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ΜΗΧΑΝΙΚΗ. — **Spectral decomposition of the compliance fourth-rank tensor for orthotropic materials**, by Academician *Pericles S. Theocaris* and *Dimitrios Sokolis*, Research Associate of the Academy of Athens*.

Keywords: Spectral decomposition, compliance tensor, orthotropic medium, Euler angles, elastic strain energy.

ABSTRACT

The compliance tensor related with orthotropic media is spectrally decomposed, and its characteristic values are determined. Further, its eigentensors are estimated, giving rise to energy orthogonal states of stress and thus decomposing the elastic potential in discrete elements. It is also shown that the essential parameters required for a complete characterization of the elastic characteristics of an orthotropic medium are the six eigenvalues of the compliance tensor, together with a set of three dimensionless parameters, the eigenangles θ, φ and ω. Further, it is shown that eigenangles θ, φ and ω, which are defined as independent parameters specifying the orientation of anisotropy of the orthotropic material, may be used as generalized coordinates to express in the Lagrangian formulation the orthotropic system and its variation during loading of the structure.

1. INTRODUCTION

Fourth-rank tensors are necessary for an efficient analysis of the properties of crustals or other anisotropic media. Fourth-rank tensors were presented (Nye, 1957) initially for the characterization of either elastic stiffness and

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compliance, elasto-optical and piezo-optical coefficients, or electrostriction and magnetostriction in crystalline media. In addition, fourth-rank tensors were decomposed (Srinivasan and Nigam, 1969) into independent elementary tensors, offering insight into the tensor structure and facilitating the calculations of tensors. Next, the algebra of fourth-rank tensors of the thirty-two crystal classes was broken down to irreducible subalgebras (Walpole, 1984), which were simpler than the initial one, for the purpose of simplifying the calculations of sums, products and inverses between the tensors.

On the other hand, it was proven (Rychlewski, 1984a,b) that the spectral decomposition was the simplest possible decomposition of the compliance **S** or stiffness **C** fourth-rank tensors. Futhermore, this decomposition was preferable because of its ability to split a fourth-rank tensor into idempotent tensors, which defined energy orthogonal components of both stresses and strains. Moreover, in view of the fact that the failure tensor **H** is the limiting case for the stress states of the compliance tensor **S**, this tensor also maintains the energy-orthogonality property. Last but not least, this decomposition offered a simple means of splitting the total elastic energy of the deformed body into its terms associated to each eigentensor of the compliance **S** tensor. Hence, it was shown that a generalization of the decomposition of the elastic strain energy density into dilatational and distortional components (Olszak and Urbanowski, 1956; Olszak and Maciejewska, 1985), valid only for the isotropic medium, was impossible for the anisotropic one.

Although Rychlewski ascertained the use of the spectral decomposition principle on the class of symmetric fourth-rank tensors valid for orthotropic materials, he did not evaluate the eigenvalues and eigentensors of the corresponding tensors. These were established subsequently (Theocaris and Phillipidis, 1989a,b; 1990; 1991), and combined with a characteristic angle, the eigenangle ω were the necessary parameters for an invariant specification for the elastic features of a transversely isotropic medium. Moreover, the eigenangle ω determined the arrangement and orientation of the eigentensors in the principal stress space. Then, the three-dimensional spectral decomposition was extended, to incorporate the two-dimensional equivalent case (Theocaris and Sokolis, 1998). Hence, a simplified procedure was offered, as well as a possibility of illustrating the elastic properties of a transversely isotropic body under plane-stress conditions, together with a suggestion of the way to separate the elastic energy of plane laminae into distinct elements (Theocaris, 1989). This two-dimensional spectral decomposition yielded three characteristic values, corresponding to the three energy-orthogonal eigenstates, together with a dimensionless parameter, the plane eigenagle ω_p, which influences the orientation of the eigentensors on the principal stress plane.

In this paper, the compliance **S** fourth-rank tensor, valid for orthotropic media, is decomposed spectrally for the first time, and its characteristic values are estimated. Furthermore, the eigentensors are established, thus giving rise to energy orthogonal states of stress, and decomposing the elastic potential into distinct constituents. It is also shown that the constitutive parameters, requested for an invariant description of the elastic behaviour of an orthotropic body, are now the six eigenvalues of its compliance **S** tensor, as well as a set of three dimensionless quantities, namely the eigenangles θ, φ and ω. These are responsible for the alignment of the eigentensors of the compliance tensor **S** in the principal stress space. It is subsequently shown that the eigenangles θ, φ and ω are equivalent to the Euler-angles, the parameters which are essential for the description of the motion of rigid bodies in the Lagrangian formulation of mechanics.

2. SPECTRAL DECOMPOSITION OF THE COMPLIANCE TENSOR FOR THE ORTHOTROPIC BODY

Consider the spectral decomposition of the compliance fourth-rank tensor **S** of an orthotropic linear elastic solid. We assume the Cartesian coordinate system, where the stress and strain tensors are referred to, being oriented along the principal material direstions. Recalling the classical analysis for orthotropic media, the usual elastic moduli E_1, E_2 and E_3 are defined in the 11, 22 and 33-directions respectively, as well as the Poisson's ratios ν_{ij}, for transverse strain in the j-direction when stressed in the i-direction, and the shear moduli G_{23}, G_{31} and G_{12} in the 2-3, 3-1 and 1-2 planes respectively. Hence, the following basic stress-strain relationships are obtained:

$$\varepsilon_1 = \frac{\sigma_1}{E_1} - \frac{\nu_{21}}{E_2}\sigma_2 - \frac{\nu_{31}}{E_3}\sigma_3 \qquad (1a)$$

$$\varepsilon_2 = -\frac{\nu_{12}}{E_1}\sigma_1 + \frac{\sigma_2}{E_2} - \frac{\nu_{32}}{E_3}\sigma_3 \qquad (1b)$$

$$\varepsilon_3 = -\frac{\nu_{13}}{E_1}\sigma_1 - \frac{\nu_{23}}{E_2}\sigma_2 + \frac{\sigma_3}{E_3} \qquad (1c)$$

$$2\varepsilon_{12} = \frac{1}{G_{12}}\sigma_{12} \quad , \quad 2\varepsilon_{13} = \frac{1}{G_{13}}\sigma_{13} \quad , \quad 2\varepsilon_{23} = \frac{1}{G_{23}}\sigma_{23} \qquad (1d)$$

The compliance tensor **S** is associated to the following 6 × 6 square matrix:

$$\mathbf{S} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4G_{12}} \end{bmatrix} \qquad (2)$$

For an orthotropic material, the compliance tensor **S** is comprised of nine independent constituents, since it is symmetric, that is:

$$S_{ij} = S_{ij} \qquad i, j = 1,...,6 \qquad (3)$$

Substituting engineering constants in relation (3) gives:

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ij}}{E_j} \qquad i, j = 1,...,3 \qquad (4)$$

Equation (4) implies a set of three reciprocal relations for an orthotropic material.

The eigenvalues of the square matrix of rank six associated to tensor **S** were determined from the characteristic equation:

$$(\lambda^3 + A\lambda^2 + B\lambda + C)\left(\lambda - \frac{1}{2G_{12}}\right)\left(\lambda - \frac{1}{2G_{13}}\right)\left(\lambda - \frac{1}{2G_{23}}\right) = 0 \qquad (5)$$

in which

$$A = -\left(\frac{1}{E_1} + \frac{1}{E_2} + \frac{1}{E_3}\right) \qquad (6a)$$

$$B = \frac{1}{E_1}\left(\frac{1}{E_2} + \frac{1}{E_3}\right) + \frac{1}{E_2 E_3} - \left(\frac{\nu_{12}^2}{E_1^2} + \frac{\nu_{13}^2}{E_1^2} + \frac{\nu_{23}^2}{E_2^2}\right) \qquad (6b)$$

$$C = \frac{\nu_{13}^2 + 2\nu_{13}\nu_{12}\nu_{23}}{E_1^2 E_2} + \frac{\nu_{12}^2}{E_1^2 E_3} + \frac{\nu_{23}^2}{E_1 E_2^2} - \frac{1}{E_1 E_2 E_3} \qquad (6c)$$

The polynomial inside the first parenthesis of relation (5) is a cubic and, therefore, has to be transformed to its reduced form, in order for the eigenvalues λ_1, λ_2 and λ_3 to be determined. Substituting $\lambda = y - \frac{A}{3}$, the cubic equation becomes:

$$y^3 + Py + Q = 0 \qquad (7)$$

in which

$$P = B - \frac{A^2}{3} \qquad Q = \frac{2A^3}{27} - \frac{AB}{3} + C \qquad (8)$$

Then, setting $y = z + \frac{k}{z}$, Eq. (7) is expressed as:

$$z^6 + Qz^3 + k^3 = 0 \qquad (9)$$

in which

$$k = -\frac{P}{3} \qquad (10)$$

Finally, setting $u = z^3$, Eq. (9) becomes a quadratic equation, which is readily solved:

$$u^2 + Qu + k^3 = 0 \tag{11}$$

The eigenvalues λ_m, $m = 1,...,6$ of the associated square matrix of rank six to tensor **S** defined above were then evaluated to be:

$$\lambda_1 = -\left[-\frac{Q}{2} + \left(\frac{Q^2}{4} - k^3\right)^{1/2}\right]^{1/3} \frac{(1 + i\sqrt{3})}{2} + \frac{k(-1 + i\sqrt{3})}{2\left[-\frac{Q}{2} + \left(\frac{Q^2}{4} - k^3\right)^{1/2}\right]^{1/3}} - \frac{A}{3} \tag{12a}$$

$$\lambda_2 = \left[-\frac{Q}{2} + \left(\frac{Q^2}{4} - k^3\right)^{1/2}\right]^{1/3} \frac{(-1 + i\sqrt{3})}{3} + \frac{k(-1 - i\sqrt{3})}{2\left[-\frac{Q}{2} + \left(\frac{Q^2}{4} - k^3\right)^{1/2}\right]^{1/3}} - \frac{A}{3} \tag{12b}$$

$$\lambda_3 = \left[-\frac{Q}{2} + \left(\frac{Q^2}{4} - k^3\right)^{1/2}\right]^{1/3} + \frac{k}{\left[-\frac{Q}{2} + \left(\frac{Q^2}{4} - k^3\right)^{1/2}\right]^{1/3}} - \frac{A}{3} \tag{12c}$$

$$\lambda_4 = \frac{1}{2G_{23}} \tag{12d}$$

$$\lambda_5 = \frac{1}{2G_{13}} \tag{12e}$$

$$\lambda_6 = \frac{1}{2G_{12}} \tag{12f}$$

The corresponding six idempotent tensors $\mathbf{E}_m$, $m = 1,...,6$ of the spectral decomposition of **S** were also evaluated to be:

$$\mathbf{E}_1 = E^1_{ijkl} = \mathbf{h} \otimes \mathbf{h} = h_{ij}h_{kl} \tag{13a}$$

$$\mathbf{E}_2 = E^2_{ijkl} = \mathbf{j} \otimes \mathbf{j} = j_{ij}j_{kl} \tag{13b}$$

$$\mathbf{E}_3 = E^3_{ijkl} = \mathbf{r} \otimes \mathbf{r} = r_{ij}r_{kl} \qquad \mathbf{a,b,c,h,j,r} \in \mathbf{L} \tag{13c}$$

$$\mathbf{E}_4 = E^4_{ijkl} = \frac{1}{2}(a_{ik}\, b_{jl} + a_{il}\, b_{jk} + a_{jk}\, b_{il} + a_{jl}\, b_{ik}) \tag{13d}$$

$$\mathbf{E}_5 = E^5_{ijkl} = \frac{1}{2}(c_{ik}\, a_{jl} + c_{il}\, a_{jk} + c_{jk}\, a_{il} + c_{jl}\, a_{ik}) \tag{13e}$$

$$\mathbf{E}_6 = E^6_{ijkl} = \frac{1}{2}(b_{ik} c_{jl} + b_{il}\, c_{jk} + b_{jk}\, c_{il} + b_{jl}\, c_{ik}) \tag{13f}$$

where $\mathbf{L}$ is the second-rank symmetric tensor space over $\mathbf{R}^3$.

The second-rank symmetric tensors **a**, **b** and **c**, appearing in relations (13d,e,f) in the expressions for the idempotent tensors $\mathbf{E}_4$, $\mathbf{E}_5$ and $\mathbf{E}_6$, are defined as follows:

$$\mathbf{a} = \mathbf{k} \otimes \mathbf{k},\ \mathbf{b} = \mathbf{l} \otimes \mathbf{l},\ \mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{1} \tag{14}$$

with **k**, **l** and **m** being the unit vectors of $\mathbf{R}^3$, associated wth the 33, 22 and 11 - directions of the Cartesian coordinate system. Tensors **h**, **j** and **r** are also second-rank symmetric tensors, expressed as:

$$\mathbf{h} = \sin\theta\sin\varphi\mathbf{c} - \sin\theta\cos\varphi\mathbf{b} + \cos\theta\mathbf{a} \tag{15a}$$

$$\mathbf{j} = -\sin\omega\mathbf{f} + \cos\omega\mathbf{g} \tag{15b}$$

$$\mathbf{r} = \cos\omega\mathbf{f} + \sin\omega\mathbf{g} \tag{15c}$$

in which the second-rank symmetric tensors **f, g** are defined as follows:

$$\mathbf{f} = \cos\varphi\mathbf{c} + \sin\varphi\mathbf{b} \tag{16a}$$

$$\mathbf{g} = -\cos\theta\sin\varphi\mathbf{c} + \cos\theta\cos\varphi\mathbf{b} + \sin\theta\mathbf{a} \tag{16b}$$

At this point, an examination of the characteristics of the compliance tensor **S** and its associated matrix is of considerable interest. From the defining Eq. (3), it is readily noticed that the components of the **S** tensor are symmetrical. Moreover, since the components are also real, it follows that the **S** tensor is equal to its adjoint, meaning that **S** is self-adjoint or hermitean.

It should be noted that the eigenvalues λ_m, and eigentensors $\mathbf{E}_m$, $m = 1, \ldots, 6$ of the spectral decomposition of **S** have not been assured to be real. However, the proof that all the eigenvalues of **S** are real, and that the three real directions of the eigentensors are mutually orthogonal is obtained directly from the hermitean nature of the compliance **S** fourth-rank tensor.

In addition, the three angles θ, φ and ω appearing in relations (15) and (16) are expressed as:

$$\tan\theta = \frac{\left\{B_1^2 + E_1^2\left(\frac{1}{E_1} - \lambda_1\right)^2 C_1^2\right\}^{1/2}}{A_1\left(\frac{1}{E_1} - \lambda_1\right)E_1} \tag{17a}$$

$$\tan\varphi = \frac{\left[\frac{\nu_{12}\,\nu_{23}}{E_2} + \nu_{13}\left(\frac{1}{E_2} - \lambda_1\right)\right]}{\left[\frac{\nu_{12}\,\nu_{13}}{E_1^2} + \frac{\nu_{23}}{E_2}\left(\frac{1}{E_1} - \lambda_1\right)\right]E_1} \tag{16b}$$

$$\tan\omega = \frac{\left[\dfrac{B_1^2 + E_1^2\left(\frac{1}{E_1} - \lambda_1\right)^2 C_1^2}{B_1^2 + E_1^2\left(\frac{1}{E_1} - \lambda_1\right)^2 (A_1^2 + C_1^2)} \; \dfrac{E_1^2\left(\frac{1}{E_1} - \lambda_2\right)^2 A_2^2}{B_2^2 + E_1^2\left(\frac{1}{E_1} - \lambda_2\right)^2 (A_2^2 + C_2^2)}\right]^{1/2}}{\dfrac{E_1\left(\frac{1}{E_1} - \lambda_2\right)A_2}{\left[B_2^2 + E_1^2\left(\frac{1}{E_1} - \lambda_2\right)^2 (A_2^2 + C_2^2)\right]^{1/2}}} \tag{17c}$$

in which

$$A_i = \left(\frac{1}{E_2} - \lambda_i\right)\left(\frac{1}{E_1} - \lambda_i\right) - \frac{\nu_{12}^2}{E_1^2} \tag{18a}$$

$$B_i = \frac{\nu_{12}\nu_{23}}{E_2}\left(\frac{1}{E_1} - \lambda_i\right) + \nu_{13}\left(\frac{1}{E_1} - \lambda_i\right)\left(\frac{1}{E_2} - \lambda_i\right) \tag{18b}$$

$$C_i = \frac{\nu_{12}\nu_{13}}{E_1^2} + \frac{\nu_{23}}{E_2}\left(\frac{1}{E_1} - \lambda_i\right) \tag{18c}$$

and the subscript i takes the values 1 or 2.

For the eigenvalues λ_m, $m = 1,...,6$ defined by relations (12) and the associated idempotent tensors $\mathbf{E}_m$, $m = 1,...,6$ defined by relations (13), it is valid that:

$$\mathbf{S} = \lambda_1\mathbf{E}_1 + \lambda_2\mathbf{E}_2 + \lambda_3\mathbf{E}_3 + \lambda_4\mathbf{E}_4 + \lambda_5\mathbf{E}_5 + \lambda_6\mathbf{E}_6 \tag{19}$$

Furthermore, the idempotent tensors $\mathbf{E}_m$, $m = 1,...,6$ decompose the unit element $\mathbf{I}$ of the fourth-rank symmetric tensor space and satisfy the following set of equations:

$$\mathbf{I} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{E}_4 + \mathbf{E}_5 + \mathbf{E}_6 \tag{20a}$$

$$\mathbf{E}_m \cdot \mathbf{E}_n = 0, \ m \neq n \tag{20b}$$

$$\mathbf{E}_m \cdot \mathbf{E}_m = \mathbf{E}_m \tag{20c}$$

3. REDUCTION OF THE EIGENVALUES FOR THE SYMMETRIC STATES OF ANISOTROPY

It is important noting that relations (12), (13) and (18) are general formulae expressing the spectral decomposition of the compliance tensor $\mathbf{S}$ for the general orthotropic material and, therefore, they are valid also for both the transversely isotropic (Theocaris and Philippidis, 1989) and the isotropic media.

Considering first the class of transversely isotropic media, we assume that the Cartesian coordinate system is oriented with its 33-axis normal to the isotropic (transverse) plane. Then, it is valid that: $E_1 = E_2 = E_T$, $E_3 = E_L$, $\nu_{13} = \nu_{31} = \nu_{23} = \nu_{32} = \nu_L$, $\nu_{12} = \nu_{21} = \nu_T$, $G_{13} = G_{23} = G_L$, and $G_{12} = G_T$. In this case, relations (19) and (20a) may be expressed as follows:

$$\mathbf{I} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{E}_4 \tag{21a}$$

$$\mathbf{S} = \lambda_1\mathbf{E}_1 + \lambda_2\mathbf{E}_2 + \lambda_3\mathbf{E}_3 + \lambda_4\mathbf{E}_4 \tag{21b}$$

where the eigenvalues $\lambda_1,...,\lambda_4$ obtained from the characteristic equation satisfied by **S** are given by:

$$\lambda_1 = \frac{1+\nu_T}{E_T} = \frac{1}{2G_T} \tag{22a}$$

$$\lambda_2 = \frac{(1-\nu_T)}{2E_T} + \frac{1}{2E_L} + \left\{\left[\frac{(1-\nu_T)}{2E_T} - \frac{1}{2E_L}\right]^2 + \frac{2\nu_L^2}{E_L^2}\right\}^{1/2} \tag{22b}$$

$$\lambda_3 = \frac{(1-\nu_T)}{2E_T} + \frac{1}{2E_L} - \left\{\left[\frac{(1-\nu_T)}{2E_T} - \frac{1}{2E_L}\right]^2 + \frac{2\nu_L^2}{E_L^2}\right\}^{1/2} \tag{22c}$$

$$\lambda_4 = \frac{1}{2G_L} \tag{22d}$$

and the idempotent tensors $\mathbf{E}_m$, m = 1,...,4 are defined as follows:

$$\mathbf{E}_1 = E^1_{ijkl} = 1/2\,(b_{ik}\, b_{jl} + b_{jk}\, b_{il} - b_{ij}\, b_{kl}) \tag{23a}$$

$$\mathbf{E}_2 = E^2_{ijkl} = \mathbf{j} \otimes \mathbf{j} = j_{ij}\, j_{kl} \tag{23b}$$

$$\mathbf{E}_3 = E^3_{ijkl} = \mathbf{r} \otimes \mathbf{r} = r_{ij} r_{kl} \qquad \mathbf{a}, \mathbf{b}, \mathbf{j}, \mathbf{r} \in \mathbf{L} \tag{23c}$$

$$\mathbf{E}_4 = E^4_{ijkl} = 1/2\,(b_{ik}a_{jl} + b_{il}a_{jk} + b_{jl}a_{ik} + b_{jk}a_{il}) \tag{23d}$$

Tensors **j** and **r** are axisymmetric and depend on the components of the compliance tensor **S**. They are given by:

$$\mathbf{j} = -\frac{1}{\sqrt{2}}\sin\omega\mathbf{b} + \cos\omega\mathbf{a} \tag{24a}$$

$$\mathbf{r} = \frac{1}{\sqrt{2}}\cos\omega\mathbf{b} + \sin\omega\mathbf{a} \tag{24b}$$

where the eigenangle ω is expressed by:

$$\cot 2\omega = -\frac{2\sqrt{2}\nu_L}{E_L} \Big/ \left(\frac{1-\nu_T}{E_T} - \frac{1}{E_L}\right) \tag{25}$$

and the values of the eigenangles θ and φ are equal to $\pi/2$ and $\pi/4$ respectively. The second-rank symmetric tensors **a**, **b** are defined as follows:

$$\mathbf{a} = \mathbf{k} \otimes \mathbf{k}, \quad \mathbf{a} + \mathbf{b} = \mathbf{1} \tag{26}$$

Finally, in the case of an isotropic elastic body, it is valid that: $E_1 = E_2 = E_3 = E$, $G_{12} = G_{23} = G_{13} = G$ and $\nu_{12} = \nu_{21} = \nu_{13} = \nu_{31} = \nu_{32} = \nu_{23} = \nu$. Then, relations (19) and (20a) may be written as:

$$\mathbf{I} = \mathbf{E}_1 + \mathbf{E}_3 \tag{27a}$$

$$\mathbf{S} = \lambda_1 \mathbf{E}_1 + \lambda_3 \mathbf{E}_3 \tag{27b}$$

in which

$$\lambda_1 = \lambda_2 = \frac{1}{2G}, \quad \lambda_3 = \frac{1}{3K}, \quad K = \frac{E}{3(1-2\nu)}, \quad \mathbf{E}_3 = \frac{1}{3}(\mathbf{1} \otimes \mathbf{1}) \tag{28}$$

4. STATES OF STRESS FOR AN ORTHOTROPIC MATERIAL

If the stress states $\overline{\boldsymbol{\sigma}_m}$ constitute the eigenstates of tensor **S** they should satisfy the eigenvalue equation:

$$\mathbf{S} \cdot \overline{\boldsymbol{\sigma}_m} = \lambda_m \overline{\boldsymbol{\sigma}_m} \tag{29}$$

in which the index m varies between 1 and 6, and the λ_m values are described in terms of relations (12). Therefore, the eigentensors of the orthotropic material compliance tensor, S, are derived by the orthogonal projection of a second-rank symmetric tensor $\boldsymbol{\sigma}$ on subspaces $\mathbf{L}_{\lambda_m}$, produced by the linear operators $\mathbf{E}_m$, as follows:

$$\overline{\boldsymbol{\sigma}_m} = \mathbf{E}_m : \boldsymbol{\sigma}, \qquad m = 1,\ldots,6 \tag{30}$$

Denoting by $\boldsymbol{\sigma}$ the contracted stress tensor in the form of a 6-D vector, this tensor is given by:

$$\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6]^T \tag{31}$$

Carrying out the calculations implied by relations (30), it was found that:

$$\overline{\boldsymbol{\sigma}_1} = (\sin\theta\sin\varphi(\sigma_1) - \sin\theta\cos\varphi(\sigma_2) + \cos\theta(\sigma_3)) \times [\sin\theta\sin\varphi, -\sin\theta\cos\varphi, \cos\theta, 0,0,0]^T \quad (32a)$$

$$\overline{\boldsymbol{\sigma}_2} = \{(-\sin\omega\cos\varphi - \cos\omega\cos\theta\sin\varphi)(\sigma_1) + (-\sin\omega\sin\varphi + \cos\omega\cos\theta\cos\varphi)(\sigma_2) + \cos\omega\sin\theta(\sigma_3)\} \times[-\sin\omega\cos\varphi - \cos\omega\cos\theta\sin\varphi, -\sin\omega\sin\varphi + \cos\omega\cos\theta\cos\varphi, \cos\omega\sin\theta, 0,0,0]^T \quad (32b)$$

$$\overline{\boldsymbol{\sigma}_3} = \{(\cos\omega\cos\varphi - \sin\omega\cos\theta\sin\varphi)(\sigma_1) + (\cos\omega\sin\varphi + \sin\omega\cos\theta\cos\varphi)(\sigma_2) + \sin\omega\sin\theta(\sigma_3)\} \times[\cos\omega\cos\varphi, -\sin\omega\cos\theta\sin\varphi, \cos\omega\sin\varphi + \sin\omega\cos\theta\cos\varphi, \sin\omega\sin\theta, 0,0,0]^T \quad (32c)$$

$$\overline{\boldsymbol{\sigma}_4} = [0,0,0,\sigma_4,0,0]^T \quad (32d)$$

$$\overline{\boldsymbol{\sigma}_5} = [0,0,0,0,\sigma_5,0]^T \quad (32e)$$

$$\overline{\boldsymbol{\sigma}_6} = [0,0,0,0,0,\sigma_6]^T \quad (32f)$$

It should be noted that relations (32) state that the stress eigenstates corresponding to a spectral decomposition of the compliance tensor **S** for an orthotropic medium, break down the generic stress tensor **σ** into six elements, that is:

$$\boldsymbol{\sigma} = \overline{\boldsymbol{\sigma}_1} + \overline{\boldsymbol{\sigma}_2} + \overline{\boldsymbol{\sigma}_3} + \overline{\boldsymbol{\sigma}_4} + \overline{\boldsymbol{\sigma}_5} + \overline{\boldsymbol{\sigma}_6} \quad (33)$$

As can observed, eigenvectors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ are dependent on the value of the eigenangles θ, φ and ω, given by relations (17), and the engineering elastic constants of the material. On the contrary, the remaining three eigenvectors $\overline{\boldsymbol{\sigma}_4}$, $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$ are independent of the eigenangles θ, φ and ω and the material properties, thus remaining the same for all orthotropic materials. Therefore, the six eigenvalues λ_m, m = 1,...,6, together with the eigenangles θ, φ and ω constitute the nine invariant quantities necessary for the description of the elastic behaviour of orthotropic materials.

If we now consider the definition of the strain energy density we have that:

$$\begin{aligned}2T(\boldsymbol{\sigma}) &= \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon} = \boldsymbol{\sigma}\cdot\mathbf{S}\cdot\boldsymbol{\sigma} = \\ &= (\overline{\boldsymbol{\sigma}_1} + ... + \overline{\boldsymbol{\sigma}_6}) \cdot (\lambda_1\mathbf{E}_1 + ... + \lambda_6\mathbf{E}_6) \cdot (\overline{\boldsymbol{\sigma}_1} + ... + \overline{\boldsymbol{\sigma}_6}) \\ &= \lambda_1\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_1 + ... + \lambda_6\boldsymbol{\sigma}_6 \cdot \boldsymbol{\sigma}_6 \end{aligned} \tag{34}$$

Therefore, the strain energy density is given by:

$$2T(\boldsymbol{\sigma}) = T(\overline{\boldsymbol{\sigma}_1}) + T(\overline{\boldsymbol{\sigma}_2}) + T(\overline{\boldsymbol{\sigma}_3}) + T(\overline{\boldsymbol{\sigma}_4}) + T(\overline{\boldsymbol{\sigma}_5}) + T(\overline{\boldsymbol{\sigma}_6}) \tag{35}$$

that is the elastic potential is decomposed in distinct energy components, each associated to the same eigenstress tensor. We denote by $T(\overline{\boldsymbol{\sigma}_m})$ the following quantity:

$$T(\overline{\boldsymbol{\sigma}_m}) = \lambda_m(\overline{\boldsymbol{\sigma}_m} \cdot \overline{\boldsymbol{\sigma}_m}), \qquad m = 1,...,6 \tag{36}$$

Thus, any eigenstate $\overline{\boldsymbol{\sigma}_m}$ has its own potential $T(\overline{\boldsymbol{\sigma}_m})$, which does not depend on the action of the other $\overline{\boldsymbol{\sigma}_m}$'s. From relations (34), (35) and (36) it can be seen that in order for the strain energy density T to be positive definite, the eigenvalues λ_m $(m = 1,...,6)$ of the compliance tensor **S** have to be positive definite.

Substituting relations (12) and (32) into relation (36), the following expressions are obtained for the strain energy density parts of an orthotropic medium:

$$T(\overline{\boldsymbol{\sigma}_1}) = \left\{\left[-\frac{Q}{2} + \left(\frac{Q^2}{4} - k^3\right)^{1/2}\right]^{1/3} \frac{(1 + i\sqrt{3})}{2} + \frac{k(-1 + \sqrt{3})}{2\left[-\frac{Q}{2} + \left(\frac{Q^2}{4} - k^3\right)^{1/2}\right]^{1/3}} - \frac{A}{3}\right\} \tag{37a}$$
$$\times\, [\sin\theta\sin\varphi(\sigma_1) - \sin\theta\cos\varphi(\sigma_2) + \cos\theta(\sigma_3)]^2$$

$$T(\overline{\boldsymbol{\sigma}_2}) = \left\{\left[-\frac{Q}{2} + \left(\frac{Q^2}{4} - k^3\right)^{1/2}\right]^{1/3} \frac{(-1 + i\sqrt{3})}{2} + \frac{k(-1 - i\sqrt{3})}{2\left[-\frac{Q}{2} + \left(\frac{Q^2}{4} - k^3\right)^{1/2}\right]^{1/3}} - \frac{A}{3}\right\} \tag{37b}$$
$$\times \begin{bmatrix} (-\sin\omega\cos\varphi - \cos\omega\cos\theta\sin\varphi)(\sigma_1) \\ + (-\sin\omega\sin\varphi + \cos\omega\cos\theta\cos\varphi)(\sigma_2) + \cos\omega\sin\theta(\sigma_3) \end{bmatrix}^2$$

$$T(\overline{\boldsymbol{\sigma}_3}) = \left\{ \left[-\frac{Q}{2} + \left(\frac{Q^2}{4} - k^3 \right)^{1/2} \right]^{1/3} + \frac{k}{\left[-\frac{Q}{2} + \left(\frac{Q^2}{4} - k^3 \right)^{1/2} \right]^{1/3}} - \frac{A}{3} \right\} \times \begin{bmatrix} (\cos\omega\cos\varphi - \sin\omega\cos\theta\sin\varphi)(\sigma_1) + \\ (\cos\omega\sin\varphi + \sin\omega\cos\theta\cos\varphi)(\sigma_2) + \sin\omega\sin\theta(\sigma_3) \end{bmatrix}^2 \tag{37c}$$

$$T(\overline{\boldsymbol{\sigma}_4}) = \frac{1}{2G_{23}} (\sigma_4^2) \tag{37d}$$

$$T(\overline{\boldsymbol{\sigma}_5}) = \frac{1}{2G_{13}} (\sigma_5^2) \tag{37e}$$

$$T(\overline{\boldsymbol{\sigma}_6}) = \frac{1}{2G_{12}} (\sigma_6^2) \tag{37f}$$

in which the quantities Q and k are defined in terms of Eqs. (6a), (8) and (10).

It is observed from relations (37) that the three energy components $T(\overline{\boldsymbol{\sigma}_1})$, $T(\overline{\boldsymbol{\sigma}_2})$ and $T(\overline{\boldsymbol{\sigma}_3})$ are dependent upon the values of the three eigenangles θ, φ and ω defined by relations (17), and are associated with both distortion and volume change of the medium. The last three evergy components $T(\overline{\boldsymbol{\sigma}_4})$, $T(\overline{\boldsymbol{\sigma}_5})$ and $T(\overline{\boldsymbol{\sigma}_6})$ are independent of the values of the set of eigenangles θ, φ and ω and are related exclusively to shape distortion of the medium.

5. GEOMETRIC REPRESENTATION OF THE STRESS - EIGENSTATES IN THE PRINCIPAL STRESS SPACE (AN ANALOGY WITH EULER ANGLES OF RIGID BODY MOTION)

When, the eigentensors $\overline{\boldsymbol{\sigma}_m}$, m = 1,...,6 are projected in the principal stress space $(\sigma_1,\sigma_2,\sigma_3)$, the projection of eigentensors $\overline{\boldsymbol{\sigma}_4}$, $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$ vanishes. In addition, the projections of stress eigentensors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ are represented by a set of three orthogonal vectors with associated unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ having as direction cosines:

$$\mathbf{e}_1 = [\sin\theta\sin\varphi, -\sin\theta\cos\varphi, \cos\theta]^T \tag{38a}$$

$$\mathbf{e}_2 = [-\cos\theta\sin\varphi\cos\omega - \cos\varphi\sin\omega, \cos\theta\cos\varphi\cos\omega - \sin\varphi\sin\omega, \sin\theta\cos\omega]^T \tag{38b}$$

$$\mathbf{e}_3 = [\,-\cos\theta\sin\varphi\sin\omega + \cos\varphi\cos\omega,\ \cos\theta\cos\varphi\sin\omega + \sin\varphi\cos\omega,\ \sin\theta\sin\omega]^T \quad (38c)$$

Relations (38) indicate that vectors $\mathbf{e}_i$ $(i = 1,2,3)$ are obtained by means of three separate rotations, which can be expressed in matrix form $\mathbf{A}_i$ $(i = 1, 2,3)$ as follows:

$$\mathbf{A}_1 = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (39a)$$

$$\mathbf{A}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad (39b)$$

$$\mathbf{A}_3 = \begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (39c)$$

Therefore, the product matrix $\mathbf{S}$ is expressed by:

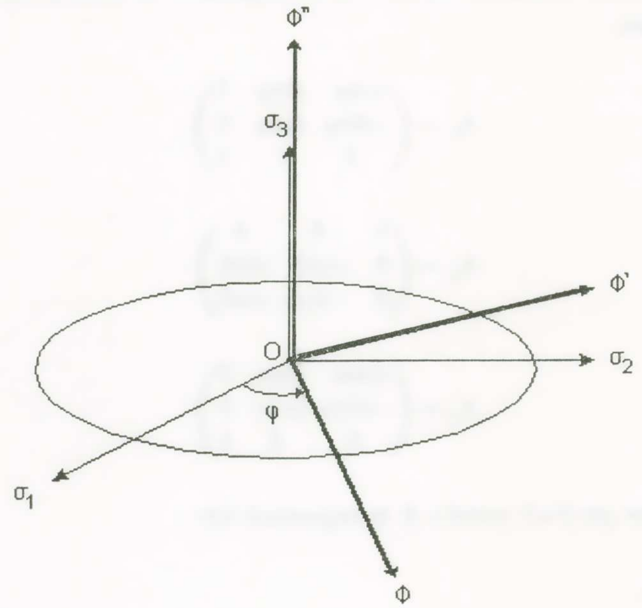
$$\mathbf{S} = \mathbf{A}_3\,\mathbf{A}_2\,\mathbf{A}_1$$

$$= \begin{bmatrix} \cos\varphi\cos\omega - \cos\theta\sin\varphi\sin\omega & \sin\varphi\cos\omega + \cos\theta\cos\varphi\sin\omega & \sin\theta\sin\omega \\ -\cos\varphi\sin\omega - \cos\theta\sin\varphi\cos\omega & -\sin\varphi\sin\omega + \cos\theta\cos\varphi\cos\omega & \sin\theta\cos\omega \\ \sin\varphi\sin\theta & -\cos\varphi\sin\theta & \cos\theta \end{bmatrix} \quad (40)$$

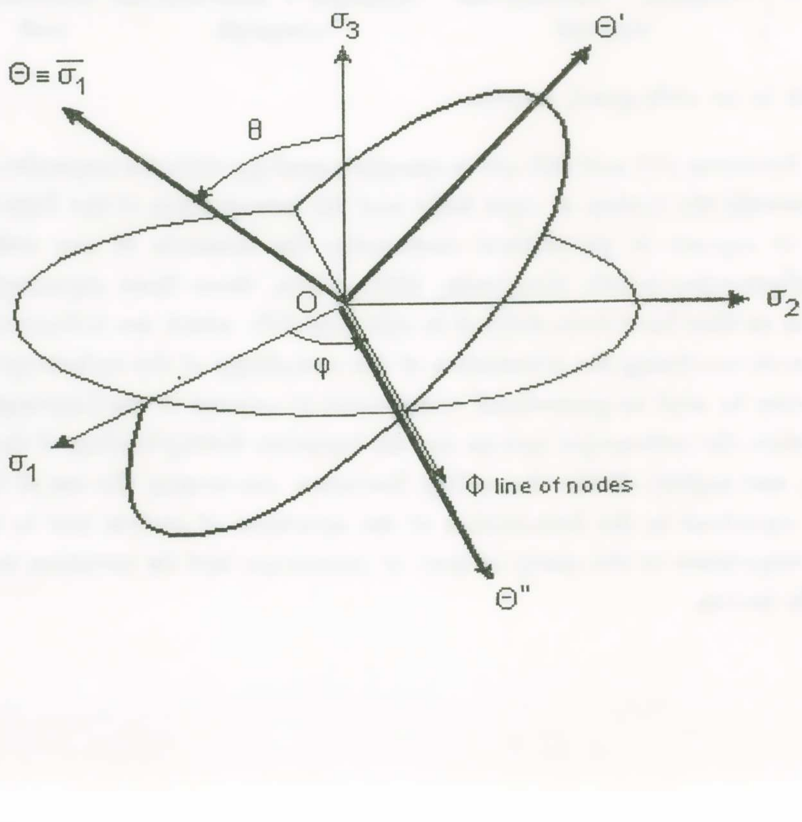
which is an orthogonal matrix.

Relations (39) and (40) are in complete analogy with the respective theory concerning the motion of rigid body and the introduction of the Eulerian angles to express in generalized coordinates the elements of any orthogonal transformation matrix (Goldstein, 1980). Then, these three eigenangles θ, φ and ω as they have been defined in relations (17), which are independent parameters specifying the orientation of the anisotropy of the orthotropic material may be used as generalized coordinates to express in the Lagrangian formulation the orthotropic system and its variation during loading of the structure, and exploit all the abounding literature concerning the use of Lagrangian equations in the formulation of the equations of motion and to transfer this experience to the study of laws of anisotropy and its variation in orthotropic bodies.

Indeed, it is possible to carry out the transformation from a given Cartesian coordinate system to another by means of three successive angular displacements φ, θ, ω performed in a specific sequence. The sequence will be started



(a)



(b)

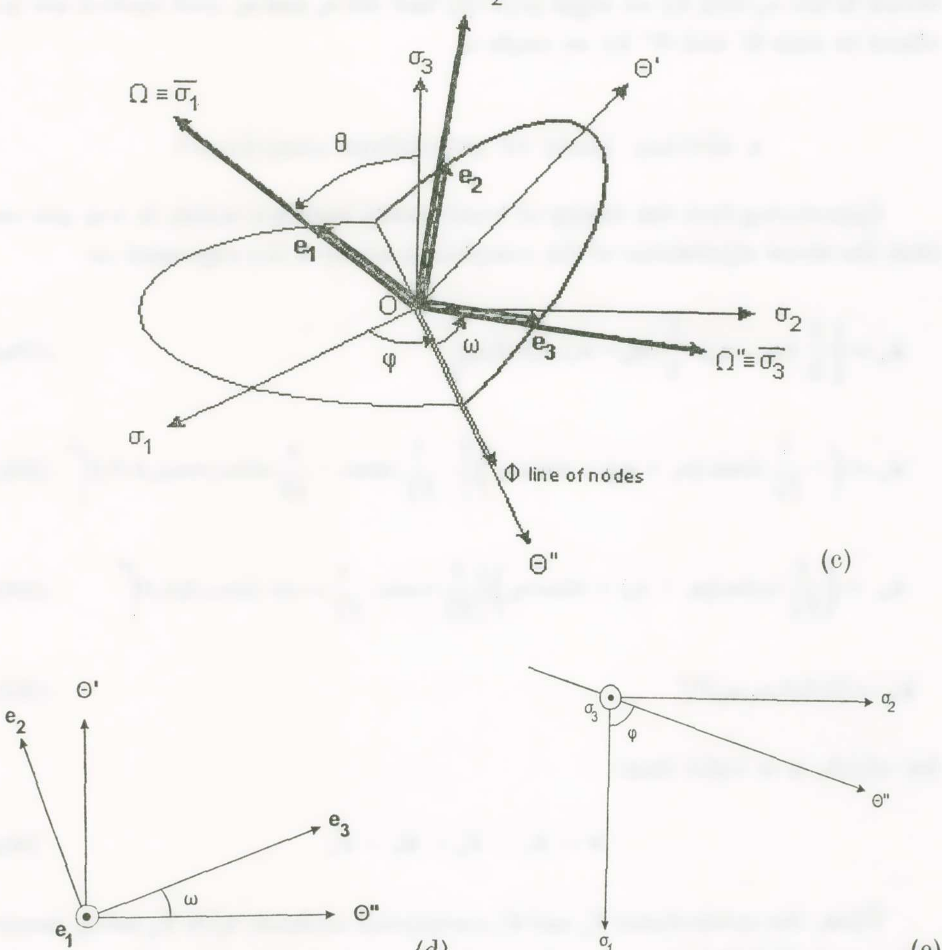


Fig. 1. Geometric representation of the eigenvectors of the orthotropic compliance tensor in the principal stress place (σ_1, σ_2, σ_3).

by rotating the initial frame $O\sigma_1\sigma_2\sigma_3$ by an angle φ counterclockwise about the $O\sigma_3 \equiv O\Phi''$ - axis. The resultant $O\Phi\Phi'\Phi''$ - coordinate system is now rotated about the $O\Phi \equiv O\Theta''$ - axis by an angle θ thus forming the subsequent system $O\Theta\Theta'\Theta''$, which is finally rotated about the $O\Omega$-axis by an angle ω thus forming the final frame $O\overline{\boldsymbol{\sigma}_1}\ \overline{\boldsymbol{\sigma}_2}\ \overline{\boldsymbol{\sigma}_3} \equiv O\Omega\Omega'\Omega''$, displaced to a new position. Hence, as seen in Fig. 1, the unit vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ lie on plane $O\Theta'\Theta''$, which is inclined to plane $O\sigma_3\Theta''$ by an angle $(\pi/2-\theta)$. In addition, the Θ''-axis is inclined to the σ_2-axis by an angle $(\pi/2-\varphi)$, and the $\mathbf{e}_2$ and $\mathbf{e}_3$ unit vectors are inclined to axes Θ' and Θ'' by an angle ω.

6. SPECIAL CASES OF SYMMETRIC ANISOTROPY

Considering first the family of transversely isotropic solids, it was proven that the stress eigenstates of the compliance tensor **S** are expressed as:

$$\overline{\boldsymbol{\sigma}_1} = \left[\frac{1}{2}(\sigma_1 - \sigma_2), \frac{1}{2}(\sigma_2 - \sigma_1), 0,0,0,\sigma_6\right]^T \tag{39α}$$

$$\boldsymbol{\sigma}_2 = \left(-\frac{1}{\sqrt{2}}\sin\omega\,(\sigma_1 + \sigma_2) + \cos\omega\sigma_3\right)\left[-\frac{1}{\sqrt{2}}\sin\omega, -\frac{1}{\sqrt{2}}\sin\omega, \cos\omega, 0,0,0\right]^T \tag{39b}$$

$$\boldsymbol{\sigma}_3 = \left(\frac{1}{\sqrt{2}}\cos\omega\,(\sigma_1 + \sigma_2) + \sin\omega\sigma_3\right)\left[\frac{1}{\sqrt{2}}\cos\omega, \frac{1}{\sqrt{2}}\cos\omega, \sin\omega, 0,0,0\right]^T \tag{39c}$$

$$\boldsymbol{\sigma}_4 = [0,0,0,\sigma_4,\sigma_5,0]^T \tag{39d}$$

for which, it is valid that:

$$\boldsymbol{\sigma} = \overline{\boldsymbol{\sigma}_1} + \overline{\boldsymbol{\sigma}_2} + \overline{\boldsymbol{\sigma}_3} + \overline{\boldsymbol{\sigma}_4} \tag{40}$$

Then, the stress states $\overline{\boldsymbol{\sigma}_1}$ and $\overline{\boldsymbol{\sigma}_4}$ correspond to shear, with $\overline{\boldsymbol{\sigma}_4}$ being simple shear, and $\overline{\boldsymbol{\sigma}_1}$ being a superposition of pure and simple shear. Moreover, the $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ states constitute an equilateral stressing in the transverse plane, together with a tension or compression along the longitudinal axis of symmetry. In addition, if we consider the projections of the stress eigenstates $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$, $\overline{\boldsymbol{\sigma}_3}$ and $\overline{\boldsymbol{\sigma}_4}$ in the principal stress space $(\sigma_1, \sigma_2, \sigma_3)$, then the $\overline{\boldsymbol{\sigma}_4}$ state vanishes whereas the stress states $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ are represented by three mutually orthogonal unit vectors, with the following direction cosines:

$$\mathbf{e}_1 = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right]^T \tag{41a}$$

$$\mathbf{e}_2 = \left[-\frac{\sin\omega}{\sqrt{2}}, -\frac{\sin\omega}{\sqrt{2}}, \cos\omega \right]^T \tag{41b}$$

$$\mathbf{e}_3 = \left[\frac{\cos\omega}{\sqrt{2}}, \frac{\cos\omega}{\sqrt{2}}, \sin\omega \right]^T \tag{41c}$$

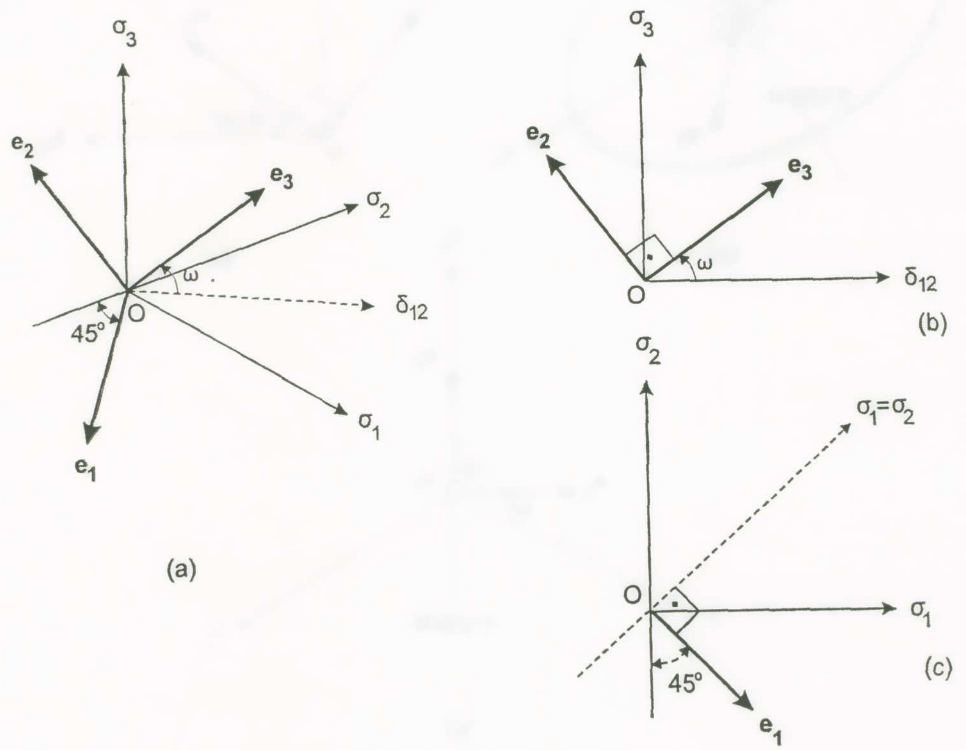


Fig. 2. Geometric representation of the eigenvectors of the transversely isotropic compliance tensor in the principal stress space $(\sigma_1, \sigma_2, \sigma_3)$.

It is noted in Fig. 2 that the unit vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ are equally inclined to axes σ_1 and σ_2, thus lying on the principal diagonal plane (σ_3, δ_{12}). In addition, vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ subtend angles equal to $(\omega - \pi/2)$ and $(\pi - \omega)$ with respect to the σ_3 axis, whereas vector $\mathbf{e}_1$ is perpendicular to the σ_3-axis, thus lies on the intersection of the deviatoric π-plane and the plane $\sigma_3 = 0$.

Finally, for the isotropic body, it was shown that according to relations (41), the unit vector $\mathbf{e}_3$ lies on the positive direction of the hydrostatic axis,

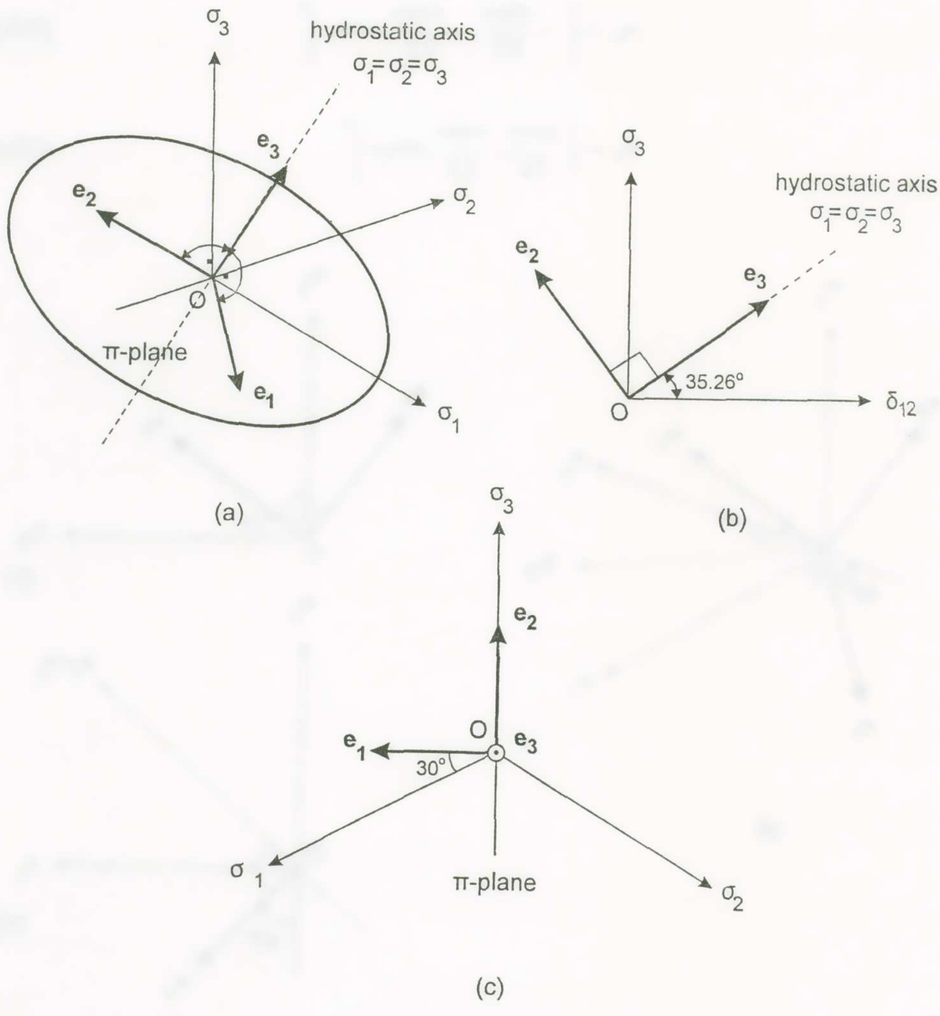


Fig. 3. Geometric representation of the eigenvectors of the transversely isotropic compliance tensor in the principal stress place $(\sigma_1, \sigma_2, \sigma_3)$.

whereas the $\mathbf{e}_2$ vector lies on the deviatoric plane, and both these vectors lie on the main diagonal plane $\sigma_1 = \sigma_2$.

7. DISCUSSION

The spectral decomposition of the elastic compliance fourth-rank tensor **S** for orthotropic materials permits the separation of the stress and strain tensors in energy-orthogonal components. Furthermore, the six eigenvalues of the compliance tensor **S**, together with the values of the three eigenangles θ, φ and ω, may be used for an invariant description of the elastic behaviour of orthotropic media.

The decomposition of the stress tensor **σ** obtained for orthotropic materials, yielded six energy-orthogonal stress states, which separate the elastic strain energy directly. The stress tensor may thus be effectively described by eigentensors $\overline{\boldsymbol{\sigma}_m}$, m = 1,...,6. Additionally, the normality of the eigentensors of stress and strain corresponding to a different than the former stress-eigentensor was shown.

However, the decomposition of the elastic potential which is valid for the isotropic body, is not valid for the anisotropic medium:

$$T(\boldsymbol{\sigma}) = \frac{1}{18K}(\mathrm{tr}\boldsymbol{\sigma})^2 + \frac{1}{2G}\left[\mathrm{tr}\boldsymbol{\sigma}^2 - \frac{1}{3}(\mathrm{tr}\boldsymbol{\sigma})^2\right] \qquad (42)$$

Hence, it was shown that an extension of the decomposition of the elastic strain energy density into dilatational and distortional components (Olszak and Urbanowski, 1956; Olszak and Maciejewska, 1985), valid for the isotropic medium, as well as for cubic crystals is impossible for the orthotropic medium, since the strain eigentensors of its compliance tensor **S** do not include the spherical tensor **1**. Hence, the Huber-Mises-Hencky criterion cannot be generalized to hold for anisotropic media and the distortional component of the strain energy density cannot be established as the critical failure quantity.

The projection of the stress eigentensors $\overline{\boldsymbol{\sigma}_m}$, m = 1,...,6 in the principal stress space (σ_1, σ_2, σ_3) yields a zero value for eigentensors $\overline{\boldsymbol{\sigma}_4}$, $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$, whereas the projections of stress eigentensors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ are represented by a set of three orthonormal vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. It was shown that the unit vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ lie on plane OΘ′Θ″, which is inclined to plane $O\sigma_3\Theta''$ by an angle (π/2 − θ). Furthermore, the Θ″-axis is inclined to the σ_2-axis by an angle (π/2 − φ), and the $\mathbf{e}_2$ and $\mathbf{e}_3$ unit vectors are inclined to axes Θ′ and Θ″ by an angle ω.

It was then proved that the eigenangles θ, φ and ω for an isotropic body are equal to π/2, π/4 and 35.26°, whereas for a transversely isotropic medium the values of θ and φ are equal to π/2, π/4 respectively. In the latter case, the unit vector $\mathbf{e}_1$ is vertical to both the principal diagonal plane (σ_3,δ_{12}) and the

σ_3-axis. Hence, it lies on the intersection of the deviatoric π-plane and the principal stress plane (σ_1,σ_2) and this is true for the whole class of transversely isotropic bodies. In addition, the unit vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ lie on the principal diagonal plane (σ_3,δ_{12}), and subtend angles equal to $(\omega-\pi/2)$ and $(\pi-\omega)$ with respect to the σ_3 axis. Ultimately, for the isotropic body, the $\mathbf{e}_2$ vector belongs to the deviatoric plane, whereas the unit vector $\mathbf{e}_3$ lies parallel to the hydrostatic axis, and both lie on the principal diagonal plane (σ_3,δ_{12}).

Finally, it was shown that the eigenangles θ, φ and ω, describing the alignment of the anisotropy of the orthotropic material are of equal value to the Euler-angles, which are essential for the characterization of rigid body motion in the Lagrangian formulation of mechanics. Consequently, these three eigenangles may be utilized as generalized coordinates valid for the representation of the orthotropic system and its variation during loading of the structure. Hence, the laws of anisotropy and its variation in orthotropic bodies may be investigated taking into consideration the acquired experience concerning the use of Lagrangian equations in the formulation of the equations of motion.

ΠΕΡΙΛΗΨΙΣ

Φασματικὴ ἀνάλυσις τοῦ τανυστοῦ τετάρτης τάξεως ἐνδόσεως S ὀρθοτρόπων μέσων

Ἡ θεωρία τῆς φασματικῆς ἀναλύσεως τοῦ τανυστοῦ ἐνδόσεως **S** ἐφαρμόζεται εἰς τὸ ἄρθρον αὐτὸ διὰ τὸ ὀρθότροπον στερεόν. Ὁρίζεται πρὸς τοῦτο καρτεσιανὸν σύστημα συντεταγμένων πρὸς τὸ ὁποῖον ἀναφέρονται αἱ συνιστῶσαι τοῦ τανυστοῦ **S** καὶ τοῦ ὁποίου αἱ διευθύνσεις ταυτίζονται μὲ τὰς κυρίας διευθύνσεις τοῦ μέσου. Δι᾽ αὐτὸ τὸ σύστημα ἀναφορᾶς εὑρίσκονται αἱ συνιστῶσαι τοῦ τανυστοῦ ἐνδόσεως S_{ijkl} ὡς συναρτήσεις τῶν ἐλαστικῶν μέτρων καὶ τῶν λόγων Poisson τοῦ μέσου, καὶ ὑπολογίζονται αἱ ἐκφράσεις τῶν ἕξι ἰδιοτιμῶν τοῦ τανυστοῦ ἐνδόσεως **S**.

Περαιτέρω δίδεται τὸ σύνολον τῶν τανυστῶν $\{\mathbf{E}_N\}$ οἱ ὁποῖοι ἀναλύουν τὸν μοναδιαῖον τανυστὴν **I**. Οἱ τανυσταὶ αὐτοὶ ἀνέρχονται εἰς ἕξι καὶ ὁρίζονται ἀπὸ τοὺς συμμετρικοὺς τανυστὰς **a**, **b** καὶ **c**, καθὼς ἐπίσης καὶ ἀπὸ τοὺς συμμετρικοὺς τανυστὰς **h**, **j** καὶ **r**, ἐξηρτημένους ἀπὸ τὰς συνιστῶσας τοῦ τανυστοῦ ἐνδόσεως **S**. Τέλος, δεικνύεται ὅτι αἱ ἰδιοτιμαὶ καὶ οἱ ἰδιοτανυσταὶ $\{\mathbf{E}_N\}$ τοῦ τετάρτης τάξεως τανυστοῦ ἐνδόσεως εἶναι πραγματικοὶ ἀριθμοὶ καὶ ἐπιπλέον ὅτι αἱ πραγματικαὶ διευθύνσεις τῶν

ἰδιοτανυστῶν $\{\mathbf{E}_N\}$ εἶναι κάθετοι ἐπ' ἀλλήλοις βάσει τοῦ ἑρμητιανοῦ χαρακτῆρος τοῦ τανυστοῦ ἐνδόσεως.

Κατ' αὐτὸν τὸν τρόπον ὁρίζεται πλήρως ἡ φασματικὴ ἀνάλυσις τοῦ τανυστοῦ ἐνδόσεως **S**. Ἐπιπλέον, ὁρίζονται τρεῖς παράμετροι, θ, φ καὶ ω, αἱ ὀνομαζόμεναι ὡς **ἰδιογωνίαι**, αἱ ὁποῖαι ἐκφράζονται συναρτήσει τῶν ἐλαστικῶν σταθερῶν τοῦ μέσου. Τέλος ἀποδεικνύεται ὅτι αἱ ἀναγκαῖαι ποσότητες διὰ τὴν ὁλοκληρωμένην περιγραφὴν τῶν ἐλαστικῶν χαρακτηριστικῶν παντὸς ὀρθοτρόπου μέσου εἶναι αἱ ἕξι ἰδιοτιμαὶ τοῦ τανυστοῦ ἐνδόσεως, μετὰ τῶν ἰδιογωνιῶν θ, φ καὶ ω.

Αἱ ἐκφράσεις τῶν ἰδιοτιμῶν καὶ τῶν ἰδιοτανυστῶν τοῦ τανυστοῦ ἐνδόσεως **S** τοῦ ὀρθοτρόπου σώματος εἶναι γενικαὶ ἐκφράσεις, ἰσχύουσαι καὶ διὰ τὴν κατηγορίαν τῶν ἐγκαρσίως ἰσοτρόπων καὶ τῶν ἰσοτρόπων ἐλαστικῶν μέσων. Ἀρχικῶς, εἰς τὴν περίπτωσιν τῶν ἐγκαρσίως ἰσοτρόπων ὑλικῶν, αἱ ἰδιοτιμαὶ ἀνέρχονται εἰς τέσσαρας, ἐκ τῶν ὁποίων δύο εἶναι διπλῆς πολλαπλότητος. Περαιτέρω τὸ σύνολον τῶν τανυστῶν $\{\mathbf{E}_N\}$ ἀνέρχονται πάλιν εἰς τέσσαρας καὶ ὁρίζονται ἀπὸ τοὺς συμμετρικοὺς τανυστὰς **a** καὶ **b** καθὼς καὶ ἀπὸ τοὺς τανυστὰς **j** καὶ **r**, ἐξηρτημένους ἀπὸ τὰς συνιστώσας τοῦ τανυστοῦ ἐνδόσεως **S**. Τελικῶς, εἰς τὴν κατηγορίαν τῶν ἰσοτρόπων σωμάτων, αἱ ἰδιοτιμαὶ καὶ οἱ ἰδιοτανυσταὶ $\{\mathbf{E}_N\}$ ἀριθμοῦν δύο εἰς τὸ σύνολον.

Διὰ τὸ ὀρθότροπον ἐλαστικὸν στερεόν, τυχαῖος τανυστὴς τάσεως δύναται νὰ ἀναλυθῆ σὲ ἕξι ἰδιοτανυστὰς $\overline{\boldsymbol{\sigma}}_1$, $\overline{\boldsymbol{\sigma}}_2$, $\overline{\boldsymbol{\sigma}}_3$, $\overline{\boldsymbol{\sigma}}_4$, $\overline{\boldsymbol{\sigma}}_5$ καὶ $\overline{\boldsymbol{\sigma}}'_6$, οἱ ὁποῖοι ὁρίζονται συναρτήσει τῶν σ_1, σ_2, σ_3, σ_4, σ_5 καὶ σ_6 συνιστωσῶν τῶν τάσεων καὶ τῶν ἰδιογωνιῶν θ, φ καὶ ω. Ἀποδεικνύεται ὅτι οἱ ἰδιοτανυσταὶ $\overline{\boldsymbol{\sigma}}_1$, $\overline{\boldsymbol{\sigma}}_2$ καὶ $\overline{\boldsymbol{\sigma}}_3$ ἐξαρτῶνται ἐκ τῶν τριῶν γωνιῶν θ, φ καὶ ω, ἐνῶ ἀντιθέτως οἱ ἰδιοτανυσταὶ $\overline{\boldsymbol{\sigma}}_4$, $\overline{\boldsymbol{\sigma}}_5$ καὶ $\overline{\boldsymbol{\sigma}}_6$ εἶναι ἀποκλίνοντες καὶ παραμένουν σταθεροὶ δι' ὅλα τὰ ὀρθότροπα σώματα. Ἑπομένως, ἀποδεικνύεται ὅτι ἡ φασματικὴ ἀνάλυσις τοῦ τανυστοῦ ἐνδόσεως δίδει τρεῖς ἰδιοτανυστὰς $\overline{\boldsymbol{\sigma}}_1$, $\boldsymbol{\sigma}_2$ καὶ $\overline{\boldsymbol{\sigma}}_3$ ποὺ παριστοῦν συνδυασμοὺς διογκωτικῆς καὶ στροφικῆς ἐλαστικῆς ἐνεργείας, καὶ ἄλλους τρεῖς τανυστὰς $\overline{\boldsymbol{\sigma}}_4$, $\overline{\boldsymbol{\sigma}}_5$ καὶ $\overline{\boldsymbol{\sigma}}_6$, οἱ ὁποῖοι συνδέονται ἀποκλειστικῶς μὲ τὴν στροφικὴν ἐνέργειαν.

Ἐν συνεχείᾳ δεικνύεται ὅτι ἡ προβολὴ τῶν ἕξι ἰδιοτανυστῶν τοῦ ὀρθοτρόπου μέσου στὸν τρισδιάστατον χῶρον τῶν κυρίων τάσεων μηδενίζει τοὺς τανυστὰς $\overline{\boldsymbol{\sigma}}_4$, $\overline{\boldsymbol{\sigma}}_5$ καὶ $\overline{\boldsymbol{\sigma}}_6$. Ἐπιπλέον, αἱ προβολαὶ τῶν τανυστῶν $\overline{\boldsymbol{\sigma}}_1$, $\overline{\boldsymbol{\sigma}}_2$ καὶ $\overline{\boldsymbol{\sigma}}_3$ παριστοῦν τρία κάθετα μεταξύ των διανύσματα, αἱ διευθύνσεις τῶν ὁποίων καθορίζονται ἐκ τῶν μοναδιαίων διανυσμάτων $\mathbf{e}_1$, $\mathbf{e}_2$ καὶ $\mathbf{e}_3$. Τοιουτοτρόπως ἀποδεικνύεται ὅτι τὰ μοναδιαῖα διανύσματα $\mathbf{e}_2$ καὶ $\mathbf{e}_3$ εὑρίσκονται εἰς τὸ ἐπίπεδον ΟΘ'Θ'', τὸ ὁποῖον σχηματίζει μὲ τὸ ἐπίπεδον $O\sigma_3\Theta''$ γωνίαν ἴσην μὲ $(\pi/2-0)$. Ἐπιπροσθέτως, ὁ Θ''-ἄξων σχηματίζει μὲ τὸν σ_2-ἄξονα γωνίαν ἴσην μὲ $(\pi/2-\varphi)$, καὶ τὰ μοναδιαῖα διανύσματα $\mathbf{e}_2$ καὶ $\mathbf{e}_3$ σχηματίζουν γωνίαν ω μὲ τοὺς ἄξονας Θ' καὶ Θ''.

Αἱ τιμαὶ τῶν ἰδιογωνιῶν θ, φ καὶ ω αἱ ὁποῖαι εὑρέθησαν διὰ τὸ ἰσότροπον σῶμα ἰσοῦνται μὲ π/2, π/4 καὶ 35.26^0 ἀντιστοίχως, ἐνῶ διὰ τὸ ἐγκαρσίως ἰσότροπον στερεὸν αἱ τιμαὶ τῶν γωνιῶν θ καὶ φ εἶναι π/2 καὶ π/4. Εἰς τὴν τελευταίαν περίπτωσιν, τὸ μοναδιαῖον διάνυσμα $\mathbf{e}_1$ εἶναι κάθετον τόσον εἰς τὸ ἐπίπεδον (σ_3,δ_{12}) ὅσον καὶ εἰς τὸν ἄξονα σ_3. Κατὰ συνέπειαν, κεῖται ἐπὶ τῆς τομῆς τοῦ ἀποκλίνοντος π-ἐπιπέδου καὶ τοῦ ἐπιπέδου (σ_1,σ_2), γεγονὸς τὸ ὁποῖον ἰσχύει διὰ κάθε ἐγκαρσίως ἰσότροπον μέσον καθὼς ἐπίσης καὶ διὰ τὸ ἰσότροπον στερεόν. Ἐπιπλέον, τὰ μοναδιαῖα διανύσματα $\mathbf{e}_2$ καὶ $\mathbf{e}_3$ εὑρίσκονται ἐπὶ τοῦ ἐπιπέδου (σ_3,δ_{12}), καὶ σχηματίζουν γωνίας ἴσας μὲ $(\omega - \pi/2)$ καὶ $(\pi - \omega)$ μὲ τὸν ἄξονα σ_3. Τέλος, διὰ τὸ ἰσότροπον σῶμα, τὸ μὲν διάνυσμα $\mathbf{e}_2$ εὑρίσκεται ἐπὶ τοῦ ἀποκλίνοντος ἐπιπέδου, τὸ δὲ μοναδιαῖον διάνυσμα $\mathbf{e}_3$ κατέχει τὴν θετικὴν διεύθυνσιν τοῦ ὑδροστατικοῦ ἄξονος, καὶ τὰ δύο εὑρίσκονται ἐπὶ τοῦ ἐπιπέδου (σ_3,δ_{12}).

Τέλος, ἀποδεικνύεται ὅτι αἱ ἰδιογωνίαι θ, φ καὶ ω, αἱ ὁποῖαι ὁρίζονται ὡς αἱ ἀναγκαῖαι παράμετροι διὰ τὴν περιγραφὴν τῆς ἀνισοτροπίας τοῦ ὀρθοτρόπου σώματος, δύναται νὰ χρησιμοποιηθοῦν ὡς γενικευμέναι συντεταγμέναι διὰ τὴν περιγραφὴν κατὰ Lagrange τοῦ ὀρθοτρόπου συστήματος καὶ τῆς μεταβολῆς αὐτοῦ κατὰ τὴν φόρτισίν του. Συνεπῶς καθίσταται δυνατὴ ἡ ἐφαρμογὴ τῶν ἐξισώσεων τοῦ Lagrange ποὺ περιγράφουν τὰς ἐξισώσεις κινήσεως ἑνὸς στερεοῦ σώματος διὰ τὴν μελέτην τῶν νόμων τῆς ἀνισοτροπίας, καθὼς καὶ τῆς μεταβολῆς αὐτῆς εἰς τὰ ὀρθότροπα στερεὰ σώματα.

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