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ΜΑΘΗΜΑΤΙΚΑ.— **«On Quantum Geometrodynamics»** *, *by Demetrios Christodoulou, California Institute of Technology, Pasadena, California* **. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φιλ. Βασιλείου.

INTRODUCTION

Pythagoras, in the 6th century B. C., first of all men asked the fundamental question of science «what is the physical world in essence?». Pythagoras was also the first to answer it; «number is the essence of the cosmos», he said. Plato, in the 4th century B. C., developed the Pythagorean ideas and understood the elementary particles of matter to be metric forms imposed on formless space (1).

The ideas of these thinkers were misunderstood and forgotten until the 19th century when Riemann (2) developed the geometry of a manifold with a general metric form. In 1916 Einstein (3) with general relativity theory understood the gravitational field to be the curvature of spacetime. However, general relativity theory, as it stood, offered no way of understanding the rest of physics as geometry.

On the other hand in 1927, Heisenberg (4) formulated the uncertainty principle, later elucidated by Bohr, that founded quantum mechanics and the understanding of atomic and subatomic phenomena. The

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** ΔΗΜΗΤΡΙΟΥ ΧΡΙΣΤΟΔΟΥΛΟΥ, «Κβαντικὴ Γεωμετροδυναμική».

conclusion of these and later investigations was that a physical theory that is consistent with measurement must be probabilistic and not deterministic in the configuration of the relevant physical system as specified by a set of continuous variables. The consistent theory must furthermore satisfy the quantum principle, the first complete formulation of which was given by R. P. Feynman [5] by assigning probability amplitudes to entire histories of the configuration of the physical system described.

The above conclusions when applied to general relativity theory show that it must be inconsistent with measurement. A consistent theory was therefore sought. J. A. Wheeler [6] realized that the dynamical variable specifying configuration in general relativity must be the 3-geometry of space. He therefore termed general relativity theory «classical geometrodynamics» and the theory sought «quantum geometrodynamics». The investigations toward realizing such a theory were then led to first bringing the classical theory to Hamilton's canonical form and then using some form of correspondence of classical dynamical variables and quantum mechanical operators to get the quantum theory. This work was carried out chiefly [7] by P. A. M. Dirac, by R. Arnowitt, S. Deser and C. W. Misner, and by B. Dewitt. However all attempts were unsuccessful. This failure can be retraced to two basic reasons: 1) These investigators used many assumptions motivated from traditional field theory. There is a fundamental difference between field theory and geometrodynamics; whereas in the former case the dynamical field developes in the fixed arena of space, in the latter case the 3-geometry of space is itself both the dynamics and the arena. 2) All linear classical theories give local quantum theories and all quantum field theories known todate are of this type. Non-linear classical theories, like general relativity theory, give however non-local quantum theories. The existing mathematical framework was insufficient to handle such global theories.

In this paper the theory of quantum geometrodynamics is founded on the relativity and quantum principles alone, using no additional assumptions except that of the form of the action of general relativity. In section (1) the physical interpretation of general relativity theory is given. Einstein's equations are distinguished into two sets: 1) Equations intrinsic to classical geometrodynamics, namely the dynamical equations that give a sequence of spacial 3-geometries, once the boundary conditions are specified. 2) Equations that express fundamental laws of geometrodynamics and must therefore remain valid in quantum geometrodynamics. These are laws determining the way that known spacial 3-geometries are intercorresponded to form the 4-geometry of spacetime, and the law that determines the temporal separation of known 3-geometries of space. In section (2) the concept of superspace, the configuration space of 3-geometries, is introduced following J. A. Wheeler [(6)]. Here, however, the element of distance in superspace is consistently defined, derived from the action integral as supplemented by the intercorrespondence conditions. The quantum principle, as first formulated by R. P. Feynman [(5)], is suited to geometrodynamics. In section (3) the structure of superspace is investigated. In section (4) the absolute differential calculus of global functionals is introduced. Finally, in section (5) the laws intrinsic to quantum geometrodynamics are derived, expressed by Eq. (42), a functional differential equation for the wave functional of the geometrical world.

(1) The Relativity Principle and Classical Geometrodynamics

In this section we shall begin by briefly summarizing the considerations which led Einstein [(3)] to the discovery of the theory of general relativity and then we schall proceed to give the physical interpretation of that theory.

Consider an observer in a small laboratory anywhere in the universe. He observes the motion of objects in the laboratory and concludes that forces are being exerted on them. These forces can be attributed either to the action of inertia or the action of gravitation or a combination of the two. In the former case he concludes that the system is being accelerated and in the latter case that it is not. Thus the observer can never tell the state of motion of the laboratory from observations performed on that laboratory alone. Thus the laws of physics must be the same in two reference frames with respect to which the laboratory has two arbitrarily different states of motion. The above considerations lead to the following statement which constitutes the relativity principle:

I. The laws of physics including gravitation remain unchanged under general coördinate transformations in spacetime.

Consider now a laboratory freely falling under the action of a gravitational field. According to the previous considerations, the physics in that laborarory go in the same way as in a laboratory moving with no acceleration in the absense of gravitational fields. The reference frames of both laboratories will be called «inertial» since no forces are present in either frame. For the latter laboratory, the statement «moving with no acceleration», in view of the relativity principle, must be given a meaning that is invariant under general coördinate transformations in the Minkowski spacetime that is relevant in that case. Such a meaning can only be that given by the statement «following a geodesic of the spacetime» The equivalence of the frames of the two laboratories requires the previous statement also to hold true in the former case of the laboratory freely falling in the gravitational field. But in that case, this geodesic, being different from an ordinary straight line, does not belong to the flat Minkowski spacetime, but rather to a spacetime endowed with curvature. This curvature however is revealed not through observations of the motion of objects in one freely falling laboratory-local inertial frame-, but rather through the correlation of the motions, or in other words through the observation of the geodesic deviation of two nearby local inertial frames. Thus gravitation manifests itself as the curvature of the spacetime continuum.

The laws determining the metric structure of classical spacetime were discovered by Einstein[3] by imposing, in addition to the relativity principle, the following hypotheses on the form of these laws, motivated from the analog of the Newtonian theory of gravitation.

1. They must be derivable from a variational principle.
2. They must contain no differential coefficients of the metric components ${}^{(4)}g_{\mu\nu}$ higher than the second.
3. They must be linear in these second differential coefficients.

The only laws satisfying the above conditions were found to be

$$ {}^{(4)}G^{\mu\nu} = 0, \tag{1}$$

where ${}^{(4)}G^{\mu\nu} = {}^{(4)}R^{\mu\nu} - \frac{1}{2}\,{}^{(4)}g^{\mu\nu}\,{}^{(4)}R$, the ${}^{(4)}R^{\mu\nu}$ being the components of the (contravariant) Ricci curvature tensor and ${}^{(4)}R$ being the scalar curvature invariant ${}^{(4)}R^{\mu}{}_{\mu}$, all quantities refering to the 4-dimensional spacetime continuum.

The above 10 equations satisfy 4 identities, the so-called Bianchi identities

$$^{(4)}G^{\mu\nu}{}_{;\,\nu} = 0. \tag{2}$$

The classical laws expressed by Eq. (1) represent therefore 6 independent differential conditions on the 6 independent metric components that give the 4-geometry of the classical spacetime. They can be derived by extremizing the action

$$S = \frac{1}{16\pi}\int {}^{(4)}R\sqrt{-{}^{(4)}g}\; d^4x \tag{3}$$

with respect to the metric components $^{(4)}g_{\mu\nu}$.

To give the physical interpretation of these equations we have to look at the way that the relevant quantities are measured. Cover the space densely with timelike inertial observers. Let these observers have an arbitrary but continuous velocity distribution. Continuity implies that there exists a global frame of reference in which the velocities of all the observers are zero. In this «global inertial frame» there is also a global standard of simultaneity. Only in this frame the coördinate x^4 has the psysical significance of being, for every point in the frame, the age of an observer at rest at that point in the frame. It can be prooved that for a general (non-flat) spacetime only one such frame exists. Given the 4-geometry of spacetime, the element of proper distance that belongs to it is interpreted physically as being the proper time interval between events happenning to nearby observers at nearby ages in the prescribed frame :

$$^{(4)}ds^2 = -\,d\tau^2 + g_{mn}\,({}^3x,\,\tau)\,dx^m\,dx^n\,; \tag{4}$$

where dx^i is the difference in the frame labels of the observers and $d\tau$ is their age difference. In accordance with our prescribed theme we have to look at how this 4-geometry is measured.

The observers measure their relative distances when they are all of a certain age τ. Such a set of measurements is tacitly assumed in classical physics to give a unique set of real numbers that constitutes a unique 3-geometry of space. The measurements are then repeated at constant age intervals $d\tau$ and a sequence of spacial 3-geometries is obtained. In the limit of infinitesimal $d\tau$, can we reconstruct, from the data thus obtained, the 4-geometry of spacetime? In the spacetime we are seeking to construct, each 3-geometry of the aforementioned sequence must belong to a spacelike hypersurface. The world lines of the observers must be normal to the hypersurfaces, the length of the segment of any such world line between any two successive hypersurfaces being equal to $d\tau$.

Make a one to one correspondence between points on each space and its successor in the sequence. Make the correspondences in such a way that the difference of the distance between any two nearby points on any one space from the distance between the corresponding points on the next space in the sequence, divided by $d\tau$, gives a bounded result. Give the same label x^{*i} to all points belonging to different spaces that are brought into correspondence with each other. The lines $x^{*i} = \text{const.}$ that do the correspondence in this case are then, in the spacetime wanted, the world lines of the observers that performed the measurements. The element of proper distance of the spacetime thus obtained by the given correspondence, is then given by:

$$^{(4)}ds^{*2} = - d\tau^2 + \overset{*}{g}_{ij}\,(^3x^*, \tau)\, dx^{*i}\, dx^{*j} \tag{5}$$

The way of doing the correspondence is however not unique, since we could have used for that purpose the lines $x^m = \text{const.}$, where

$$x^{*i} = f^i\,(^3x, \tau)\,, \tag{6}$$

provided that the functions f^i are differentiable. The lines $x^m = \text{const.}$ will then be the world lines of the measuring observers in a spacetime obtained by the new correspondence, with element of proper distance given by:

$$\begin{aligned} ^{(4)}ds^2 &= - d\tau^2 + g_{mn}\,(^3x, \tau)\, dx^m\, dx^n = - d\tau^2 + \\ &+ \overset{*}{g}_{ij}\,(dx^{*i} - v^{*i}\, d\tau)\,(dx^{*j} - v^{*j}\, d\tau) \end{aligned} \tag{7}$$

Here we have defined $v^{*i} = \partial x^{*i} / \partial \tau$, and the following relations hold

$$\left.\begin{aligned} g_{mn} &= \frac{\partial x^{*i}}{\partial x^m}\,\frac{\partial x^{*j}}{\partial x^n}\,\overset{*}{g}_{ij} \\ \frac{dg_{mn}}{d\tau} &= \frac{\partial x^{*i}}{\partial x^m}\,\frac{\partial x^{*j}}{\partial x^n}\left(\frac{d\overset{*}{g}_{ij}}{d\tau} + \overset{*}{v}_{i;*j} + \overset{*}{v}_{j;*i}\right) \end{aligned}\right\} \tag{8}$$

The proper distance elements of Eqs. (5) and (7) belong to different 4-geometries. In conclusion, from a sequence of spacial 3-geometries as measured by the observers at constant age intervals, one can construct not one but a whole class of 4-geometries of the spacetime. Each member of the class is obtainable from any other member by transformations of the type given by Eq. (6). Such transformations will be called «velocity transformations» in the following.

Let us now turn our attention back to Einstein's equations and their physical interpretation. Consider first the equations $G^{4i} = 0$ and $G^{44} = 0$, equations containig only first time derivatives. In the global inertial frame the equations $G^{4i} = 0$ have the following form

$$G^{ijmn} \left(\frac{dg_{mn}}{d\tau} \right)_{;j} = 0 \,, \tag{9}$$

where $$G^{ijmn} = \frac{1}{4} \sqrt{g} \, [g^{ij} g^{mn} - (g^{im} g^{jn} + g^{in} g^{jm}) / 2] \tag{10}$$

Let us be given a sequence of spaces with determined 3-geometries. We make an intercorrespondence of the spaces and form a 4-geometry of spacetime described by the functions $g^*_{ij}\,({}^3x^*, \tau)$ that donot, in general, satisfy Eq. (9). We may however perform a velocity transformation and obtain a new 4-geometry described by functions $g_{ij}\,({}^3x, \tau)$ that do satisfy Eq. (9), provided that the following equations can always be satisfied

$$G^{*ijmn} \left(\frac{dg^*_{mn}}{d\tau} + v^*_{m;*n} + v^*_{n;*m} \right)_{;*j} = 0 \,. \tag{11}$$

The above equations represent 3 differential conditions on the 3 components v^{*i}, the existence and uniqueness of the solution of which can be prooved for a general (non-flat) spacetime. Therefore, once a sequence of spacial 3-geometries is given from measurement, Eq. (9) gives a unique way of making the intercorrespondence, thus forming a unique 4-geometry. The meaning of this particular way of corresponding will be seen in the next section.

The equation $G^{44} = 0$ is partly redundant, since it can be shown [8] that if the conditions expressed by Eq. (9) are fulfiled and furthermore the integrated condition $\int G^{44} \sqrt{g}\, d^3x = 0$, which is

$$\int \left(G^{ijmn} \frac{dg_{ij}}{d\tau} \frac{dg_{mn}}{d\tau} - R\sqrt{g} \right) d^3x = 0 \tag{12}$$

expressed in the global inertial frame, is fulfiled, then the equation $G^{44} = 0$ is identically satisfied. If a sequence of spacial 3-geometries is given from measurement but the temporal separations of the spaces are left undetermined, then Eq. (12) determines these temporal separations.

The rest of Einstein's equations, namely the equations $G^{ij} = 0$, contain second time derivatives. These equations, in the global inertial frame, assume the following form

$$G^{ijmn} \frac{d^2 g_{mn}}{d\tau^2} + C^{rskl,ij} \frac{dg_{rs}}{d\tau} \frac{dg_{kl}}{d\tau} = \frac{1}{2}\left(R^{ij} + \frac{1}{2} g^{ij} R\right)\sqrt{g}, \quad (13)$$

where

$$C^{rskl,ij} = \frac{1}{2}\left(\frac{\partial G^{klij}}{\partial g_{rs}} + \frac{\partial G^{rsij}}{\partial g_{kl}} - \frac{\partial G^{rskl}}{\partial g_{ij}}\right) \quad (14)$$

Take the covariant divergence of both sides of Eq. (13). The covariant divergence of the left hand side is equal to the time derivative of the expression in the left hand side of Eq. (9) and therefore it vanishes identically if the conditions expressed by Eq. (9) are fulfiled. On the other hand, the covariant divergence of the right hand side of Eq. (13) also vanishes identically since that represents the Bianchi identities. Therefore 3 of the 6 equations Eq. (13) are redundant. There remain 3 independent differential conditions for the 3 independent metric components of the 3-geometry of space. These are the dynamical equations of classical geometrodynamics that determine the sequence of 3-geometries verified by the measuring observers.

(2) The Quantum Principle and the Concept of Superspace

In the previous section we assumed that measurement of the relative distances of the observers gives a unique 3-geometry of space. This assumption is here reconsidered and found not to be correct. Not a single 3-geometry, but indenumerably many 3-geometries follow from measurement. Each alternative 3-geometry is assigned with a determined probability. Thus, we must abandon the proceedure of extremizing the action integral and obtaining deterministic dynamical equations for the 3-geometry of space; classical geometrodynamics is inconsistent with measurement. We must now look for a consistent theory and such a theory must be probabilistic in the 3-geometry of space. The criterion of consistency of a theory with respect to the measurement of the configuration of the 3-geometry of space is found to be the so-called quantum principle, a condition on the way of assigning the probabilities. We require the new theory to satisfy this principle and therefore the new theory is called quantum geometrodynamics. Central to the formulation of the quantum principle is the action integral, the link between classical and quantum physics. We will first analyze the action integral and then we will proceed to the formulation of the quantum principle.

The action integral given by Eq. (3) assumes the following form, expressed in the global inertial frame

$$\mathcal{S} = \frac{1}{16\pi}\int d\tau \int d^3x \left\{-2g_{ij}\frac{d}{d\tau}\left(G^{ijmn}\frac{dg_{mn}}{d\tau}\right) - G^{ijmn}\frac{dg_{ij}}{d\tau}\frac{dg_{mn}}{d\tau} + R\sqrt{g}\right\} \tag{15}$$

The above integral represents the action in the «momentum» formulation of geometrodynamics; the term in round brackets indicated that it is the quantity $G^{ijmn}(dg_{mn}/d\tau)$, involving the rate of change of the 3-geometry («geometrodynamical field momentum»), rather than the 3-geometry itself that is kept fixed at the limits of integration in τ. To go to the «coördinate» formulation of geometrodynamics, which is the formulation needed here, where the 3-geometry is fixed at the limits, add the quantity

$$\frac{1}{16\pi}\left[\int 2G^{ijmn} g_{ij}\frac{dg_{mn}}{d\tau} d^3x\right]_{\text{at limits}}$$

and arrive at the desired action

$$\mathcal{S}' = \frac{1}{16\pi}\int d\tau \left\{\int d^3x \left[G^{ijmn}\frac{dg_{ij}}{d\tau}\frac{dg_{mn}}{d\tau} + R\sqrt{g}\right]\right\} \tag{16}$$

Noting the resemblance of the action of geometrodynamics with that of a particle in a potential, we call the first term in the integral in Eq. (16) «kinetic energy» term and the second term «potential energy» term. The «kinetic energy» term involves the square of the rate of change of «distance» with time of the universe in a configuration space of 3-geometries, a manifold each point of which represents a 3-geometry. This manifold has been named «superspace» by J. A. Wheeler [6]. The distance in superspace, $d\mathcal{L}$, of two 3-geometries infinitesimally different from each other, as deduced from the above considerations is then given by

$$d\mathcal{L} = \left(\int G^{ijmn}\, dg_{ij}\, dg_{mn}\, d^3x \right)^{1/2} \qquad (17)$$

Here we have done a one to one correspondence between points of the spaces to which the 3-geometries belong, in the manner described in the previons section, and we have taken the difference dg_{ij}, between the metric components at the corresponded points. The intercorrespondence of given spacial 3-geometries can however be done in a multiplicity of ways, as discussed in the previous section. Each way of doing the intercorrespondence is equivalent to any other way plus a velocity transformation. The metric component differences dg^*_{is}, obtained through a new intercorrespondence, are related to the original metric component differences dg_{ij}, in the following manner (from Eq. (8))

$$dg_{mn} = \frac{\partial f^i}{\partial x^m} \frac{\partial f^i}{\partial x_n} \left[dg^*_{ij} + (v^*_{i;*j} + v^*_{j;*i})\, d\tau \right]$$

Substitution of the transformed quantities dg^*_{ij} in Eq. (17) will give a new result for $d\mathcal{L}$. Different ways of corresponding given 3-geometries will give different results for $d\mathcal{L}$ as computed from Eq. (17). But if $d\mathcal{L}$ is to be the element of distance between points in superspace, it must be unique for two given points in superspace corresponding to two given 3-geometries, infinitesimally different from each other. Therefore we must require a particular way of corresponding to be singled out. In addition we must require the distance of two identical 3-geometries, to be zero in superspace. The previous two requirements can be met only if we define the actual distance element to be, the minimum value of the expression on the right hand side of Eq. (17) as we go through all the alternative ways of corresponding the given 3-geometries. Therefore, given one way of doing the intercorrespondence, we must perform a velocity transformation and look for the transformation vector v^{*i} that minimizes expression (17). This gives the conditions

$$G^{*ijmn} [dg^*_{mn} + (v^*_{m;*n} + v^*_{n;*m})\, d\tau]_{;*j} = G^{ijmn} (dg_{mn})_{;j} = 0$$

to be fulfiled by v^{*i}, identical to the equations $G^{4i} = 0$ (Eq. (9)). In conclusion, Eq. (17) has to be supplemented by the above intercorrespondence conditions to give a consistent definition of distance in superspace. Thus, the full meaning of the equations $G^{4i} = 0$ is now realized.

Now we are ready to formulate the quantum principle. Form the product manifold of superspace with the real line of time and call it «superspace-time». This manifold is the evolution space of 3-geometries. A history of the 3-geometry of space is a path in superspace-time. Such a path is also a 4-geometry of spacetime. However not every 4-geometry is a path in superspace-time; only those that are described by functions $g_{ij}(^3x, \tau)$, in the global inertial frame, satisfying Eq. (9), represent paths in superspace-time. The quantum principle, as formulated first by R. P. Feynman [5] and suited here to geometrodynamics, consists of the following two statements:

II. 1. The probability that the universe has a path, namely a history of 3-geometry, lying in a region of superspace-time, is the absolute square of a sum of complex contributions one from each path in the region.

2. The paths contribute equally in magnitude, but the phase of their contribution is the action integral corresponding to each path.

The probability amplitude $\varphi[U]$ that the history of the 3-geometry of the universe passes through a region U in superspace-time can therefore be expressed in the form

$$\varphi[U] = \lim_{\delta\tau \to 0} \int_U \exp\left\{ i \sum_{i=1}^{k-1} S'\left(\mathcal{G}_{i+1}, \mathcal{G}_i\right)\right\} \mathcal{D}\mathcal{G}_k \, \mathcal{D}\mathcal{G}_{k-1} \cdots \mathcal{D}\mathcal{G}_1, \quad (18)$$

up to a normalization factor chosen so that the probability of a certain event is 1 in the limit $\delta\tau \to 0$. In the expression above and in the following the symbol $\mathcal{G}$ is used to denote the 3-geometry of space. The time interval of the region U of superspace-time has been divided into (k—1) subintervals of duration $\delta\tau$, and each continuous path passing through the region has been replaced by a series of «straight line» (see following section) segments, one for each temporal subinterval. Thus continuous histories of the universe are replaced by skeleton histories. The argument of the action means that the action integral is taken along a «straight line» segment between the geometrical configuration $\mathcal{G}_i$, of the beginning of the i^{th} temporal subinterval, and the geometrical configuration $\mathcal{G}_{i+1}$, of the beginning of the $(i+1)^{th}$ temporal subinterval.

Consider now a region U′ of superspace-time restricted to $\tau \leqslant \tau_k$ and such that for $\tau < \tau_k$ it covers all superspace but as $\tau \rightarrow \tau_k$ it converges to the point in superspace corresponding to the geometrical configuration $\mathcal{G}_k$. Then $\varphi[U']$ assumes the form of a functional $\varphi[\mathcal{G}_k, \tau_k]$ giving the amplitude that the universe arrives at the geometrical configuration $\mathcal{G}_k$ with the value τ_k of time. This functional, given by the integral expression (applying Eq. (18)).

$$\varphi[\mathcal{G}_k, \tau_k] = \lim_{\delta\tau \rightarrow 0} \int_{\substack{\text{all}\\ \text{superspace}}} \exp\left\{ i \sum_i^{k-1} S'(\mathcal{G}_{i+1}, \mathcal{G}_i) \right\} \mathcal{D}\mathcal{G}_{k-1} \mathcal{D}\mathcal{G}_{k-2} \cdots, \quad (19)$$

up to a normalization factor, is therefore the wave-functional of the geometrical world.

In the limit of large actions only the paths for which the action is nearly an extremum contribute to the integral in Eq. (18). The amplitudes of paths for which the corresponding action is away from the extremum interfere destructively. Hence, in the classical limit the history of the 3-geometry of the universe is restricted to $\delta S' = 0$, which yields the dynamical equations $G^{ij} = 0$, in addition to the prerequisite conditions $G^{4i} = 0$. The dynamical equations give a first integral

$$\int \left(G^{ijmn} \frac{dg_{ij}}{d\tau} \frac{dg_{mn}}{d\tau} - R\sqrt{g} \right) d^3x = \lambda \text{ (constant of the motion)} \quad (20)$$

From the discussion in the previous section we infer that an additional condition need be applied in order to obtain general relativity theory in the limit of large actions. It is the condition expressed by Eq. (12), namely the constraint $\lambda = 0$. This condition has the following meaning: given any two infinitesimally different 3-geometries their temporal separation is not arbitrary but it is determined by

$$d\tau = \frac{d\mathcal{L}}{\mathcal{R}^{1/2}}, \quad (21)$$

where $d\mathcal{L}$ is the distance of the 3-geometries in superspace and $\mathcal{R}$ is the functional $\int R\sqrt{g}\, d^3x$. This condition expresses a fundamental law of geometrodynamics that must be retained in the quantum theory so that in the classical limit general relativity theory is obtained. In the following we will temporarily disregard this condition and we will find the wave equation that $\varphi[\mathcal{G}, \tau]$ must satisfy. Then the condition will be applied as an eigenvalue condition to get from the wave functional $\varphi[\mathcal{G}, \tau]$ to a new wave functional $\psi[\mathcal{G}]$ depending only on the 3-geometry, the temporal separations determined from the law expressed by condition (21).

(3) The structure of superspace

In this section, the structure of superspace is investigated, superspace being considered as a set of histories of 3-geometries. Thus, the concepts introduced will be concepts intrinsic to paths only. We saw however in the previous section that quantum theory requires the introduction of a volume element in superspace and such a concept is foreign to concepts intrinsic to paths. The determination of the volume element in superspace will be demonstrated in section (5).

Points in superspace can be labeled with the «coördinates» g_{ij}, namely the metric components specified at all points of ordinary space. This way of labeling points in superspace constitutes the «metric coördinate system» of superspace. Clearly this is not the only allowable coördinate system in superspace. Any geometric symmetric tensor — function of the metric tensor and its derivatives up to n'th order — completely describing the 3-geometry when specified at all points of ordinary space, forms a basis for labeling points in superspace. Thus in the following, unless so specified, the symbol g_{ij} will denote not only the metric tensor but an arbitrary geometric symmetric tensor, the basis of a general coördinate system in superspace and not necessarily the metric coördinate system. A coördinate transformation in superspace is a change of the way of describing the 3-geometry and it is clear that the formalism must be invariant under such a change.

The action integral, or rather the «kinetic energy» part of that integral, defines, as we have seen in the previous section, the element of arc length in superspace $d\mathcal{L}$, a superspace invariant by definition, its square given by

$$d\mathcal{L}^2 = \int G^{ijmn}\, dg_{ij}\, dg_{mn}\, d^3x,$$

supplemented by the intercorrespondence conditions $G^{ijmn}(dg_{mn})_{;j} = 0$, in the metric coördinate system in superspace. The form of the line element in an arbitrary coördinate system in superspace can then be derived from its form in the metric coördinate system and a functional $d\mathcal{L}^2 = d\mathcal{L}^2\,[g_{ij}, dg_{ij}]$ will in general result, depending on the geometry $\mathcal{G}$ described by the functions g_{ij} and the infiniterimal change in the geometry, $d\mathcal{G}$, from the configurarion $\mathcal{G}$, described by the functions dg_{ij}. Furthermore, $d\mathcal{L}^2$ is always of second order in dg_{ij}. The functions dg_{ij} have the properties: (1) the resultant of two successive infinitesimal changes in 3-geometry is described by the sum of the two corresponding dg_{ij}'s and (2) a change in 3-geometry that is proportional to another such change is described by functions dg_{ij} that bare the same proportion to the corresponding functions describing the other change in 3-geometry. The functions dg_{ij} are thus the components of a vector $d\mathcal{G}$ in superspace, a supervector, representing the infinitesimal change in 3-geometry from the configuration $\mathcal{G}$, and defined at the point in superspace corresponding to that configuration. Under a superspace coördinate transformation the components of the supervector $d\mathcal{G}$ transform in the following manner

$$dg'_{mn} = \frac{\partial g'_{mn}}{\partial g_{ij}}\, dg_{ij} + \frac{\partial g'_{mn}}{\partial g_{ij,k}}\, (dg_{ij})_{,k} + \frac{\partial g'_{mn}}{\partial g_{ij,kl}}\, (dg_{ij})_{,kl} + \dots \qquad (22)$$

Here the unprimed functions refer to the original coördinate system and the primed functions refer to the new coördinate system.

We must note here that the distance in superspace of two geometrical configurations belonging to different topologies cannot be defined. As a result, any two regions of superspace, such that the families of geometries corresponding to the regions belong to different topologies, are disjoint in superspace.

The concept of a vector field cannot be defined in superspace. Only «tangent» vectors $d\mathcal{G}$ can be defined attached to a point $\mathcal{G}$ in superspace. Consequently a metric tensor does not exist in superspace. However in the metric coördinate system we have the functions G^{ijmn} that play the role of a metric tensor. These functions will thus be called «the components of the metric tensor in superspace», keeping in mind that they are defined only in the metric description of 3-geometry and they do not have any invariant meaning.

We introduce here the notion of «parallelism» of two supervectors defined at two distinct but nearly points in superspace. We construct at the point in superspace corresponding to the configuration $\mathcal{G} + d\mathcal{G}$ the super vector $(d\mathcal{G}^*)_{\mathcal{G}+d\mathcal{G}}$ parallel to a supervector $(d\mathcal{G}^*)_{\mathcal{G}}$ defined at the point in superspace corresponding to the configuration $\mathcal{G}$ by assuming that the differences in the components of the two supervectors

$$(dg^*_{ij})_{g_{ij}+dg_{ij}} - (dg^*_{ij})_{g_{ij}} = - C_{ij}\,(g_{ij},\ dg_{ij},\ dg^*_{ij} \text{ and derivatives}), \quad (23)$$

are functions C_{ij} depending on the coördinates g_{ij} of superspace and their derivatives and on the supervector components dg_{ij} and dg_{ij} and their derivatives defined at the point $\mathcal{G}$ in superspace. Furthermore, the functions C_{ij} are of first order in both dg_{ij} and dg^*_{ij}.

If the form of the element of are length $d\mathcal{L}$ in superspace is given the functions C_{ij} are determined by demanding only equality in length of the two supervectors, imposed however for arbitrary supervectors $d\mathcal{G}$ and $d\mathcal{G}^*$.

$$d\mathcal{L}\left[g_{ij},\ (dg^*_{ij})_{g_{ij}}\right] = d\mathcal{L}\left[g_{ij} + dg_{ij},\ (dg^*_{ij})_{g_{ij}+dg_{ij}}\right] \quad (24)$$

In the metric coördinate system of superspace the functions C_{ij} assume then the form

$$C_{ij} = C^{mnpq}_{ij}\, dg_{mn}\, dg^*_{pq}$$

Here $$C^{mnpq}_{ij} = \frac{1}{2}\,(G^{-1})_{ijkl}\left[\frac{\partial G^{klmn}}{\partial g_{pq}} + \frac{\partial G^{klpq}}{\partial g_{mn}} - \frac{\partial G^{pqmn}}{\partial g_{kl}}\right], \quad (25)$$

where $(G^{-1})_{ijkl} = \frac{2}{\sqrt{g}}\left[g_{ij}\, g_{kl} - (g_{ik}\, g_{jl} + g_{il}\, g_{jk})\right]$ are the components of the matrix that is the inverse of the matrix with components G^{klmn}, that is to say $(G^{-1})_{ijkl}\, G^{klmn} = \frac{1}{2}\,(\delta^m_i\, \delta^n_j + \delta^m_j\, \delta^n_i)$. The functions C^{mnpq}_{ij} depending on the metric g_{ij} of the configuration $\mathcal{G}$ only, play the role of «the affine connection in superspace», and they will be called this way, although it is always necessary to keep in mind that they are defined only in the metric coördinate system of superspace.

Consider now a supervector $d\mathcal{G}^*$ defined at a point in superspace corresponding to the geometrical configuration $\mathcal{G}$, and a path $\mathcal{G} = \mathcal{G}(\tau)$ in superspace passing through the point $\mathcal{G}$. We can «parallel transport» the supervector $d\mathcal{G}^*$ along the path $\mathcal{G} = \mathcal{G}(\tau)$ by division of the path into infinitesimal segments and repeated application for each segment of definition (11) of the supervector parallel to a given supervector defined at a nearby point in superspace. If $(dg_{ij})_{\text{parallel}}$ denote the components of the parallel transported supervector $d\mathcal{G}^*$, then

$$\frac{d}{d\tau}(dg^*_{ij})_{\text{parallel}} + C_{ij}\left(g_{ij}(\tau),\ \frac{dg_{ij}}{d\tau}(\tau),\ (dg_{ij})_{\text{parallel}}\right) = 0 \tag{26}$$

Let us now parallel transport along the path $\mathcal{G} = \mathcal{G}(\tau)$ not an arbitrary supervector $d\mathcal{G}^*$, but the supervector $d\mathcal{G}/d\tau$ tangent to the path $\mathcal{G}(\tau)$ at the point $\mathcal{G}$. Consider then a path $\mathcal{G} = \mathcal{G}(\tau)$ in superspace such that the parallel transported supervector is equal to the tangent supervector to the path at the same point. It can be prooved that such a path, satisfying the equation

$$\frac{d^2 g_{ij}}{d\tau^2} + C_{ij}\left(g_{ij},\ \frac{dg_{ij}}{d\tau},\ \frac{dg_{ij}}{d\tau}\right) = 0, \tag{27}$$

minimizes the distance of any two points on the path. Hence such paths (geodesics in superspace) will be called «straight lines» in the following.

(4) Absolute Differential Calculus of Functionals

If the limit

$$\frac{\mathcal{D}\varphi[\mathcal{G}]}{\mathcal{D}\mathcal{G},\, d\mathcal{G}} = \lim_{\varepsilon \to 0}\left\{\frac{1}{\varepsilon}\left(\varphi[\mathcal{G} + \varepsilon d\mathcal{G}] - \varphi[\mathcal{G}]\right)\right\} = \chi\,[\mathcal{G}, d\mathcal{G}] \tag{28}$$

exists for the functional $\varphi[\mathcal{G}]$, at the point in superspace corresponding to the geometrical configuration $\mathcal{G}$, for the approach $d\mathcal{G}$ of the limit, we define it to be «the partial derivative of the functional $\varphi[\mathcal{G}]$ with respect to the approach $d\mathcal{G}$» at the point in superspace considered. Then the functional $\varphi[\mathcal{G}]$ will be called «differentiable at $\mathcal{G}$ for the limiting approach $d\mathcal{G}$». The partial derivative is an invariant functional $\chi[\mathcal{G}, d\mathcal{G}]$ depending not only on the geometry $\mathcal{G}$, but also on the supervector $d\mathcal{G}$. Furthermore it is of first order in the super vector $d\mathcal{G}$.

The partial derivative of a general functional $\chi[\mathcal{G}, d\mathcal{G}]$ of the geometry $\mathcal{G}$ and the supervector $d\mathcal{G}$, with respect to the approach $d\mathcal{G}$ is defined as follows

$$\frac{\mathcal{D}\chi[\mathcal{G}, d\mathcal{G}]}{\mathcal{D}\mathcal{G}, d\mathcal{G}} = \lim_{\varepsilon \to 0}\left\{\frac{1}{\varepsilon}\left(\chi[\mathcal{G}+\varepsilon d\mathcal{G}, (d\mathcal{G})_{\mathcal{G}+\varepsilon d\mathcal{G}}] - \chi[\mathcal{G}, (d\mathcal{G})_{\mathcal{G}}]\right)\right\} = \psi[\mathcal{G}, d\mathcal{G}]. \quad (29)$$

It is again an invariant functional $\psi[\mathcal{G}, d\mathcal{G}]$ of the geometry $\mathcal{G}$ and the limiting approach $d\mathcal{G}$. Here $(d\mathcal{G})_{\mathcal{G}+\varepsilon d\mathcal{G}}$ is the supervector defined at the point $\mathcal{G} + \varepsilon d\mathcal{G}$ in superspace and constructed parallel, as in the previous section, to the original supervector $(d\mathcal{G})_{\mathcal{G}}$ defined at the point $\mathcal{G}$.

The «second derivative of the functional $\varphi[\mathcal{G}]$ with respect to the approach $d\mathcal{G}$» is defined by substituting in Eq, (29) the functional $\chi[\mathcal{G}, d\mathcal{G}]$ of Eq. (28). Thus from the above definitions partial derivatives of all orders can be generated, all invariant under coördinate transformations in superspace. Partial derivatives of n'th order are of n'th order in the approach $d\mathcal{G}$. A functional $\varphi[\mathcal{G}]$ is called «differentiable to n'th order» at the point in superspace corresponding to the geometrical configuration $\mathcal{G}$, for the limiting approach $d\mathcal{G}$, if the nth partial derivative of the functional exists at the point and for the approach in question.

Let us restrict the domain of a functional $\varphi[\mathcal{G}]$ to a path $\mathcal{G}=\mathcal{G}(\tau)$ in superspace. Then the functional $\varphi[\mathcal{G}(\tau)]$ is an ordinary function of the affine parameter τ of the path. If this function is differentiable to n'th order then the value of the function at $\tau+\delta\tau$ can be expressed in terms of the function and its derivatives at τ by the Taylor expansion theorem for ordinary functions of one real variable :

$$\varphi\left[\mathcal{G}(\tau+\delta\tau)\right]=\varphi\left[\mathcal{G}(\tau)\right]+\delta\tau\frac{d\varphi}{d\tau}\left[\mathcal{G}(\tau)\right]+\frac{1}{2}(\delta\tau)^2\frac{d^2\varphi}{d\tau^2}\left[\mathcal{G}(\tau)\right]+$$

$$+\cdots\frac{1}{n!}(\delta\tau)^n\frac{d^n\varphi}{d\tau^n}\left[\mathcal{G}(\tau)\right]+\frac{\chi\left[\mathcal{G}(\tau)\right](\delta\tau)^{n+1}}{(n+1)!} \qquad (30)$$

where $\chi\left[\mathcal{G}(\tau)\right]$ is bounded by $\max\left\{\left|\frac{d^{n+1}\varphi}{d\tau^{n+1}}\left[\mathcal{G}(\tau+\lambda\delta\tau)\right]\right|:0<\lambda<1\right\}$.

If now the path $\mathcal{G}=\mathcal{G}(\tau)$ is a «straight line» then derivatives of $\varphi[\mathcal{G}(\tau)]$, with respect to the affine parameter τ, are equal to functional derivatives of the same order of $\varphi[\mathcal{G}]$, with respect to the approach $d\mathcal{G}/d\tau$, namely the tangent supervector to the «straight line» at the point where the limit is taken:

$$\frac{d^n\varphi}{d\tau^n}=\frac{\mathcal{D}^n\varphi}{\mathcal{D}\mathcal{G},\, d\mathcal{G}/d\tau^n}. \qquad (31)$$

We then deduce from (17) the following Taylor expansion theorem for differentiable functionals:

$$\varphi[\mathcal{G}+\delta\mathcal{G}]=\varphi[\mathcal{G}]+\frac{\mathcal{D}\varphi}{\mathcal{D}\mathcal{G},\, d\mathcal{G}}[\mathcal{G}]+\frac{1}{2}\frac{\mathcal{D}^2\varphi}{\mathcal{D}\mathcal{G},\, d\mathcal{G}^2}[\mathcal{G}]+\cdots$$

$$+\frac{1}{n!}\frac{\mathcal{D}^n\varphi}{\mathcal{D}\mathcal{G},\, d\mathcal{G}^n}[\mathcal{G}]+\frac{1}{(n+1)!}\chi[\mathcal{G},\, d\mathcal{G}], \qquad (32)$$

where $\chi[\mathcal{G},d\mathcal{G}]$ is bounded by $\max\left\{\left|\frac{\mathcal{D}^{n+1}\varphi}{\mathcal{D}\mathcal{G},\, d\mathcal{G}^{n+1}}\left[\mathcal{G}(\tau+\lambda\delta\tau)\right]\right|:0<\lambda<1\right\}$, expressing the value of a functional φ of the 3-geometry at the point $\mathcal{G} + \delta\mathcal{G}$ in superspace in terms of the value of the functional and its derivatives at the point $\mathcal{G}$ with respect to the approach $d\mathcal{G} = (d\mathcal{G}/d\tau)\,\delta\tau$, where $d\mathcal{G}/d\tau$ is the tangent supervector at the point $\mathcal{G}$ to a straight line with affine parameter τ joining the points $\mathcal{G}\,(=\mathcal{G}(\tau))$ and $\mathcal{G} + \delta\mathcal{G}\;(=\mathcal{G}(\tau+\delta\tau))$ in superspace.

Finally, we define the 2n'th order «total derivative» of the functional $\varphi[\mathcal{G}]$ at the point $\mathcal{G}$ in superspace, by integrating the partial derivatire of the same order with the weighting factor $\exp\left\{i\, d\mathcal{L}^2[\mathcal{G}, d\mathcal{G}]\right\}$ explained in the following section, with respect to the limiting approaches $d\mathcal{G}$:

$$\frac{\mathcal{D}^{2n}\varphi}{\mathcal{D}\mathcal{G}^{2n}} = \text{const.} \int e^{i d\mathcal{L}^2} \frac{\mathcal{D}^{2n}\varphi}{\mathcal{D}\mathcal{G},\, d\mathcal{G}^{2n}}\, \mathcal{D}(d\mathcal{G}). \qquad (33)$$

Here $\mathcal{D}(d\mathcal{G})$ is the volume element that the differentials of the supervectors $d\mathcal{G}$ define, and it is determined in the following section. The constant factor in definition (33) is $\left((i/2)^n\,[(2n-1)(2n-3)\ldots 1]\right)^{-1}$. Only total derivatives of even order can be invariantly defined. The total derivatives of all even orders are functionals of the 3-geometry $\mathcal{G}$ only and they are of course invariant under coördinate transformations in superspace, namely under a change of the way of describing de 3-geometry. They are analogous to the Laplacian derivatives $\nabla^{2n} f$ of a function $f(x^1, \ldots, x^m)$ in an ordinary m-dimensional Riemannian space.

(5) The Wave Equation

Consider the definition of the wave functional, Eq. (19). We can express it in the form

$$\varphi[\mathcal{G}, \tau] = \int_{\text{all superspace}} \exp\left\{i\, S'[\mathcal{G}(\tau), \mathcal{G}'(\tau-\delta\tau)]\right\} \varphi[\mathcal{G}', \tau-\delta\tau]\, \mathcal{D}\mathcal{G}', \qquad (34)$$

where we have made the substitutions $\mathcal{G}$, $\mathcal{G}'$, τ for $\mathcal{G}_k$, $\mathcal{G}_{k-1}$, τ_k respectively. The action in the exponential is taken over a «straight line» segment in superspace joining the points corresponding to the geometrical configurations $\mathcal{G}$ and $\mathcal{G}'$, as explained in section 2. For a «straight line» we have

$$\frac{d}{d\tau}\left(\frac{d\mathcal{L}}{d\tau}\right) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left\{ d\mathcal{L}\left[\mathcal{G}(\tau+\varepsilon),\ \frac{d\mathcal{G}}{d\tau}(\tau+\varepsilon)\right] - \right.$$

$$\left. - d\mathcal{L}\left[\mathcal{G}(\tau),\ \frac{d\mathcal{G}}{d\tau}(\tau)\right]\right\} = 0 \qquad (35)$$

from Eq. (24), since by definition the tangent supervector $\frac{d\mathcal{G}}{d\tau}$ at the point $\mathcal{G}(\tau+\varepsilon)$ in superspace is equal for a «straight line» to the tangent supervector $\frac{d\mathcal{G}}{d\tau}$ parallel transported along the «straight line» from the point corresponding to the geometrical configuration $\mathcal{G}(\tau)$. Using the form of the action given by Eq. (16) as well as the results expressed by Eqs. (31), (32), (35) we can expand the action in the exponential of expression (34) in power series in $\delta\tau$:

$$16\pi\, S'\left[\mathcal{G}(\tau),\ \mathcal{G}'(\tau-\delta\tau)\right] = \frac{1}{\delta\tau} d\mathcal{L}^2\left[\mathcal{G},\ d\mathcal{G}\right] + \delta\tau\,\mathcal{R}\left[\mathcal{G}\right] + \cdots$$

$$+ \frac{(\delta\tau)^n}{n!}(-1)^{n-1}\frac{\mathcal{D}^{n-1}\mathcal{R}}{\mathcal{D}\mathcal{G},\ d\mathcal{G}^{n-1}},\ \text{ to } n^{\text{'th}} \text{ order in } \delta\tau. \qquad (36)$$

Here $d\mathcal{G}$ is the supervector $\left(\frac{d\mathcal{G}}{d\tau}\right)_\tau \delta\tau$, where $\left(\frac{d\mathcal{G}}{d\tau}\right)_\tau$ is the tangent supervector, at the point $\mathcal{G}(\tau)$ in superspace, to the «straight line» joining the points $\mathcal{G}(\tau)$ and $\mathcal{G}'(\tau-\delta\tau)$.

The wave functional $\varphi[\mathcal{G}',\ \tau-\delta\tau]$ of the right hand side of Eq. (34) can be expressed in terms of the functional $\varphi[\mathcal{G},\ \tau]$, of the left hand side of the same equation, and its partial derivatives, by the Taylor expansion theorem for differentiable functionals expressed by Eq. (32):

$$\varphi[\mathcal{G}', \tau - \delta\tau] = \varphi[\mathcal{G}, \tau] - \delta\tau \frac{\partial \varphi}{\partial \tau}[\mathcal{G}, \tau] + \cdots +$$

$$+ (-1)^n \frac{(\delta\tau)^n}{n!} \frac{\partial^n \varphi}{\partial \tau^n}[\mathcal{G}, \tau] - \frac{\mathcal{D}\varphi}{\mathcal{D}\mathcal{G}, d\mathcal{G}}[\mathcal{G}, d\mathcal{G}, \tau] + \cdots$$

$$+ \frac{(-1)^m}{m!} \frac{\mathcal{D}^m \varphi}{\mathcal{D}\mathcal{G}, d\mathcal{G}^m}[\mathcal{G}, d\mathcal{G}, \tau], \qquad (37)$$

to n'th order in $\delta\tau$ and m'th order in $d\mathcal{G}$.

Substitution of expansions (36) and (37) in Eq. (34) leads to conditions that must be satisfield to each order in $\delta\tau$. The terms of zero'th order in $\delta\tau$ give the condition

$$\int e^{id\mathcal{L}^2} \mathcal{D}(d\mathcal{G}) = 1, \qquad (38)$$

determining the volume element $\mathcal{D}(d\mathcal{G})$ that the differentials of the supervectors $d\mathcal{G}$ define in superspace. In the metric description of 3-geometry, this volume element assumes the form

$$\mathcal{D}(d\mathcal{G}) = G^{1/2}(1) \prod_{i>j} d(dg_{ij}(1)) \times G^{1/2}(2) \prod_{i>j} d(dg_{ij}(2)) \times \ldots, \qquad (39)$$

indenumerably infinite product over all points of ordinary space. Here G is the determinant of the — pseudo — «metric tensor in superspace» G^{ijmn} given by Eq. (10). The volume element that the supervectors $d\mathcal{G}$ themselves define in superspace, namely the «volume element of superspace» $\mathcal{D}\mathcal{G}$ itself, is then determined uniquely from the volume element $\mathcal{D}(d\mathcal{G})$ by direct substitution of $d\mathcal{G}$ — namely dg_{ij}'s — for $d(d\mathcal{G})$ — namely $d(dg_{ij})$'s.

The terms of first order in $\delta\tau$ give the functional equation

$$4\pi \frac{\mathcal{D}^2 \varphi}{\mathcal{D}\mathcal{G}^2} + \frac{\mathcal{R}}{16\pi}\varphi = \frac{1}{i}\frac{\partial \varphi}{\partial \tau}, \qquad (40)$$

and terms of order in $\delta\tau$ higher than the first, give functional equations derivable from the above. As defined in section (2), $\varphi[\mathcal{G}, \tau]$ is the amplitude that the universe assumes the geometrical configuration $\mathcal{G}$ with the value τ of time. Since the time τ is causal, it is certain that for any value of τ the universe assumes s o m e geometrical configuration. Hence φ is normalized as follows

$$\int_{\text{all superspace}} |\varphi[\mathcal{G}, \tau]|^2 \mathcal{D}\mathcal{G} = 1 \tag{41}$$

it must be remarked here that the region of integration in (41) need only be the class of geometries with a given topology, since regions of different topology are disjoint in superspace (section 3). Thus the region of definition of $\varphi[\mathcal{G}, \tau]$ is the product manifold of one such region in superspace with the real line of time. We can verify from Eq. (40) that the left hand side of Eq. (41) is constant in time. Since the left hand side of Eq. (40) does not depend explicitly on τ, the equation can be solved by separation of the variables $\mathcal{G}$ and τ. The general solution has the form $\varphi[\mathcal{G}, \tau] = \int_{-\infty}^{+\infty} e^{-i\lambda\tau} \psi_\lambda[\mathcal{G}]\, d\lambda$.

Now we impose condition (21) of section (2) by requiring that we extract from φ the eigenfunction ψ corresqonding to the eigenvalue $\lambda = 0$. Thus we get the wave functional $\psi[\mathcal{G}]$ satisfying the wave equation

$$\frac{\mathcal{D}^2\psi}{\mathcal{D}\mathcal{G}^2} + \frac{\mathcal{R}}{64\pi^2} \cdot \psi = 0 \tag{42}$$

Then the wave functional $\psi[\mathcal{G}] = \int_{\text{all }\tau} \varphi[\mathcal{G}, \tau]\, d\tau$ is the amplitude that the universe assumes the geometrical configuration $\mathcal{G}$ with any value of the time τ. Here again the region of definition of $\psi[\mathcal{G}]$ is any isolated region in superspace, a region where the geometry $\mathcal{G}$ belongs to a unique topology. If φ is normalized, ψ is also normalized, unless it turns out that for a normalized φ, ψ is not defined. It can be prooved that the latter is in fact the case when the domain of ψ is a region in superspace with geometries of open topology; then ψ is not normalizable. The wave functional $\psi[\mathcal{G}]$ can be normalized however, if the domain of ψ is a region of superspace with geometries of closed topology. Then,

$$P[V] = \int_V |\psi[\mathcal{G}]|^2 \mathcal{D}\mathcal{G} \tag{43}$$

is the probability that the outcome $\mathcal{G}$, of a measurement of the 3-geometry of the universe performed without an accompanying measurement of the time, belongs to the region V of superspace. If two such measurements of the 3-geometry give identical outcomes then the measurements are tautochronous. If two measurements of the 3-geometry give outcomes differing by $d\mathcal{G}$, then the temporal separation of the measurements is (Eq. (21))

$$d\tau = \frac{d\mathcal{L}}{\mathcal{R}^{1/2}}$$

where $d\mathcal{L}$ is the length of the supervector $d\mathcal{G}$ defined at the point in superspace to which the outcome of one of the measurements corresponds. Thus, to obtain complete knowledge of the geometrical world it is necessary and sufficient to know $\psi[\mathcal{G}]$ over all its domain.

In conclusion, the laws expressed by Eqs. (42) and (43), as well as the law of the definition of time Eq. (21), complete the statement of the fundamental laws of quantum geometrodynamics.

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ΠΕΡΙΛΗΨΙΣ

Εἰς τὴν παροῦσαν ἐργασίαν ἡ θεωρία τῆς Κβαντικῆς Γεωμετροδυναμικῆς θεμελιοῦται μόνον ἐπὶ τῆς ἀρχῆς τῆς Σχετικότητος καὶ τῆς ἀρχῆς τῶν Κβάντα.

Τὸ πρῶτον μέρος τῆς ἐργασίας ἀφορᾷ εἰς φυσικὴν ἑρμηνείαν τῆς γενικῆς Σχετικότητος. Αἱ ἐξισώσεις τοῦ Einstein διακρίνονται εἰς δύο ὁμάδας:

1) Ἐξισώσεις ἀναφερομένας εἰς τὴν κλασικὴν Γεωμετροδυναμικήν,

2) Ἐξισώσεις ἐκφραζούσας θεμελιώδεις τῆς Γεωμετροδυναμικῆς νόμους καὶ δι᾽ αὐτὸ ἰσχυούσας, κατ᾽ ἀνάγκην, εἰς τὴν Κβαντικὴν Γεωμετροδυναμικήν.

Εἰς τὸ δεύτερον μέρος εἰσάγεται, κατὰ J. A. Wheeler, ἡ ἔννοια τοῦ ὑπερχώρου ὁριζομένης καὶ τῆς ἐννοίας τῆς ἀποστάσεως εἰς τὸν ὑπερχῶρον

Εἰς τὸ τρίτον μέρος ἐρευνᾶται ἡ δομὴ τοῦ ὑπερχώρου.

Εἰς τὸ τέταρτον μέρος εἰσάγεται ἀπόλυτος ἀπειροστικὸς Λογισμὸς τῶν συναρτησιακῶν.

Τέλος, εἰς τὸ πέμπτον μέρος συνάγονται οἱ συμφυεῖς τῆς Κβαντικῆς Γεωμετροδυναμικῆς νόμοι, ἐκφραζόμενοι μὲ τὴν εἰς τὸ κείμενον κυματικὴν ἐξίσωσιν (42). Ἡ κυματικὴ αὕτη ἐξίσωσις θεμελιώνει τὴν Κβαντικὴν Γεωμετροδυναμικήν.

★

Παρουσιάζων τὴν ἀνωτέρω ἀνακοίνωσιν ὁ Ἀκαδημαϊκὸς κ. **Φ. Βασιλείου,** λέγει τὰ ἑξῆς:

«Ὁ κ. Χριστοδούλου εἶναι Research Fellow in Physics καὶ μέλος τῆς Faculty τοῦ Τεχνολογικοῦ Ἰνστιτούτου τῆς Καλιφορνίας, γνωστοῦ συντόμως ὡς Caltec.

Εἰς γενικὰς γραμμάς, διὰ τῆς παρούσης ἐργασίας ὁ κ. Χριστοδούλου, βασιζόμενος ἐπὶ τῆς ἀρχῆς τῆς Σχετικότητος καὶ τῆς ἀρχῆς τῶν Κβάντα, ἐπιδιώκει τὴν ἑρμηνείαν εὐρυτέρων μερῶν τοῦ φυσικοῦ κόσμου ἀπ᾽ ὅ,τι ἡ θεωρία τῆς Σχετικότητος. Εἰς τὴν πιθανολογικὴν αὐτὴν θεωρίαν του χρησιμοποιεῖ λίαν προχωρημένας γνώσεις ἀπὸ τὰ Ἀνώτερα Μαθηματικά. Ἀξιοσημείωτον εἶναι ὅτι — κατὰ τὴν ὁμολογίαν τοῦ ἰδίου — εἰς τὴν ἐν λόγῳ ἐργασίαν ἀκολουθεῖ οὗτος τὴν διδασκαλίαν τοῦ Πλάτωνος ἀναζητῶν, διὰ τὴν ἑρμηνείαν τῶν φαινομένων τοῦ φυσικοῦ κόσμου, τὴν ὕπαρξιν ἰδεατοῦ γεωμετρικοῦ κόσμου.

Παραπέμπων τοὺς ἐνδιαφερομένους, διὰ τὰς λεπτομερείας τοῦ περιεχομένου τῆς ἐργασίας, εἰς τὰ Πρακτικὰ τῆς Ἀκαδημίας, θὰ ἐπεθύμουν νὰ ἀναφέρω ἐν ὀλίγοις τὴν πράγματι ἀσυνήθη ἐπιστημονικὴν ἐξέλιξιν τοῦ νεαροῦ συγγραφέως διανύοντος σήμερον μόλις τὸ εἰκοστὸν ἔτος τῆς ἡλικίας του. Οὗτος, ἕνα καὶ ἥμισυ ἔτος πρὸ τῆς ἀποφοιτήσεώς του ἀπὸ τὸ Γυμνάσιον εἰς Ἀθήνας καὶ κατόπιν εἰδικῆς ἐξετάσεως εἰς τὴν ὁποίαν ὑπεβλήθη εἰς Παρισίους ἀπὸ τὸν καθηγητὴν τοῦ Πανεπιστημίου τοῦ Princeton (Ἡνωμ. Πολιτειῶν) John Wheeler, καὶ τὸν καθηγητὴν Ἀχ. Παπαπέτρου, εἶχεν εἰσαχθῆ εἰς τὸ ἐν λόγῳ Πανεπιστήμιον καὶ μάλιστα ἀμέσως ὡς Graduate Student, κατὰ παρέκκλισιν τῶν ἰσχυουσῶν διατάξεων ἐγγραφῆς.

Ἡ ἔγκαιρος ἀνακάλυψις τῆς ἰδιοφυΐας τοῦ νεαροῦ γυμνασιόπαιδος ὀφείλεται εἰς τὸν τότε ἀντιπρόεδρον τοῦ Τεχνικοῦ Ἐπιμελητηρίου τῆς Ἑλλάδος κ. Σπ. Μιχαλόπουλον, ὅστις ἐπέτυχε τὴν μετὰ τοῦ Καθηγητοῦ Wheeler ἐπαφὴν καὶ τὴν εἰς Παρισίους συνάντησιν.

Μετὰ ἕνα σχεδὸν ἔτος σπουδῶν, ὁ κ. Χριστοδούλου ἔλαβε τὸ δίπλωμα Master of Science διὰ τὴν ἀπόκτησιν τοῦ ὁποίου προαπαιτεῖται κανονικῶς ὄχι μόνον τὸ πτυχίον Γυμνασίου ἀλλὰ καὶ τό, κατόπιν τουλάχιστον τετραετῶν σπουδῶν χορηγούμενον ἐκεῖ, δίπλωμα Bachelor. Ἕνα ἔτος μετὰ τὸ Master, ὁ κ. Χριστοδούλου γίνεται διδάκτωρ τῆς Φιλοσοφίας εἰς τὸν κλάδον τῆς Μαθηματικῆς Φυσικῆς, εἰς ἡλικίαν 19 ἐτῶν, μὲ τὴν σπανιώτατα προηγουμένως ἀπονεμηθεῖσαν διάκρισιν Excellent, τόσον διὰ τὰς προφορικὰς ἐξετάσεις ὅσον καὶ διὰ τὴν ἐπὶ διδακτορίᾳ διατριβήν του. Σημειωτέον ὅτι πρόεδρος τῆς ἐξεταστικῆς Ἐπιτροπῆς διὰ τὴν ἐπὶ διδακτορίᾳ ἐξέτασιν ἦτο ὁ Καθηγητὴς Eugene Wigner, βραβεῖον Nobel, μέλη δὲ ὁ προαναφερθεὶς John Wheeler, βραβεῖον Fermi, καθὼς καὶ ἄλλαι διεθνοῦς κύρους προσωπικότητες.

Τέλος, αἱ μέχρι σήμερον ἤδη δημοσιευθεῖσαι πέντε ἐπιστημονικαὶ ἐργασίαι τοῦ κ. Χριστοδούλου ἔτυχον εὐμενοῦς κριτικῆς κατέστησαν δὲ αὐτὸν εὐρύτερα γνωστόν».

Ἀκολούθως ὁ κ. **Πρόεδρος** λέγει:

«Δὲν εἶμαι ἁρμόδιος νὰ συζητήσω τὴν ἀνακοίνωσιν τοῦ κ. Χριστοδούλου τὴν γινομένην ὑπὸ τοῦ συναδέλφου κ. Βασιλείου, τὸν ὁποῖον εὐχαριστῶ.

Δὲν δύναμαι ὅμως νὰ παρέλθω ἀπαρατήρητον τὸ γεγονὸς ὅτι ὁ νεαρὸς Ἕλλην ἀνεγνωρίσθη παγκοσμίως ὡς μαθηματικὴ ἰδιοφυΐα καὶ ἐγένετο ἤδη εἰς ἡλικίαν 19 ἐτῶν διδάκτωρ τοῦ Princeton, μὲ τὸν βαθμὸν Excellent.

Δὲν εἶναι δυστυχῶς παρὼν ὁ κ. Χριστοδούλου. Ἡ Ἀκαδημία ὅμως τοῦ στέλλει δι' ἐμοῦ καὶ ἐν συνεδρίᾳ τῆς Ὁλομελείας της, ἐκεῖ μακρὰν ὅπου εὑρίσκεται, τοὺς ἑλληνικούς της χαιρετισμούς, τὰ συγχαρητήριά της καὶ τὰς εὐχάς της. Ἂς τὸν παρακολουθοῦν παντοῦ καὶ εἰς ὅλην του τὴν ζωὴν αἱ εὐλογίαι τῆς πατρίδος του, τῆς αἰωνίας Ἑλλάδος».

Τέλος, ὁμιλεῖ περὶ τῆς ἰδιοφυΐας τοῦ κ. Χριστοδούλου καὶ ὁ κ. **Ι. Ξανθάκης**: «Ἐκλήθην, λέγει, καὶ παρηκολούθησα εἰς τὸ Ἐθνικὸν Ἵδρυμα Ἐρευνῶν διάλεξιν τοῦ νεαροῦ κ. Χριστοδούλου ἐπὶ σχετικοῦ θέματος. Ὁμολογῶ ὅτι κατέβαλον ἰδιαιτέραν προσπάθειαν διὰ νὰ παρακολουθήσω τὸν ὁμιλητήν. Τοῦτο ὠφείλετο, ἀφ' ἑνὸς μὲν εἰς τὸ γεγονὸς ὅτι ὁ νέος αὐτὸς χρησιμοποιοῦσε τὴν ἐντελῶς πρόσφατον μαθηματικὴν διάλεκτον μὲ εἰδικοὺς συμβολισμούς, οἱ ὁποῖοι δὲν μοῦ ἦσαν ἐντελῶς οἰκεῖοι, ἀφ' ἑτέρου δέ, διότι ἐδυσκολεύετο νὰ ἐκφρασθῇ σαφῶς εἰς τὴν κοινῶς ὁμιλουμένην ἑλληνικὴν γλῶσσαν. Πάντως τοῦ ὑπέβαλον πολλὰς ἐρωτήσεις, ἀφ' ἑνὸς μὲν διὰ νὰ διασαφηνίσω τοὺς ἐπιστημονικοὺς ὁρισμούς, τοὺς ὁποίους ἔδιδε, ἀφ' ἑτέρου δὲ διὰ νὰ διαπιστώσω κατὰ πόσον ἦτο κάτοχος ἢ ἐνήμερος ὡρισμένων ἀστρονομικῶν φαινομένων, τὰ ὁποῖα προέβλεπε ἡ γενικὴ θεωρία τοῦ Ἀϊνστάϊν. Ἐκ τῆς συζητήσεως ταύτης διεπίστωσα ὅτι αἱ προσπάθειαι τοῦ κ. Χριστοδούλου τείνουν νὰ ὑπερκεράσουν τὴν θεωρίαν τῆς Σχετικότητος, δηλαδὴ νὰ διατυπώσῃ νέαν θεωρίαν, ἡ ὁποία νὰ περιλαμβάνῃ ἐκτὸς τῶν γνωστῶν φαινομένων τοῦ μακροκόσμου καὶ ὡρισμένα φαινόμενα τοῦ μικροκόσμου. Εὐνόητον τυγχάνει ὅτι μία τοιαύτη σύνδεσις τῆς μακροφυσικῆς καὶ τῆς μικροφυσικῆς θὰ εἶναι, ἐὰν πράγματι ἐπιτευχθῇ, ἐξαιρετικὸν γεγονὸς διὰ τὴν Ἐπιστήμην.

Τοῦτο ἄλλωστε ὑπῆρξε καὶ τὸ ἀντικείμενον τῶν ἐρευνῶν τοῦ Ἀϊνστάϊν κατὰ τὰ τελευταῖα ἔτη τῆς ζωῆς του.

Ἐν ὀλίγοις, ἡ προσωπική μου γνώμη εἶναι ὅτι ὁ νέος οὗτος κέκτηται τάλαντον ἐξαιρετικὸν ἢ ὅτι πρόκειται περὶ μαθηματικῆς ἰδιοφυΐας».