Ἡ προκαλουμένη μύσις δὲν εἶναι ἔντονος καὶ διαρκεῖ ὀλιγώτερον, ἐν σχέσει πρὸς τὸν φυσιολογικῶς ἔχοντα ὀφθαλμόν.

Τὰ πειράματα δέον νὰ γίνωνται μὲ ἰδιαιτέραν προσοχὴν λόγῳ τοῦ εὐαισθήτου τῆς μεθόδου.

SUMMARY

The possibility of causing local habituation, upon immediate application of an agent on a locus of its selective action and on an organ that could be subject to direct comparison test, was studied experimentally. In the literature available to us no report concerning local habituation could be found. The site of application was cat's conjunctiva and the agent employed physostigmine. Following the determination of the minimal dose causing miosis, – that is, one drop of a 0.05 % physostigmine salicylate solution – one drop of a 0.04 % solution had been applied on one eye for twelve consecutive days. The twelfth day one drop of a 0.05 % solution in each of the animal's eye were dropped. In the eye in which the drug had been applied previously the caused miosis was not intensive and its duration was shorter compared to the normal one.

Physostigmine, in doses less than the minimal drastic dose, applied on cat's eye for a long period of time, has been shown to cause local habihuation.

Because of the method's sensitivity, experimentation should be performed with specific care.

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ΘΕΩΡΗΤΙΚΗ ΦΥΣΙΚΗ.—**Subharmonics of any order in case of non-linear restoring force.** Part I., *by* ***Dem. G. Magiros****. Ἀνεκοινώθη ὑπό τοῦ κ. Βασ. Αἰγινήτου.

Introduction.

We discuss here the subharmonics of any order in the case of linear damping, sinusoidal external force, and cubic type restoring force, with coefficients not necessarily small. By using ideas of Van der Pol[1], Mandels-

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* ΔΗΜ. Γ. ΜΑΓΕΙΡΟΥ, Περὶ τῶν ὑποαρμονικῶν ταλαντώσεων οἱασδήποτε τάξεως.

[1] Van der Pol, *Phil. Mag.*, **3**, 1927, 65.

tam - Papalexi[1], Andronow - Witt[2], we get a proper transformation of the equation, and for its «steady state» and «transient» solutions use of the Poincaré's method for periodic solutions is made.

The conditions for the existence of the subharmonics and their stability in a steady state are discussed by considering that of the singularities of the corresponding equation.

The formulae given here can be used for investigation of the subharmonics of any order.

§ 1. *The problem.*

Many problems in Physics lead to the differential equation of the form:

$$\ddot{Q} + \bar{k}\dot{Q} + \bar{c}_1 Q + \bar{c}_2 Q^2 + \bar{c}_3 Q^3 = B \sin n\tau, \qquad n = 2, 3, \ldots, \tag{1}$$

where the coefficients $\bar{k}$, $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_3$, B are not necessarely small. The solution of (1) is know when the coefficients are small, but it is unknown in case of not necessarily small coefficients.

We intend to find the solution in this last case.

§ 2. *Proper transformation.*

We transform (1) by taking a parameter ε such that:

$$\bar{k} = \varepsilon k, \quad 1 - \bar{c}_1 = \varepsilon c_1, \quad \bar{c}_2 = \varepsilon c_2, \quad \bar{c}_3 = \varepsilon c_3 \tag{2}$$

The result of this transformation is:

$$\ddot{Q} + Q = \varepsilon f(Q, \dot{Q}) + B \sin n\tau \tag{3}$$

$$f(Q, \dot{Q}) = -k\dot{Q} + c_1 Q - c_2 Q^2 - c_3 Q^3 \tag{3α}$$

The coefficients and ε in (2) are finite.

a) In case ε=0, the solution of (1) is:

$$Q = x \sin \tau - y \cos \tau - \frac{B}{1 - n^2} \sin n\tau, \tag{4}$$

with period 2π; $n \neq 1$. The arbitrary constants x and y can be determined by using the initial conditions of (1). x and y are the components of the subharmonic of order $\frac{1}{n}$, of which $r = (x^2 + y^2)^{1/2}$ is the amplitude. The third part in (4) is the harmonic part of the solution due to the forcing term of (1).

1 L. Mandelstam - N. Papalexi, *Techn. Phys*, U. S. S R., 1935, 415.

2 A. Andronow - A. Witt, Arch. für Electrotechn., 1930.

b) In case $\varepsilon \neq 0$, we attempt to determine a periodic solution of (3) of the form (4), in which x and y are fonctions of ε and τ and such that the limits:

(5) $$\lim_{\varepsilon \to 0} x, \quad \lim_{\varepsilon \to 0} y$$

are the constants of the «generating solution» in case $\varepsilon = 0$.

§ 3. *Reduction to a first order system.*

We take a new variable q according to:

(6) $$q = Q - \frac{B}{1-n^2} \sin n\tau .$$

Substituting (6) into (3) we get:

(7) $$\ddot{q} + q = \varepsilon f(q, \dot{q})$$

where:

(7α) $$f(q, \dot{q}) = \Big[k\dot{q} + c_1 q - c_2 q^2 - c_3 q^3 - \frac{kn}{1-n^2} B \cos n\tau + c_1 \frac{B}{1-n^2} \sin n\tau - c_2 \frac{B^2}{(1-n^2)^2} \sin^2 n\tau - c_3 \frac{B^3}{(1-n^2)^3} \sin^3 n\tau - 2c_2 \frac{B}{1-n^2} q \sin n\tau - 3c_3 \frac{B}{1-n^2} q^2 \sin n\tau - 2c_3 \frac{B^2}{(1-n^2)^2} q \sin^2 n\tau \Big] .$$

Introduce into (7) new variables u_1 und u_2 defined by:

(8) $$\begin{cases} u_1 = \dot{q} \cos\tau + q \sin\tau \\ u_2 = \dot{q} \sin\tau - q \cos\tau , \end{cases}$$

from which we get:

(9) $$\begin{cases} q = u_1 \sin\tau - u_2 \cos\tau \\ \dot{q} = u_1 \cos\tau + u_2 \sin\tau , \end{cases}$$

and

(10) $$\begin{cases} \dot{u}_1 = (\ddot{q} + q) \cos\tau \\ \dot{u}_2 = (\ddot{q} + q) \sin\tau , \end{cases}$$

when according to (7), we have:

(11) $$\begin{cases} \dot{u}_1 = \varepsilon f_1(u_1, u_2, \tau) \\ \dot{u}_2 = \varepsilon f_2(u_1, u_2, \tau) , \end{cases}$$

where:

(11α) $\begin{cases} f_1(u_1, u_2, \tau) = f(u_1, u_2, \tau) \cdot \cos\tau \\ f_2(u_1, u_2, \tau) = f(u_1. u_2, \tau) \cdot \sin\tau ; \end{cases}$

The function $f(u_1, u_2, \tau)$ is given from (7α) by using (9).

The equation (3) is replaced by the first order system (11), which gives advantages in the analysis.

From (6) and the first of (9) we obtain:

(12) $$\dot{Q} = u_1 \sin\tau - u_2 \cos\tau + \frac{B}{1 - n^2} \sin n\tau .$$

The expressions (4) and (12) are of the same form, then we ask for a determination of the limits:

(5α) $$\lim_{\varepsilon \to 0} u_1 , \quad \lim_{\varepsilon \to 0} u_2$$

§ 4. *Discussion of a general system.*

We briefly discuss, as it is needed for our purpose here, the solution of the general system:

(N) $$\begin{cases} \dot{u}_1 = \varepsilon f_1(u_1, u_2, \tau) \\ \dot{u}_2 = \varepsilon f_2(u_1, u_2, \tau) \\ u_1(\tau_0) = u_{1\tau_0}, \; u_2(\tau_0) = u_{2\tau_0} , \end{cases}$$

where the functions f_1 and f_2 are analytic in u_1 and u_2, and periodic of period 2π and continuous in τ, hence continuous in u_1, u_2, τ, and therefore $|f_1|$ and $|f_2|$ have upper bounds M_1 and M_2 respectively in the domain:

(D) $$|u_1 - u_{1\tau_0}| < r_1 , \; |u_2 - u_{2\tau_0}| < r_2 , \quad \tau_0 \leq \tau \leq T ,$$

and in this domain f_1 and f_2 are expansible as power series in $(u_1 - u_{1\tau_0})$ and $(u_2 - u_{2\tau_0})$ and convergent.

The number ε is real. The system (11) of our problem is a special case of the system (N).

α) *The formal solution.*

We want to find a solution u_1 and u_2 of the system (N) such that, if $\varepsilon \to 0$, u_1 and u_2 tend to the constants x and y. For $\varepsilon \neq 0$, u_1 aud u_2 depend on ε and τ, and assume that for $\tau = \tau_0$ they differ a little from x and y,

(13) $$u_{1\tau_0} = x + \xi , \quad u_{2\tau_0} = y + \eta$$

ξ and η are very small.

Take as formal solutions u_1 and u_2 of (N) the expressions:

$$(14)\qquad \begin{cases} u_1 = X_1^{(0)} + \varepsilon X_1^{(1)} + \varepsilon^2 X_2^{(2)} + \dots \\ u_2 = X_2^{(0)} + \varepsilon X_2^{(1)} + \varepsilon^2 X_2^{(2)} + \dots \end{cases}$$

where:

$$(14\alpha)\qquad X_1^{(0)} = x + \xi(\varepsilon, \tau_0) \qquad X_2^{(0)} = y + \eta(\varepsilon, \tau_0).$$

The coefficients X of (14) are regular functions of x, y, ξ, η, τ if $|x|$, $|y|$, $|\xi|$, $|\eta|$, $|\tau-\tau_0|$ are less than certain constants.

β) *The coefficients of the formal solution.*

By presupposing the series (14) as convergent, we can find that the coefficients X of (14) are given by:

$$(15)\qquad \begin{aligned} X_1^{(1)} &= \int_{\tau_0}^{\tau} [f_1] d\tau, \\ X_2^{(1)} &= \int_{\tau_0}^{\tau} [f_2] d\tau, \\ X_1^{(2)} &= \int_{\tau_0}^{\tau} \left\{ X_1^{(1)} [f_1]_{u_1} + X_2^{(1)} [f_1]_{u_2} \right\} d\tau, \\ X_2^{(2)} &= \int_{\tau_0}^{\tau} \left\{ X_1^{(1)} [f_2]_{u_1} + X_2^{(1)} [f_2]_{u_2} \right\} d\tau, \\ X_i^{(3)} &= \int_{\tau_0}^{\tau} \sum_{j=1}^{2} [f_i]_{u_j} X_j^{(2)} + \frac{1}{2} \int_{\tau_0}^{\tau} \sum_{j=1}^{2} \sum_{l=1}^{2} [f_i]_{u_j u_l} X_j^{(1)} X_l^{(1)}, \\ &\dots\dots\dots\dots\dots\dots \\ & j, l, i = 1, 2. \end{aligned}$$

The brackets indicate that the corresponding functions and their derivatives are taken at $u_1 = u_{1\tau_0} = x + \xi$, $u_2 = u_{2\tau_0} = y + \eta$. By carrying out the integrations in the first two of (15) we find the functions $X_1^{(1)}$, $X_2^{(1)}$. Upon substituting the $X_1^{(1)} X_2^{(1)}$ into the second two of (15), the integrands become known continuous functions of τ, then $X_1^{(1)}$, $X_2^{(2)}$ are defined by quadratures. With the same procedure we can find $X_1^{(3)}$, $X_2^{(3)}$, . . .

γ) *The convergence of the formal solution.*

To prove the convergence and to find the domain of the validity of of the series (14), we use the «method of dominants». We can prove that the condition for the convergence is:

$$(16)\qquad |\varepsilon| < \frac{r}{4M(\tau-\tau_0)},$$

$$(16\alpha)\qquad M \geq M_i, \quad r \leq r_i, \quad i = 1, 2.$$

In many cases the domain of convergence of (14) is much larger than the condition (16) shows.

The inequality (16) can be satisfied by imposing restrictions upon both ε and τ, or by taking τ arbitrarily and restricting ε or by taking ε arbitrarily and then restricting τ.

§ 5. *Use of the periodicity.*

Take now the solution u_1 and u_2 of (N) as periodic one in τ with period 2π, that is:

$$u_1(\tau_0 + 2\pi) - u_1(\tau_0) = 0, \quad u_2(\tau_0 + 2\pi) - u_2(\tau_0) = 0. \tag{17}$$

Applying this periodicity condition to the series (14) we have:

$$\begin{aligned} &\left\{X_1^{(1)}(\tau_0 + 2\pi) - X_1^{(1)}(\tau_0)\right\} + \varepsilon\left\{X_1^{(2)}(\tau_0 + 2\pi) - X_1^{(2)}(\tau_0)\right\} + \ldots = 0, \\ &\left\{X_2^{(1)}(\tau_0 + 2\pi) - X_2^{(1)}(\tau_0)\right\} + \varepsilon\left\{X_2^{(2)}(\tau_0 + 2\pi) - X_2^{(2)}(\tau_0)\right\} + \ldots = 0. \end{aligned} \tag{18}$$

If the Fourier series developments in τ of f_1 and f_2 are:

$$\begin{aligned} f_1 &= A_0 + A_1 \cos\tau + B_1 \sin\tau + \ldots + A_m \cos m\tau + B_m \sin m\tau + \ldots \\ f_2 &= C_0 + C_1 \cos\tau + D_1 \sin\tau + \ldots + C_m \cos m\tau + D_m \sin m\tau + \ldots, \end{aligned} \tag{19}$$

where the coefficients A, B, C, D are functions of: $u_{1\tau_0} = x + \xi$, $u_{2\tau_0} = y + \eta$, and such that the developments of (19) are convergent. By taking into account the expression of X given by (15), the conditions (18) give:

$$\begin{aligned} &A_0(x + \xi, y + \eta) + \varepsilon\varphi_1(x + \xi, y + \eta, \tau_0) + \ldots = 0, \\ &C_0(x + \xi, y + \eta) + \varepsilon\varphi_2(x + \xi, y + \eta, \tau_0) + \ldots = 0. \end{aligned} \tag{20}$$

Provided that the jakobian of (20) is not zero, i. e.

$$\begin{vmatrix} \dfrac{\partial A_0}{\partial \xi} & \dfrac{\partial C_0}{\partial \xi} \\ \dfrac{\partial A_0}{\partial y} & \dfrac{\partial C_0}{\partial u} \end{vmatrix} \neq 0 \tag{21}$$

we can solve the system (20) in ξ and η interms of ε and τ_0:

$$\xi = \xi(\varepsilon, \tau_0), \quad \eta = \eta(\varepsilon, \tau_0), \tag{22}$$

with the conditions:

$$\xi = \xi(o, \tau_0) = 0, \quad \eta = \eta(o, \tau_0) = 0,$$

If the Jakobian is zero, we consider in (20) terms of 1, 2,... degree in ε, that is the functions φ_1, φ_2, ...[1]

From (22) and (13) we get:

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[1] P. FATOU, *Bulletin Société Math. de France*, 1928 - 30, pp. 112 - 115.

(23) $$u_{1\tau_0} = x + \xi(\varepsilon, \tau_0), \quad u_{2\tau_0} = y + \eta(\varepsilon, \tau_0);$$

ξ and η are given in series in ε, τ_0 in the interval $[0,2\pi]$. ε is under the condition (16). Then is the series (14) the only unknown are the constants x and y.

§ 6. *The components of the amplitude of the subharmonics. Conditions of existence of the subharmonics.*

The equations (20) must be satisfied in case: $\varepsilon = \xi = \eta = 0$. In this case the equation (20) give:

(24) $$A_0(x, y) = 0, \quad C_0(x, y) = 0.$$

The solutions (x, y) of the system (24) give the limits x and y, when the steady state solutions of the original equation are known in the form (4).

The conditions for real intersections of the curves (24) in the x, y-plane give the conditions of the existence of the subharmonics of our equation (3).

§ 7. *The stability of the steady state subharmonics.*

For the stability of the steady state subharmonics we study the stability of the singularities of the equation:

(25) $$\frac{du_2}{du_1} = \frac{f_2}{f_1},$$

which comes from the system (N). The difficulty is that f_1 and f_2 depend on time τ.

But by taking into account the developements of f_1 f_2 given by (19), and that *their mean values with respect to time* τ *over the period* 2π *are* A_0 *and* C_0 *respectively,* the singularities of (25) are that of the equation:

(25α) $$\frac{dy}{dx} = \frac{C_0(x, y)}{A_0(x, y)},$$

then the singularities are given by the solutions (x, y) of the system (24).

According to the corresponding theory of Poincaré[1] and Bendixson[2] the distinction between the different kinds of the singularities depends on two numbers ϱ_1 and ϱ_2, the roots of the characteristic equation:

(26) $$\begin{vmatrix} \alpha_1 - \varrho & b_1 \\ \alpha_2 & b_1 - \varrho \end{vmatrix} = 0,$$

[1] H. Poincaré, Sur les courbes définies par une équation différentielle. *Œuvres,* Gauthier-Villars, Paris, Vol. 1892.

[2] I. Bendixson, *Acta Math.*, **24**, 1901.

where:

(26α) $$\alpha_1 = \frac{\partial A_0}{\partial x}, \quad \alpha_2 = \frac{\partial A_0}{\partial y}, \quad b_1 = \frac{\partial C_0}{\partial x}, \quad b_2 = \frac{\partial C_0}{\partial y}.$$

For non zero roots ϱ_1 and ϱ_2 of (26) the «simple singularities» are classified in the following classes:

I: «*nodal points*», when ϱ_1, ϱ_2, are real and of the same sign,

II: «*saddle points*», when ϱ_1, ϱ_2, are real but of opposite sign,

III: «*spiral points*», when ϱ_1, ϱ_2, are complex conjugates, aud

IV: «*spiral points* or *centers*», when ϱ_1, ϱ_2, are pure imaginaries.

The condition of the roots being pure imaginaries, which is a necessary condition for being a center, it is not a sufficient condition. There is the Poincaré's criterion[1] for distinguishing spiral from center in this case.

We define the above singularities as «*stable*» or «*unstable*», when any point on any integral curve moves into the said singularity or not with increasing time τ, i.e. according as the real part of the roots is negative or positive respectively.

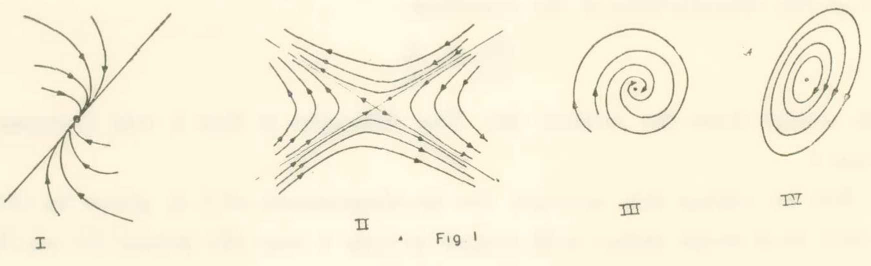

Fig 1

The singularities are shown in Fig. 1, where I is a «nodal stable», III a «stable spiral», IV a (neutral) center, and II a «saddle point intrincically unstable».

§ 8. *Application to the system* (N).

Let us apply the previous theory in the case of the equation f_1 and f_2 given by (11α).

The important here is to give to functions f_1 und f_2 a proper form from which we can get the development in Fourier series. If we replace the

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[1] J. HADAMARD, *Rice Institute Pamphlet*, **20**, 1, 1933, 9-28.

powers and products of sines and cosives of multiple angles in constructing the functions f_1 and f_2 from the function f, we get the following results, if were strict ourselves to the coefficients which are useful for the construction of the functions A_o and C_o, which depend on the number η characterizing the order of the subharmonics:

$$
\begin{aligned}
f_1(u_1, u_2, \tau) = & \left\{ -\frac{1}{2} k u_1 - \frac{1}{2} c_1 u_2 + \frac{3}{8} c_3 u_2^3 + \frac{3}{8} c_3 u_1^2 u_2 + \frac{3}{4} c_3 \frac{B}{(1-n^2)^2} u_2 \right\} + \\
& + \ldots \\
& + \left\{ -\frac{1}{2} c_2 \frac{B}{1-n^2} u_1 \right\} \cos(n-2)\tau + \\
& + \left\{ \frac{3}{4} c_3 \frac{B}{1-n^2} u_1 u_2 \right\} \cos(n-3)\tau + \ldots
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
f_2(u_1, u_2, \tau) = & \left\{ \frac{1}{2} c_1 u_1 - \frac{1}{2} k u_2 - \frac{3}{8} c_3 u_1^3 - \frac{3}{8} c_3 u_1 u_2^2 - \frac{3}{4} c_3 \frac{B^2}{(1-n^3)^2} u_1 \right\} + \\
& + \ldots \\
& + \left\{ \frac{1}{2} c_2 \frac{B}{1-n^2} u_2 \right\} \cos(n-2)\tau + \ldots \\
& + \left\{ -\frac{3}{8} c_3 \frac{B}{1-n^2} (-u_1^2 + u_2^2) \right\} \cos(n-3)\tau + \ldots
\end{aligned}
\tag{28}
$$

All these are refered to any order $\frac{1}{n}$ of subharmonics. In a next paper we shall apply the above theory for the subharmonics of order one third.

ΠΕΡΙΛΗΨΙΣ

Εἰς τὴν ἐργασίαν ταύτην μελετῶνται αἱ ὑποαρμονικαὶ ταλαντώσεις οἱασδήποτε τάξεως εἰς τὴν περίπτωσιν μὴ γραμμικῆς ἐλαστικῆς δυνάμεως. Ἡ ἀντίστοιχος διαφορικὴ ἐξίσωσις μετασχηματίζεται καταλλήλως διὰ χρησιμοποιήσεως ἰδεῶν τῶν Van der Pol, Mandelstam-Papalexi καὶ Andronow-Witt. Σπουδάζονται λύσεις «εὐσταθεῖς» καὶ «μεταβατικαὶ» διὰ χρησιμοποιήσεως τῆς μεθόδου περιοδικῶν λύσεων τοῦ Poincaré. Ἡ σπουδὴ τῶν συνθηκῶν διὰ τὴν ὕπαρξιν τῶν ὑποαρμονικῶν ταλαντώσεων καὶ τὴν εὐστάθειάν των ἀνάγεται εἰς τὴν σπουδὴν τῶν ἀνωμάλων σημείων τῆς ἀντιστοίχου ἐξισώσεως. Εἰς ἑπομένην ἀνακοίνωσιν θὰ ἐκτεθῇ ἡ ἔρευνα παρ' ἐμοῦ τῶν ὑποαρμονικῶν ταλαντώσεων τάξεως ἑνὸς πρὸς τρία, ὡς ἐφαρμογὴ τῶν γενικῶν σκέψεων τῆς παρούσης ἀνακοινώσεως.

ΜΑΘΗΜΑΤΙΚΑ.—Ἐπὶ τοῦ X βιβλίου τῶν στοιχείων τοῦ Εὐκλείδου, *ὑπὸ* **Εὐαγγ. Σταμάτη*.** Ἀνεκοινώθη ὑπὸ τοῦ κ. Μιχ. Στεφανίδου.

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* Θὰ δημοσιευθῇ κατωτέρω.