επιπροσθέτως, τα επιμέρους κριτήρια τα οποία εφαρμόζονται εις τας συνιστώσας του τανυστού ενδόσεως εκ του κλασσικού θερμοδυναμικού αξιώματος, και τα οποία αποτελούν ικανάς και αναγκαίας συνθήκας που εξασφαλίζουν θετικάς τιμάς της πυκνότητος ενεργείας παραμορφώσεων, εμελετήθησαν εις βάθος δια την τάξιν των μονοκλινών υλικών. Τέλος, τα αντίστοιχα κριτήρια δια τας υπολοίπους κρυσταλλικάς τάξεις προκύπτουν δι' απλοποιήσεως των συνθήκων, αι οποίαι ισχύουν δια την μονοκλινικήν συμμετρίαν.

Το πλέον σημαντικόν χαρακτηριστικόν της φασματικής αναλύσεως έγκειται εις την ιδιότητά της να εκθέτει, κατά φυσικόν τρόπον, την αναλογίαν μεταξύ των ελαστικών χαρακτηριστικών των ισοτρόπων και των ανισοτρόπων μέσων. Παραδείγματος χάριν, η αναλογία μεταξύ της αναλύσεως του τανυστού ενδόσεως του ισοτρόπου μέσου και της αντιστοίχου φασματικής αναλύσεως του ανισοτρόπου μέσου είναι πλέον προφανής, εάν τα μέτρα διατμήσεως και διογκώσεως υποκατασταθούν από τας ιδιοτιμάς του τανυστού ενδόσεως.

Εξ άλλου έχει γίνει ευρέως αποδεκτόν ότι οι τανυσταί τάσεων και παραμορφώσεων των ισοτρόπων ελαστικών μέσων αναλύονται εις έναν αποκλίνοντα και έναν υδροστατικόν ιδιοτανυστήν. Το χαρακτηριστικόν τούτο της ισοτρόπου ελαστικότητος συναντά το ανάλογόν του εις την φασματικήν ανάλυσιν των τανυστών τάσεων και παραμορφώσεων εις έξ διαφόρους μεταξύ των, μη αλληλεπιδρώσες φορτίσεις. Εν τούτοις, πρέπει να καταστεί σαφές ότι, ενώ οι υδροστατικοί ιδιοτανυσταί παραμένουν σταθεροί δια όλα τα ισότροπα μέσα, οι αντίστοιχοι ιδιοτανυσταί της ανισοτρόπου ελαστικότητος σχετίζονται με τας συνιστώσας του τανυστού ενδόσεως, λαμβάνοντες, επομένως, διαφόρους τιμάς διά τα διάφορα ελαστικά μέσα.

Τέλος, απεδείχθη ότι οι ιδιοτανυσταί τάσεων και παραμορφώσεων διαχωρίζουν το ελαστικόν δυναμικόν εις διαφόρους συνιστώσας, όλως αναλόγως προς ό,τι συμβαίνει εις τα ισότροπα σώματα, διά τα οποία, ως γνωστόν, το ελαστικόν δυναμικόν διαιρείται εις δύο συνιστώσας, αντιστοιχούσας εις την ενέργειαν μεταβολής του όγκου και εις την στροφικήν ενέργειαν.

Κατά συνέπειαν, η φασματική ανάλυσις του τανυστού ενδόσεως επιτρέπει διά πρώτην φοράν την γενίκευσιν των ευρέως διαδεδομένων χαρακτηριστικών των ισοτρόπων σωμάτων δια τα ανισότροπα σώματα, προσφέρουσα κατ' αυτόν τον τρόπον, εις την θεωρίαν των ανισοτρόπων μέσων θέσιν ανάλογον προς αυτήν της ισοτρόπου ελαστικότητος.