ΠΡΑΚΤΙΚΑ ΤΗΣ ΑΚΑΔΗΜΙΑΣ ΑΘΗΝΩΝ

ΣΥΝΕΔΡΙΑ ΤΗΣ 6ΗΣ ΝΟΕΜΒΡΙΟΥ 1986

ΠΡΟΕΔΡΙΑ ΚΩΝΣΤΑΝΤΙΝΟΥ ΤΡΥΠΑΝΗ

ΜΗΧΑΝΙΚΗ.– **The T-criterion and its corollaries in fracture mechanics**, *by P. S. Theocaris**.

Any fracture criterion should depend directly on the loading mode of the structure and therefore it should be a function of active components of the stress tensor defining the elastic field at the vicinity of the crack tip. Then, criteria based, directly, only on a single component of stress or strain should be considered as deficient. Thus, all criteria based either on the maximum tangential stress, or the maximum principal tangential stress or strain, cannot be considered as deriving from some particular constitutive principle of mechanics, their justification being based only on the fact that their results corroborate more or less experimental evidence. However, experimental justification of any fracture criterion is not sufficient, if this is not backed up by some basic law of physics.

The fact irself that only one component of the stress —or the strain— tensor defines explicitly a fracture criterion is a sufficient reason to doubt in advance for the soundness of this criterion, in view of the fact that, even for uniaxial loading, the stress field around the crack tip is always strongly multiaxial.

On the other hand, all criteria based on local energy density concepts, evaluated at the vicinity of the crack tip, are based on the physical princi-

* Π. Σ. ΘΕΟΧΑΡΗ, **Τὸ T-κριτήριον καὶ τὰ παράγωγα αὐτοῦ κριτήρια εἰς τὴν μηχανικὴν τῶν θραύσεων.**

ple that the energy released during the crack initiation of propagation must be an extremum. This is because, for equilibrium at the crack tip at some particular instant, the Lagrangian function of energies there, should be a minimum, in order to satisfy the instantaneous equilibrium of the system[**1**]. Then, a crack is choosing its own path, which must minimize the Lagrangian function, and therefore it should release the maximum energy for the system.

Since in the majority of fracture criteria the stress fields around the cracks are assumed as totally elastic, the stresses at the crack tip are assumed

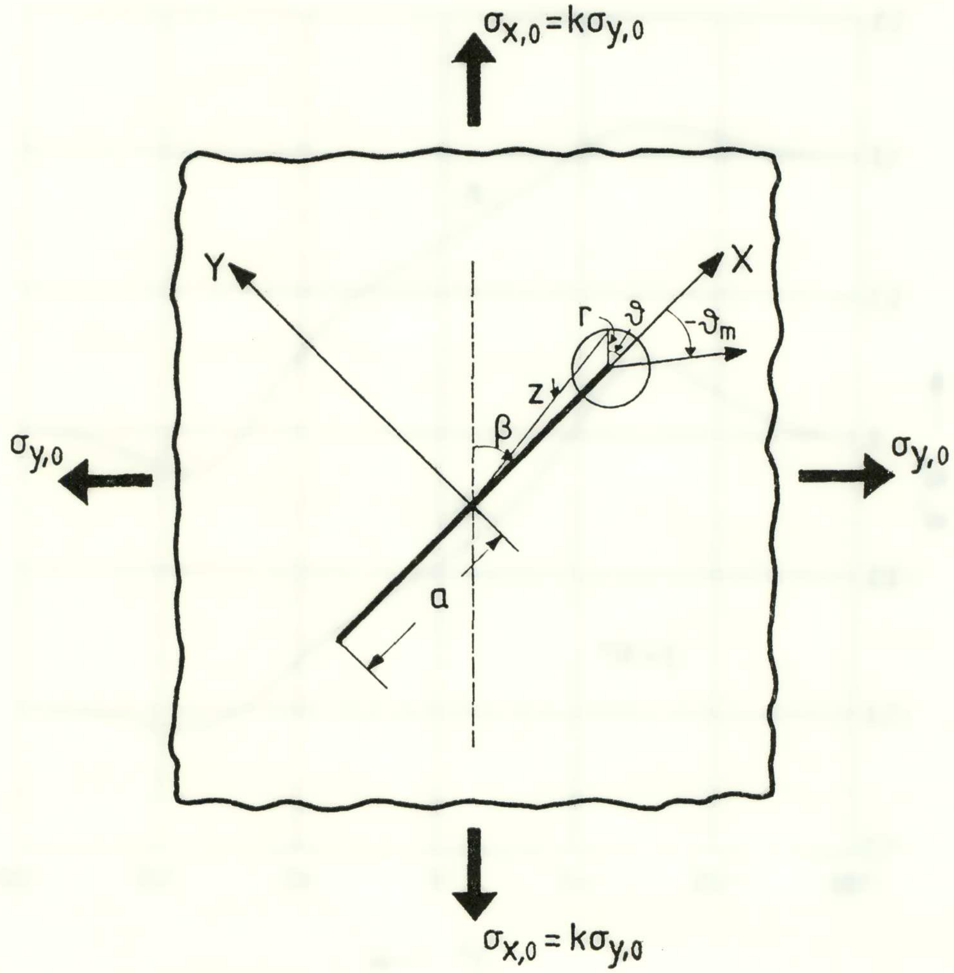

Fig. 1 The geometry of the vicinity of the crack tip and its core region.

as singular in all directions. Then, the concepts of stresses acquire a physical meaning only at some short distance from the crack tip and therefore fracture criteria require that the crack extends in a radial direction, which is defined along a circle of an infinitesimal radius, such that this *core region* may be accurately defined by the extent of dominance of the elastic stress singularity. This circle for the maximum strain energy release rate criterion is substantiated by the circular path, along which the J-integral is normally evaluated (see Fig. 1).

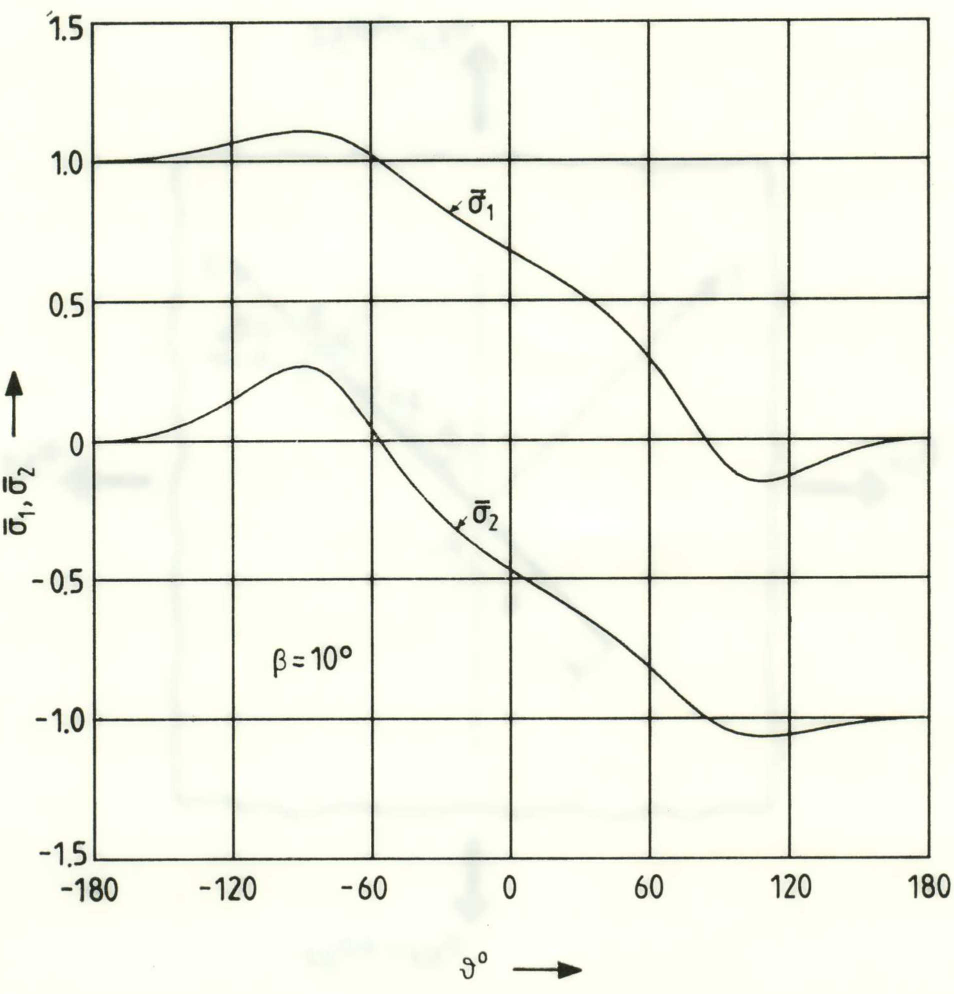

Fig. 2 Angular variation of σ_{11}, σ_{22}-principal stresses along the Mises elastic-plastic boundary for $\beta = 10°$.

Only the T-criterion [**2**] accepts the realistic view that, even for the brittlest of the materials before any initiation of crack propagation, a small plastic zone is developed around the crack tip. Considering that this small plastic enclave may be represented by the initial yield locus, it accepts this zone as the core region used in the other criteria. Then, this circular domain is transformed to the more realistic kidney-shaped plastic enclave.

By evaluating the two components of the elastic strain energy density, that is the distortional, T_D, and the dilatational, T_V, components along this contour and by assuming the validity of the Mises yield condition, it takes advantage of the fact that the T_D-component is constant along this curve and, therefore, it seeks for a maximum for the other energy density component. Then, the T_V-maximum assures the maximalization of the total strain energy density along a particular direction, which defines the instantaneous direction of propagation of the crack [**3**, **4**].

The rather small deviations of the results derived from the various fracture criteria imply that there is an eventual affinity between them, allowing only small overall discrepancies between criteria, which are accentuated only in some characteristic regions depending on the mode of loading of the cracked plate.

It is well known that the T_D- and T_V-components of elastic strain energy density around an ideal crack tip, for a plate under plane-stress conditions, are expressed by:

$$T_D = \frac{(1+\nu)}{3E}\{\sigma^2_{xx} + \sigma^2_{yy} - \sigma_{xx}\sigma_{yy} + 3\tau^2_{xy}\} \quad (1)$$

$$T_V = \frac{(1-2\nu)}{6E}\{\sigma_{xx} + \sigma_{yy}\}^2 \quad (2)$$

where σ_{ij} are the Cartesian components of the stress tensor (i, j=1, 2, 3= =x, y, z) and E and ν the elastic modulus and Poisson's ratio of the material of the plate.

Then, the T_D-component of SED may be written as:

$$T_D = \frac{(1+\nu)}{3E}\{(\sigma_{xx} + \sigma_{yy})^2 - 3(\sigma_{xx}\sigma_{yy} - \tau^2_{xy})\} \quad (3)$$

Introducing relation (3) into Eq. (2) one finds that:

$$T_V = \frac{(1-2\nu)}{2(1+\nu)} T_D + \frac{1-\nu}{2E} (\sigma_{xx}\sigma_{yy} - \tau^2_{xy}) \tag{4}$$

If one searches for a maximum of the T_V-component of SED along an elastic-plastic boundary, where the T-criterion defines the critical ϑ-angle of crack propagation and the critical load P_{cr} for initiation of fracture, one has to find

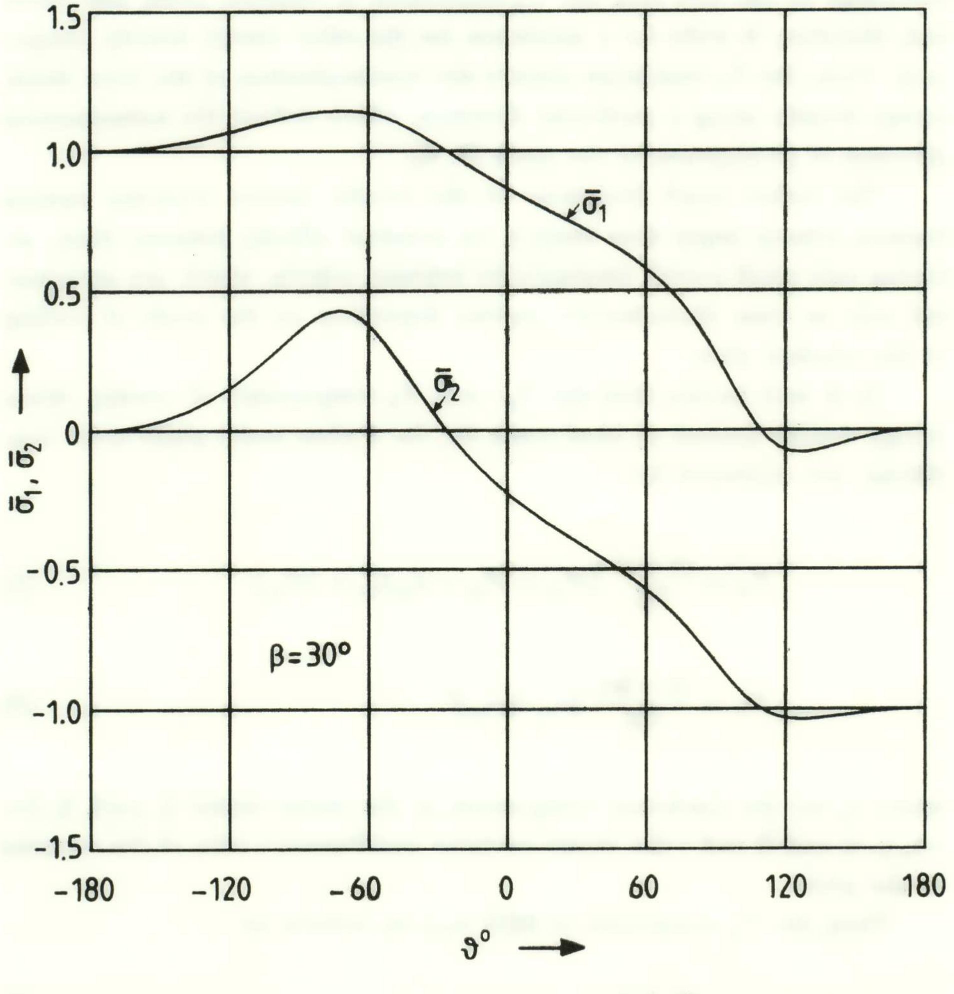

Fig. 3 Angular variation of σ_{11}, σ_{22}-principal stresses along the Mises elastic-plastic boundary for $\beta=30°$.

the $\partial T_V/\partial\vartheta$-derivative along a line where T_D=const. Performing this differentiation one finds that:

$$\frac{\partial T_V}{\partial\vartheta}=\frac{(1-2\nu)}{2E}\;\frac{\partial(\sigma_{xx}\sigma_{yy}-\tau^2_{xy})}{\partial\vartheta} \tag{5}$$

The quantity $(\sigma_{xx}\sigma_{xy}-\tau^2_{xy})$ equals the determinant of the stress components σ_{xx}, σ_{yy}, τ_{xy} in plane stress and it is well known that the following relation is valid:

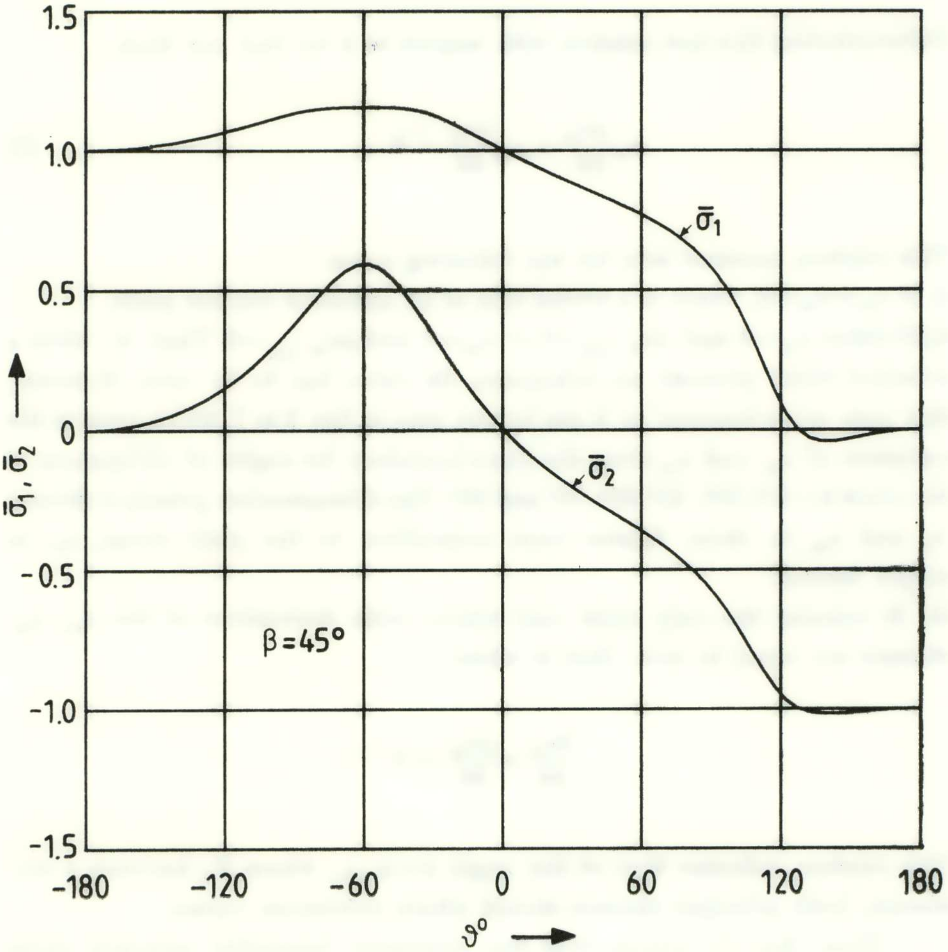

Fig. 4 Angular variation of σ_{11}. σ_{22}-principal stresses along the Mises elastic-plastic boundary for $\beta=45°$.

$$(\sigma_{xx}\sigma_{yy} - \tau^2_{xy}) = \sigma_{11}\sigma_{22} \tag{6}$$

where σ_{11}, σ_{22} are the components of principal stress tensor around the crack tip.

Then, in order to find a maximum for the dilatational component of elastic strain energy density, T_V, it is necessary to have:

$$\frac{\partial T_V}{\partial \vartheta} = \frac{(1-2_v)}{2E} \frac{\partial (\sigma_{11}\sigma_{22})}{\partial \vartheta} = 0$$

Differentiating this last relation with respect to ϑ we find out that:

$$\sigma_{11}\frac{\partial \sigma_{22}}{\partial \vartheta} + \sigma_{22}\frac{\partial \sigma_{11}}{\partial \vartheta} = 0 \tag{7}$$

This relation becomes zero for the following cases:

i) If $\sigma_{11}=\sigma_{22}=0$, which is a trivial case of an unloaded cracked plate.

ii) If either $\sigma_{11}=0$ and $\partial\sigma_{11}/_{\partial\vartheta}=0$ or $\sigma_{22}=0$ and $\partial\sigma_{22}/_{\partial\vartheta}=0$. That is, when a principal stress presents an extremum, its value has to be zero. However, this case never happens as it can be also seen in figs. 2 to 7, which present the variation of $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}$ along the Mises boundary for angles of obliqueness of the crack $\beta=10^o, 30^o, 45^o, 60^o, 70^o$ and 90^o. The dimensionless principal stresses $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}$ in these figures were normalized to the yield stress, σ_0, in simple tension.

iii) It remains the only valid case where both derivatives of the σ_{11}, σ_{22}-stresses are equal to zero, that is when:

$$\frac{\partial \sigma_{11}}{\partial \vartheta} = \frac{\partial \sigma_{22}}{\partial \vartheta} = 0$$

This relation indicates that at the angle $\vartheta=\vartheta_{max}$, where T_V becomes a maximum, both principal stresses should attain extremum values.

Then, Eq. (7) proves that the maximum tangential principal stress (MTPS), or strain (MTPSt) criteria are, in reality, alternative expressions of the T-criterion. Since a single-stress, or strain criterion is an invalid criterion, the proof that the MTPS— and the MTPSt— criteria, are different expressions of the maximum dilatational energy density criterion (T-criterion) may justify them as potential fracture criteria [**5**].

The only difference between these criteria and the T-criterion is that the single stress or strain criteria are evaluated along the circumference of a circle, whose radius, ρ_0, is arbitrarily defined, whereas the respective core region for the T-criterion is the initial yield locus, a curve defining the plastic enclave, which contributes, among others, to a relaxation of the elastic singular region around the crack tip, by the incipient plastic deformations.

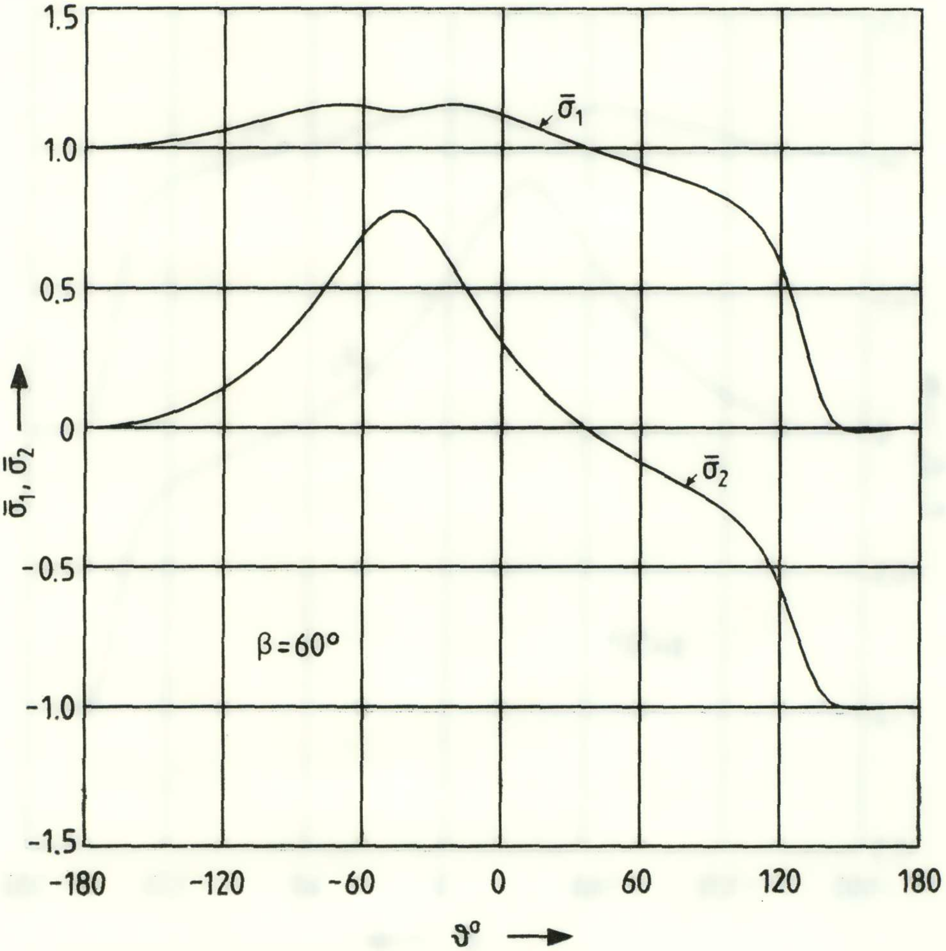

Fig. 5 Angular variation of σ_{11}, σ_{22}-principal stresses along the Mises elastic-plastic boundary for $\beta=60°$.

This difference between criteria for determining the critical ϑ_0-angle of crack propagation is the main cause of any eventual discrepancies between the two groups of results.

Another arbitrary assumption in defining the circular core region is the constancy of its radius for various angles β of inclined cracks and different loading modes of the plate. It is easy to show that the average radius of the initial yield loci for different β' s and $\overset{\infty}{\sigma}_{ij}$' s is strongly variable, since not only the shape, but also the extent of the plastic enclave is strongly

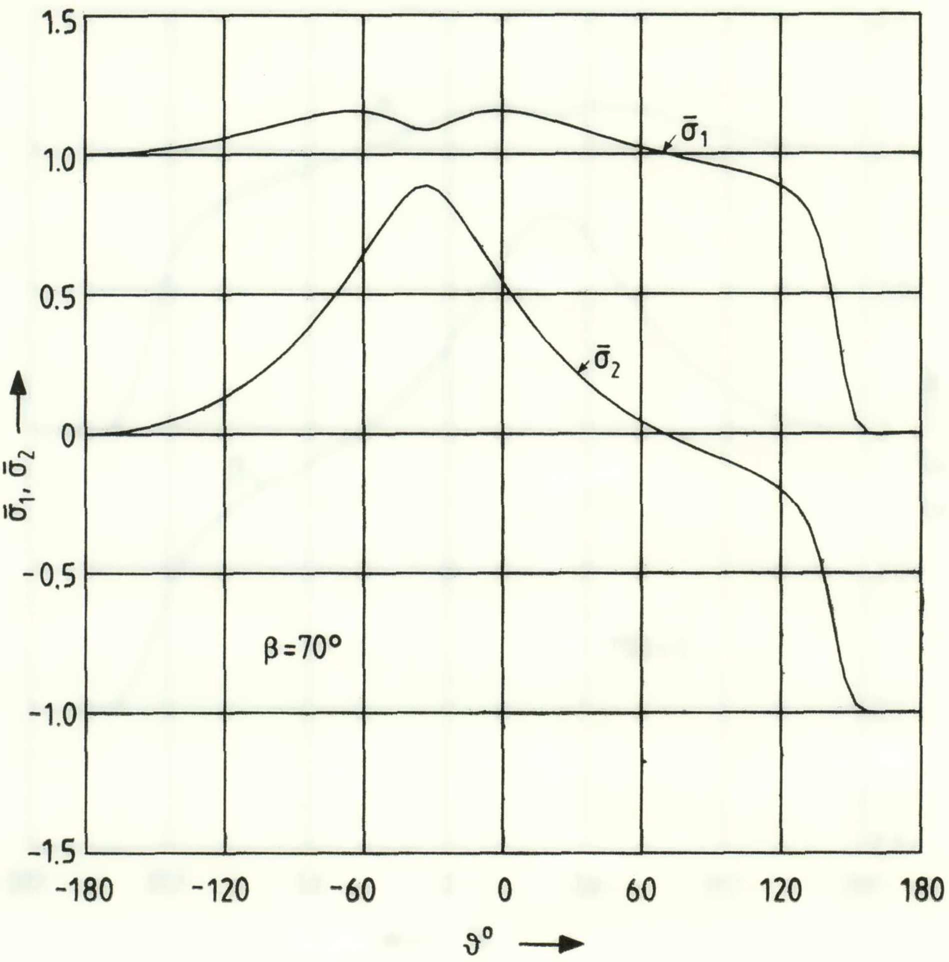

Fig. 6 Angular variation of σ_{11}, σ_{22}-principal stresses along the Mises elastic-plastic boundary for β=75°.

depending on these quantities (see Fig. 5 in ref. [**6**]). By analogy, the circular core regions around the crack tips should have variable radii r_0.

Concerning the maximum tangential stress (MTS) criterion it can be shown that this criterion coincides with the MTPS-criterion for small distances r_0 from the crack tip when the singular one-term solution is valid [**5**]. Indeed, the polar components of stresses at the vicinity of the crack tip are given by:

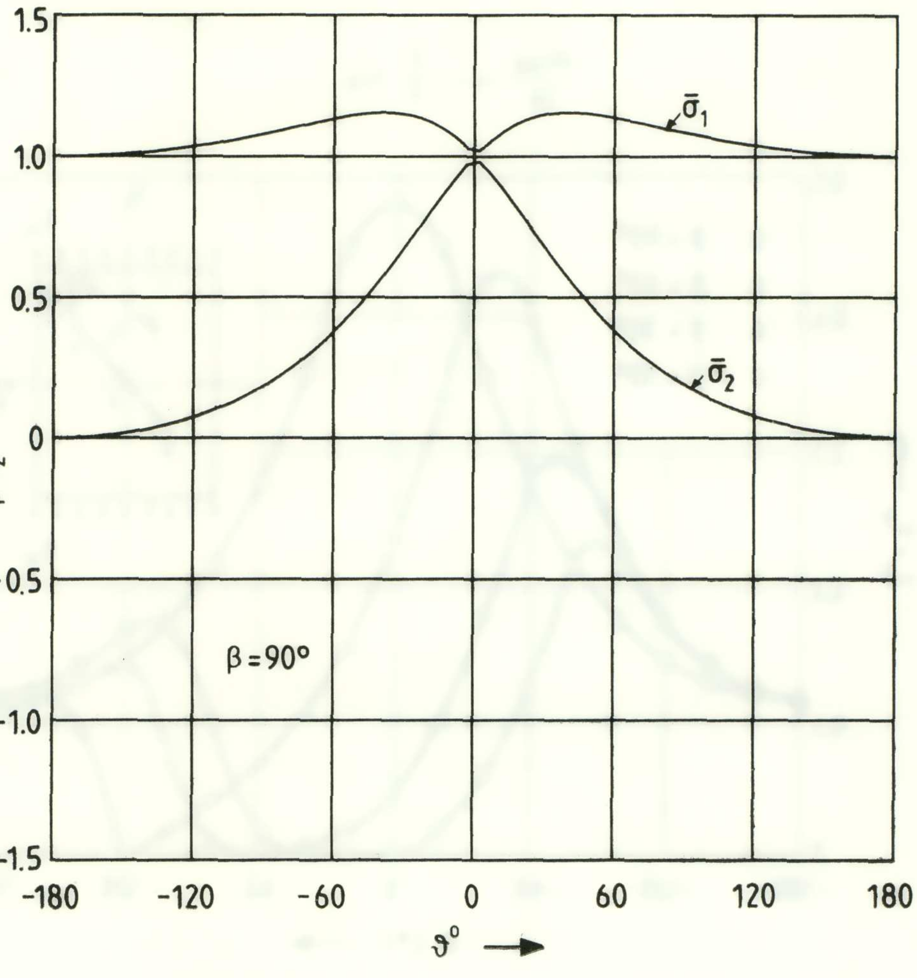

Fig. 7 Angular variation of σ_{11}, σ_{22}-principal stresses along the Mises elastic-plastic boundary for $\beta=90°$.

$$\sigma_{\vartheta\vartheta} = \frac{1}{(2\pi r)^{1/2}} \cos^2\frac{\vartheta}{2}\left\{K_I \cos\frac{\vartheta}{2} - 3K_{II}\sin\frac{\vartheta}{2}\right\}$$

and (8)

$$\tau_{r\vartheta} = \frac{1}{(2\pi r)^{1/2}} \cos^2\frac{\vartheta}{2}\left\{K_I \sin\frac{\vartheta}{2} - K_{II}\cos\frac{\vartheta}{2}\left(2\tan^2\frac{\vartheta}{2} - 1\right)\right\}$$

It can be readily shown that the following relation holds [**5**]:

$$\frac{\partial\sigma_{\vartheta\vartheta}}{\partial\vartheta} = -\frac{3}{2}\tau_{r\vartheta} \tag{9}$$

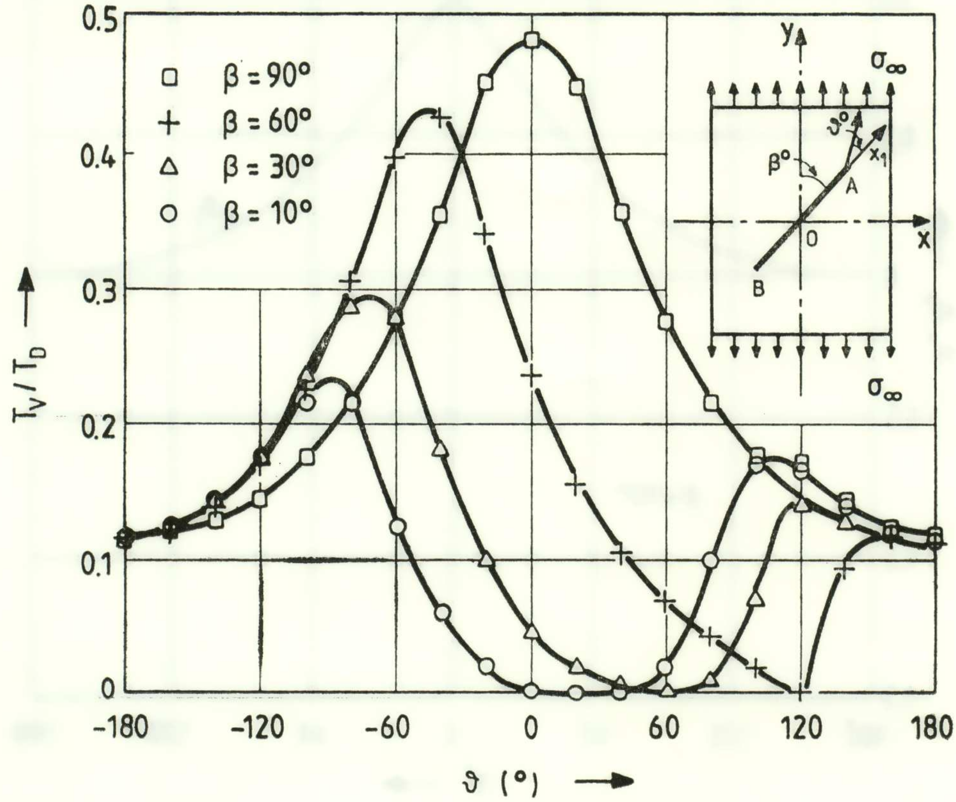

Fig. 8 The variation of the Tv/Td-ratio versus the polar angle ϑ for an internal crack in an infinite plate submitted to simple tension at infinity with $\sigma_\infty = 0.10\,E/\sigma^2_o$, inclined to different angles β, by using either the singular or the two-term solution for this problem.

which implies that for the critical angle ϑ_0 of propagation of the crack it should be valid that:

$$\frac{\partial\sigma_{\vartheta\vartheta}}{\partial\vartheta_{\vartheta=\vartheta_0}} = \tau_{r\vartheta_{\vartheta=\vartheta_0}} = 0 \qquad (10)$$

Relation (10) shows that for the singular one-term solution of the crack problem the MTPS —and MTS— criteria are identical.

Then, all fracture criteria, based on the singular solution and depending on some individual component of the stress or strain tensor may be attributed to the T-criterion, with which they are in complete agreement. The differences between the T-criterion and the S-criterion are already discussed in the open literature. It remains to state that the philosophy behind the G-criterion and the T-criterion is along the same line.

Finally, it should be stated that from all the single-stress fracture criteria the MTPS-criterion is in conformity with physical laws since it guarantees a locally symmetric stress field around the crack tip, which seems to be an inherent feature of the crack propagation process.

In a future article the validity of the T-criterion was further tested by studying the distribution of the T_D- and T_V- components of SED around the crack tip by using not only the Mises yield condition but also various generalized yielding criteria. A comparison of the distribution of the T_V/T_D-ratio around the crack tip in a cracked plate under plane stress conditions loaded in tension at infinity with a load $\sigma\infty = 0.1E/\sigma^2_0$ was undertaken for a material with $\nu = 0.34$, for parametric values of the crack inclination angle β. This ratio was evaluated for the singular and the two-term solutions. For both solutions the T_D-component was constant and equal to 0.44 (E/σ^2_0) where σ_0 was the yield stress in simple tension.

Fig. 8 presents the variation of the T_V/T_D-ratio, versus the ϑ-polar angle, for parametric values of the slantness angle β of the crack. It is clear from this figure that all maxima of the T_V/T_D-ratio and therefore of the T_V-dilatational component of SED appear in the interval $-90^o < \vartheta_0 \leqq 0^o$. Secondary maxima in the Tv-component appear for all β-angles except for $\beta = 90^o$, where the T_V-distribution is symmetric about the $\vartheta = 0^o$ axis. However, these secondary maxima, appearing in the interval between $0^o < \vartheta_0 < 180^o$, are much smaller than the principal ones. The zone of appearance of the principal maxima for the T_V-component is the appropriate one corroborating with extensive experimental evidence. Finally, in-between the zones of maxima, the T_V-component passes always through an interval where $T_V=0$ indicating the sectors where the plate is submitted only to a shearing loading, creating the T_D-component.

The variation of the same ratio was also calculated by using a paraboloid type of criterion, where the inherent anisotropy of the material during loading was taken into account through the strength differential factor $R=\sigma_{oc}/\sigma_{ot}$, where σ_{oc} and σ_{ot} are the yield stresses in simple compression and tension respectively. The criterion in this case takes the form:

$$3J_2 + I_1 (R-1)\sigma_0 = R\sigma^2_0 \tag{11}$$

where I_1 and J_2 are the well known first and second invariants of the stress tensor. In this case the T_D-component varied with the polar angle ϑ. However, for internal anisotropies varying between $R=1.00$ (isotropic material) to $R=1.30$, the T_V/T_D-ratio versus angle ϑ remained exactly the same as for the simple case of the Mises yield locus ($R=1.00$). Since the range between $R=1.00$ and $R=1.30$ includes all the ductile metals and all polymers, the constancy of the variation of the T_V-component of SED indicates an independence of the T-criterion from the inherent characteristic properties of materials.

In conclusion, it has been shown in this note that the T-criterion is a general criterion encompassing all other criteria for fracture and failure of the materials and it is in agreement with the existing extensive experimental evidence and the physical laws.

REFERENCES

1. C. B. Biegeno and R. Grammel, «*Technische Dynamik*», Springer Verlag Publ. Berlin, Chapt. **2**, pp. 79-96, 1939.
2. P. S. Theocaris and N. P. Adrianopoulos, «The Mises Elastic-Plastic Boundary as the Core Region in Fracture Criteria», *Engrg. Fract. Mech.* **16**(**3**) 425-432, 1982.
3. P. S. Theocaris, «A Higher-Order Approximation for the T-Criterion of Fracture in Biaxial Fields», *Engrg. Fract. Mech.*, **19**(**6**) 975-991, 1984.
4. P. S. Theocaris and N. P. Adrianopoulos, «The T-Criterion Applied to Ductile Fracture», *Intern. Jnl. Fract.* **20** R125-R130, 1982.
5. S. K. Maiti and R. A. Smith, «Comparison of the Criteria for Mixed Mode Brittle Fracture Based on the Preinstability Stress-Strain Field; Part I: Slit and Elliptical Cracks Under Uniaxial Tensile Loading», *Intern. Jnl. of Fract.* **23**(**3**) 281-295, 1983.
6. P. S. Theocaris and N. P. Adrianopoulos, «Failure Theories for Glassy Polymers: The T-Criterion for Various Yield Loci», Pure and Appl. Chemistry, **55**(**5**) 845-852, 1983.

ΠΕΡΙΛΗΨΙΣ

Τὸ T-κριτήριον καὶ τὰ παράγωγα αὐτοῦ κριτήρια εἰς τὴν μηχανικὴν τῶν θραύσεων

Οἱονδήποτε κριτήριον θραύσεως πρέπει νὰ ἐξαρτᾶται ἀμέσως ἀπὸ τὸν τρόπον φορτίσεως τῆς κατασκευῆς καὶ ἑπομένως πρέπει νὰ εἶναι συνάρτησις τῶν ἐνεργῶν συνιστωσῶν τοῦ τανυστοῦ τῶν τάσεων τοῦ ὁρίζοντος τὸ ἐλαστικὸν πεδίον εἰς τὴν γειτονίαν τῆς ἀκμῆς τῆς ρωγμῆς. Κατὰ συνέπειαν κριτήρια βασιζόμενα ἀμέσως εἰς μίαν μόνον συνιστῶσαν τῶν τάσεων ἢ τῶν παραμορφώσεων πρέπει νὰ θεωροῦνται ἐλαττωματικὰ καὶ ἐλλιπῆ.

Ἄρα ὅλα τὰ κριτήρια τὰ βασιζόμενα εἴτε εἰς τὴν μεγίστην ἐφαπτομενικὴν τάσιν ἢ παραμόρφωσιν εἴτε εἰς τὴν μεγίστην κυρίαν ἐφαπτομενικὴν τάσιν ἢ παραμόρφωσιν δὲν εἶναι δυνατὸν νὰ θεωρηθοῦν ὅτι προκύπτουν καὶ βασίζονται εἰς ὡρισμένην βασικὴν ἀρχὴν τῆς μηχανικῆς, καὶ ἡ μόνη ἑρμηνεία τῆς ὑπάρξεώς των στηρίζεται ἐπὶ τοῦ γεγονότος ὅτι τὰ ἀποτελέσματα τὰ ὁποῖα συνάγονται ἀπὸ τὰ παλαιὰ αὐτὰ κριτήρια συμφωνοῦν, ἄλλοτε ἱκανοποιητικῶς, ἄλλοτε μὲ ἀποκλίσεις μὲ τὰ ὑπάρχοντα πειραματικὰ ἀποτελέσματα. Ἐν τούτοις ὅμως μόνη ἡ πειραματικὴ συμφωνία οἱωνδήποτε κριτηρίων δὲν εἶναι ἀρκετὴ καὶ ἱκανοποιητικὴ διὰ τὴν παραδοχὴν οἱουδήποτε νόμου τῆς φυσικῆς.

Τὸ γεγονὸς ὅτι μόνον μία συνιστῶσα τοῦ τανυστοῦ τῶν τάσεων ἢ παραμορφώσεων καθορίζει ὁλοκληρωτικῶς οἱονδήποτε κριτήριον θραύσεως εἶναι ἱκανὸς λόγος νὰ ἀμφιβάλλῃ κανεὶς διὰ τὴν ὀρθότητα τοῦ κριτηρίου, δεδομένου ὅτι, ἀκόμη καὶ διὰ τὴν μοναξονικὴν φόρτισιν ρηγματωμένης πλακός, τὸ τασικὸν πεδίον περὶ τὴν γειτονίαν τῆς αἰχμῆς τῆς ρωγμῆς εἶναι ἐντόνως πολυαξονικόν.

Ἐξ ἄλλου, ὅλα τὰ κριτήρια τὰ βασιζόμενα εἰς τὴν ἀρχὴν τῆς κατανομῆς τῆς πυκνότητος τῆς τοπικῆς ἐνεργείας, ὑπολογιζόμενα εἰς τὴν γειτονίαν τῆς αἰχμῆς τῆς ρωγμῆς, βασίζονται ἐπὶ τῆς φυσικῆς ἀρχῆς ὅτι ἡ πυκνότης ἐνεργείας ἡ ἐκλυομένη κατὰ τὴν ἔναρξιν τῆς ρηγματώσεως καὶ τὴν διάδοσιν τῆς ρωγμῆς πρέπει νὰ λαμβάνῃ ἀκροτάτας τιμάς. Ἡ ἀρχὴ αὐτὴ στηρίζεται ἐπὶ τοῦ γεγονότος ὅτι εἰς τὴν αἰχμὴν τῆς ρωγμῆς, κατά τινα στιγμὴν τῆς διαδικασίας τῆς θραύσεως, ἡ λαγκρανζιανὴ συνάρτησις τῶν συνιστωσῶν τῆς ἐνεργείας πρέπει νὰ λαμβάνῃ ἐλαχίστην τιμὴν διὰ νὰ ἱκανοποιῇ τὴν στιγμιαίαν κατάστασιν ἰσορροπίας τοῦ συστήματος. Κατὰ συνέπειαν, ἡ ρωγμὴ ἐπιλέγει τὸν δρόμον ἐξελίξεώς της, ὥστε νὰ ἐλαχιστοποιῇ τὴν λαγκρανζιανὴν συνάρτησιν τῶν ἐνεργειῶν καὶ ἑπομένως πρέπει νὰ ἐκλύῃ τὴν μεγίστην δυνατὴν ἐνέργειαν ἐκ τοῦ συστήματος.

Εἰς τὴν ἀνακοίνωσιν αὐτὴν θὰ δείξωμεν ὅτι ὅλα τὰ παλαιὰ κριτήρια τὰ βασιζόμενα εἰς μοναδικὰς συνιστώσας τῶν τάσεων ἢ παραμορφώσεων δίδουν καλὰ ἀποτελέσματα, συμφωνοῦντα μὲ τὴν ὑπάρχουσαν πειραματικὴν ἐμπειρίαν, ἐπειδὴ προκύπτουν ἀπὸ τὸ βασικὸν κριτήριον τῆς μεγίστης πυκνότητος ἐνεργείας τῆς ἀπαιτουμένης διὰ τὴν μεταβολὴν μόνον τοῦ ὄγκου τῆς κατασκευῆς.

Δεδομένου ὅτι τὸ κριτήριον αὐτό, εἰσαχθὲν ὑφ᾽ ἡμῶν, δίδει καθ᾽ ἡμᾶς τὸν ὀρθὸν τρόπον ἀντιμετωπίσεως τοῦ προβλήματος τοῦ ἐλέγχου τῆς θραύσεως τῶν ὑλικῶν, δικαιολογεῖ τὴν ἰσχὺν καὶ τῶν ἰδιομόρφων αὐτῶν κριτηρίων καὶ ἑρμηνεύει τὴν σύμπτωσιν τῶν ἀποτελεσμάτων των μὲ τὰ πειράματα.

Εἰς τὴν ἐργασίαν αὐτὴν δίδονται τόσον ἡ θεωρητικὴ ἀπόδειξις τῆς συγγενείας τῶν κριτηρίων αὐτῶν, ὅσον καὶ ἡ συμφωνία μὲ τὰς ὑπαρχούσας βασικὰς θεωρητικὰς λύσεις τοῦ προβλήματος διανομῆς τῶν τάσεων περὶ ἐλαστικὴν ρωγμήν.