ΠΡΑΚΤΙΚΑ ΤΗΣ ΑΚΑΔΗΜΙΑΣ ΑΘΗΝΩΝ

ΣΥΝΕΔΡΙΑ ΤΗΣ 25ΗΣ ΜΑΪΟΥ 1989

ΠΡΟΕΔΡΙΑ ΣΟΛΩΝΟΣ ΚΥΔΩΝΙΑΤΟΥ

ΜΑΘΗΜΑΤΙΚΑ.— **More on univalent starlike functions,** *by Nicolas K. Artemiadis**, Regular member of the Academy of Athens.

1. INTRODUCTION

Let $\mathcal{S}$ be the class of functions $f(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n$ which are analytic and univalent in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. For $f \in \mathcal{S}$ the set f(D) is a non-empty open connected proper subset of the complex plane $\mathbb{C}$. A point $w \in f(D)$ is called a star center point (s.c.p) of f(D) if and only if:

$$tf(z) + (1 - t)\, w \in f(D), \qquad z \in D, \quad 0 \leqslant t \leqslant 1$$

For $f \in \mathcal{S}$, let S_f be the set of all s.c.p. of f(D). Define

$$\mathcal{S}_0 = \{f \in \mathcal{S} : 0 \in \mathring{S}_f\}$$

where $\mathring{S}_f$ is the interior of S_f.

In this paper the influence that the size of $\mathring{S}_f$ has on the Taylor coefficients, α_n, of a function in $\mathcal{S}_0$ is examined.

We first prove three lemmas which will be used later.

In Theorem 1 we obtain estimates of $|\alpha_n|$, depending only on the entire set $\mathring{S}_f$ for $f \in \mathcal{S}_0$.

In Theorem 2 it is shown that is $f_1, f_2 \in \mathcal{S}_0$ and $\mathring{S}_{f_1} \subset \mathring{S}_{f_2}$, then

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$$B(f_2, n) \leqslant B(f_1, n), \qquad n = 1, 2, \ldots,$$

where $B(f_1, n)$, $B(f_2, n)$ are the estimates obtained in Th. 1 for the n^{th} coefficients of f_1 and f_2 respectively. In other words, Th. 2 asserts that the larger $\mathring{S}_f$ gets the more restrictive are the coefficient bounds given by Th. 1.

Finally we give examples of functions in $\mathcal{S}_0$ and discuss the obtained results.

2. PRELIMINARIES

Lemma 1. The set of all star center points of a function in $\mathcal{S}$ is convex, therefore simply connected.

Proof. Let $g \in \mathcal{S}$, $z_1, z_2 \in D$ such that $g(z_1)$, $g(z_2)$ belong to S_g. We show that the segment $[g(z_1), g(z_2)]$ is contained in S_g. Suppose $[g(z_1), g(z_2)] \not\subset S_g$ and let $w \in (g(z_1), g(z_2))$ such that $w \notin S_g$. Since $g(z_1)$, $g(z_2)$ are s.c.p of $g(D)$ we have $w \in g(D)$.

By the hypothesis on w there is $z_0 \in D$ such that $[g(z_0), w] \not\subset g(D)$. Observe that if the points $g(z_0)$, $g(z_1)$, $g(z_2)$ are colinear then there is nothing to prove. Otherwise there is $w_1 \in (g(z_0), w)$ such that $w_1 \notin g(D)$. We have

$$[g(z_1), g(z_0)] \subset g(D)$$

because $g(z_1) \in S_g$ and $g(z_0) \in g(D)$. Let w_2 be the intersection of the segment $[g(z_1), g(z_0)]$ and the staight line determined by the points $g(z_2)$, w_1. These two sets intersect because w_1 is an interior point of the triangle $\{g(z_0), g(z_1), g(z_2).\}$. We have $w_2 \in g(D)$. Since $g(z_2) \in S_g$ it follows that $w_1 \in g(D)$ which is absurd because it contradicts $w_1 \notin g(D)$. Hence S_g is convex. This proves the lemma.

Lemma 2. Let $f \in \mathcal{S}_0$, $\xi: D \to \mathring{S}_f$ be a one-one analytic function such that $\xi(0) = 0$, $\xi(D) = \mathring{S}_f$, and let z_0, z_1 be complex numbers such that $|z_0| < |z_1| = r < 1$. Then the segment $[f(z_1), \xi(z_0)]$ is contained in $f(\overline{D}_r)$, where $\overline{D}_r = \{z: |z| \leqslant r\}$.

Proof. For $\xi(z_0) = 0$ the lemma is known (see [2], p. 220). Let ρ and θ be two real numbers such that $0 < \rho < 1$, $-\pi \leqslant \theta \leqslant \pi$, $\rho e^{i\theta} z_1 = z_0$. Put

$$\Phi(z) = tf(z) + (1 - t)\, \xi(\rho e^{i\theta} z), \qquad z \in D, 0 \leqslant t \leqslant 1.$$

Clearly Φ is analytic in D. $\Phi(0) = f(0) = 0$, and for each z the point $\xi(\rho e^{i\theta} z)$ is a s.c.p of f(D). Hence Φ is subordinate to f, so that $\Phi(z) = f(\varphi(z))$, where φ is analytic in D, $\varphi(0) = 0$, and $|\varphi(z)| \leqslant |z|$. We have

$$\Phi(z_1) = tf(z_1) + (1 - t)\,\xi(\rho e^{i\theta} z_1) = tf(z_1) + (1 - t)\,\xi(z_0) = f\varphi(z_1)$$

and $|\varphi(z_1)| \leqslant |z_1|$. Hence $\Phi(z_1) \in f(\overline{D}_r)$. This proves the lemma.

Lemma 3. Let n be an integer greater than two, and let x be a real number such that $(1/2) \leqslant x \leqslant 1$.

Put
$$\prod_{k=p}^{q} (k - x) = (p - x)(p + 1 - x) \ldots (q - x)$$

where p, q are natural numbers such that $p \leqslant q$

Then

$$(*) \qquad \Gamma_n(x) \leqslant 0$$

where
$$\Gamma_n(x) = -\,n!n + n\prod_{k=2}^{n}(k - x) + 2x\Big[\prod_{k=3}^{n}(k - x) + 2!2\prod_{k=4}^{n}(k - x) + \ldots + (n - 2)!\,(n - 2)\prod_{k=n}^{n}(k - x) + (n - 1)!\,(n - 1)\Big]$$

Proof. We proceed by induction on n. Clearly (*) holds for $n = 3$. Assume that it holds for n. To prove that it holds for $n + 1$ it suffices to show hat

$$\Gamma_{n+1}(x) \leqslant \Gamma_n(x)$$

or equivalently

$$(1) \quad (n + 1)\prod_{k=2}^{n+1}(k - x) - n\prod_{k=2}^{n}(k - x) + 2x.\,n!\,n + 2x\,(n - x)\Big[\prod_{k=3}^{n}(k - x) + 2!2\prod_{k=4}^{n}(k - x) + \ldots + (n - 1)!\,(n - 1)\Big] \leqslant (n + 1)\,!\,(n + 1) - n\,!\,n$$

Now by the induction hypothesis we have

$$2x\Big[\prod_{k=3}^{n}(k - x) + 2!2\prod_{k=4}^{n}(k - x) + \ldots + (n - 1)!\,(n - 1)\Big] \leqslant n!n - n\prod_{k=2}^{n}(k - x)$$

Hence (1) will hold if the following (2) holds

$$(2) \quad (n + 1)\prod_{k=2}^{n+1}(k - x) - n\prod_{k=2}^{n}(k - x) + 2xn!n + \Big(n!n - n\prod_{k=2}^{n}(k - x)\Big) \leqslant (n + 1)\,!\,(n + 1) - n!n$$

which is equivalent to

$$(3) \qquad \Phi(x) = \prod_{k=2}^{n+1}(k - x) + n!x\,.\,x - (n + 1)! \leqslant 0$$

Observe that $\Phi(1) = 0$. Hence (3) will be proven if we show that Φ is nondecreasing, i.e. if $\Phi'(x) \geqslant 0$. We have

$$\Phi'(x) = n!n + \Big(\prod_{k=2}^{n+1} (k - x)\Big)' = n!n - \prod_{k=2}^{n+1} (k - x) \cdot \sum_{k=2}^{n+1} \frac{1}{k - x}$$

To show that $\Phi'(x) \geqslant 0$ for $(1/2) \leqslant x \leqslant 1$ it suffices to show that

$$n!n - \prod_{k=2}^{n+1} \left(k - \frac{1}{2}\right) \cdot \sum_{k=2}^{n+1} \frac{1}{k - \frac{1}{2}} \geqslant 0 \tag{4}$$

We, again, proceed by induction on n. It is easily seen that (4) holds for $n = 3$. Assume that it holds for n. To prove that (4) holds for $n + 1$ we show that

$$(n + 1)!\,(n + 1) - \prod_{k=2}^{n+2} \left(k - \frac{1}{2}\right) \cdot \sum_{k=2}^{n+2} \frac{1}{k - \frac{1}{2}} \geqslant n!n - \prod_{k=2}^{n+1} \left(k - \frac{1}{2}\right) \cdot \sum_{k=2}^{n+1} \frac{1}{k - \frac{1}{2}} \tag{5}$$

or equivalently

$$(n + 1)!\,(n + 1) - n!n \geqslant \prod_{k=2}^{n+1} \left(k - \frac{1}{2}\right) \cdot \left[\left(n + \frac{1}{2}\right) \cdot \sum_{k=2}^{n+1} \frac{1}{k - \frac{1}{2}} + 1\right] \tag{6}$$

If in (6) the expression $\prod_{k=2}^{n+1} \left(k - \frac{1}{2}\right)$ is replaced by $n!n \Big/ \sum_{k=2}^{n+1} \frac{1}{k - \frac{1}{2}}$ we get

$$(n + 1)!\,(n + 1) - n!n \geqslant \left[\frac{n!n}{\sum_{k=2}^{n+1} \frac{1}{k - \frac{1}{2}}}\right] \cdot \left[\left(n + \frac{1}{2}\right) \sum_{k=2}^{n+1} \frac{1}{k - \frac{1}{2}} + 1\right]. \tag{7}$$

Since by hypothesis (4) holds for n, it follows that (6) will hold if (7) holds. But (7) is equivalent to

(8) $$\frac{n}{n+1} \geqslant \sum_{k=1}^{n} \frac{1}{2k+1}, \qquad n \geqslant 3$$

which is easily seen to be true by induction on n. It follows that (6) holds, and this proves the lemma.

3. THE MAIN RESULTS

We wish to give coefficient estimates for the Taylor expansion of a function in $\mathscr{S}_0$.

Let $f \in \mathscr{S}_0$. From Lemma 1 it follows that $\mathring{S}_f$ is a simply connected region. Also $\mathring{S}_f \neq \mathbb{C}$ since $f(D) \neq \mathbb{C}$.

Let α be any point of $\mathring{S}_f$. Riemann's Mapping Theorem asserts that there is a unique analytic function

(9) $$g_\alpha : \mathring{S}_f \to D$$

having the properties:

(a) $g_\alpha(\alpha) = 0$ and $g'_\alpha(\alpha) > 0$

(b) g_α is one-one

(c) $g_\alpha(\mathring{S}_f) = D$

Put $$\mu(f, \alpha) = [1 - |g_\alpha(0)|^2] / g'_\alpha(0)$$

Theorem 1.

Let $f(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n$ be a function in $\mathscr{S}_0$ and let α be a point of $\mathring{S}_f$. Then

(i) $0 < \mu(f, \alpha) \leqslant 1$

(ii) If $\mu(f, \alpha) = 1$ then $|\alpha_n| \leqslant 1$, $n = 1, 2, \ldots$

(iii) $\mu(f, \alpha) = 1$ if and only if $\mathring{S}_f = f(D)$

(iv) If $\mu(f, \alpha) < 1$ then $|\alpha_n| \leqslant A_n(f, \alpha) + R_{n-1}(\sigma) = M_n(f, \alpha)$, $n \geqslant 2$

where $A_n(f, \alpha) = 1 + (n-1) \prod_{k=2}^{n} \frac{k-1}{k-\sigma}$, $\sigma = 1 / (1 + \mu(f, \alpha))$,

$$R_n(\sigma) = \frac{-n!n}{\prod_{k=2}^{n+1}(k-\sigma)} + \frac{n}{n+1-\sigma} + \frac{2\sigma}{n+1-\sigma}\cdot$$

$$\cdot\left[\frac{1}{2-\sigma} + \frac{2!2}{(2-\sigma)(3-\sigma)} + \ldots + \frac{(n-2)!(n-2)}{(2-\sigma)\ldots(n-1-\sigma)} + \frac{(n-1)!(n-1)}{(2-\sigma)\ldots(n-\sigma)}\right]$$

(v) $|\alpha_n| \leqslant B(f, n)$, $n \geqslant 2$, where $B(f, n) = \inf\limits_{\alpha \in \mathring{S}_f} (M_n(f, \alpha))$

Proof. Put $g = g_\alpha^{-1}$ where g_α is the function defined in (9). Then $g : D \to \mathring{S}_f$ is analytic in D and has the following properties:

(a′) $g(0) = \alpha$, $g'(0) = 1/g'_\alpha(\alpha) > 0$

(b′) g in one-one

(c′) $g(D) = \mathring{S}_f$

Let $g_\alpha(0) = \beta$. Then $\beta \in D$, $g(\beta) = 0$

Put

(10) $$G(z) = g\left(\frac{z+\beta}{1+\bar{\beta}z}\right), \quad z \in D$$

The function $G : D \to \mathring{S}_f$ is analytic in D and has the following properties:

(a″) $G(0) = g(\beta) = 0$; $G'(0) = g'(\beta)(1-|\beta|^2) = (1-|\beta|^2)/g'_\alpha(0) = (1-|g_\alpha(0)|^2)/g'_\alpha(0)$

(b″) G in one-one

(c″) $G(D) = \mathring{S}_f$

Clearly G is subordinate to f. It follows that

(11) $$G(z) = f(\omega(z))$$

where ω is analytic on D, $\omega(0) = 0$ and $|\omega(z)| \leqslant |z|$. Put

$$G(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in D.$$

We have since $G'(0)$ does not vanish

(12) $$0 < b_1 = G'(0) = \omega'(0) = [1-|g_\alpha(0)|^2]/g'_\alpha(0) = \mu(f, \alpha) \leqslant 1$$

This proves assertion (i) of Th.1.

The function $G(z)/b_1 = \sum_{n=1}^{\infty} (b_n/b_1) z^n$ belongs to the class $\mathcal{S}_0$ and maps D onto the region $(1/b_1)\mathring{S}_f = \{w/b_1 : w \in \mathring{S}_f\}$ which is convex since $\mathring{S}_f$ is convex. It follows that

$$(13) \qquad |b_n / b_1| \leqslant 1, \qquad n = 1, 2, ...$$

Observe that ω is univalent in D because the composition of two univalent functions is univalent.

Summarizing the properties of ω we have:

(i) ω is univalent in D.

(ii) $|\omega(z)| \leqslant |z|$ so that $\omega(D) \subset D$

(iii) $\omega(0) = 0$

(iv) $0 < \omega'(0) = b_1 \leqslant 1$

If in addition we had $\omega(D) = D$ then we would have

$$\omega(z) = z, \qquad \omega'(0) = b_1 = 1$$

and it would follow from (11) and (13) that $G(z) = f(z)$ so that $\alpha_n = b_n$, $|\alpha_n| \leq 1$. This proves assertion (ii) of Th. 1.

Next assume that $\omega(D)$ is a proper subset of D. Then it follows from the condition for equality in Schwarz's lemma that $\omega'(0) < 1$.

The above imply:

$$(14) \quad \begin{cases} \text{(i)} \quad \omega(D) = D \quad \text{iff} \quad \omega'(0) = 1 \\ \text{(ii)} \quad \text{If} \quad \omega'(0) < 1 \quad \text{then} \quad 0 < b_1 < 1 \\ \text{(iii)} \quad |b_n| \leqslant |b_1| \leqslant 1 \end{cases}$$

and assertion (iii) of Th. 1 follows from (14) (i).

Let $z, z_0 \in D$ such that $|z_0| < |z| = r < 1$. Put $G(z_0) = w \in \mathring{S}_f$, $f(z) - w = Re^{i\tau}$, $z = re^{i\theta}$

It follows from Lemma 2 that w is a s.c.p of f $(\overline{D}_r)$. Therefore

$$\frac{\partial}{\partial\theta} \arg \, [f(z) - w] = \frac{\partial\tau}{\partial\theta} \geqslant 0$$

We have $\qquad \log\,[f(z) - w] = \log R + i\tau$

so that $\qquad \mathcal{I}_m \left[\frac{\partial}{\partial\theta} \log\,(f(z) - w)\right] \geqslant 0$

In view of $\qquad \frac{\partial}{\partial\theta} = ire^{i\theta} \frac{d}{dz} = iz \frac{d}{dz}$

we get $\qquad R_e\,[z f'(z) / (f(z) - G(z_0))] \geqslant 0$

The last inequality holds for all z, z_0 in D provided that $|z| > |z_0|$. Therefore if λ is a real number such that $0 \leqslant \lambda < 1$, we have

$$R_e\left[zf'(z) / (f(z) - G(-\lambda z))\right] \geqslant 0, \quad z \in D$$

Put

$$(15) \qquad F(z) = [zf'(z) / (f(z) - G(-\lambda z))] = \sum_{n=0}^{\infty} c_n z^n, \ z \in D$$

It is easily seen that F is analytic in D and that $c_0 = 1 / (1 + b_1\lambda)$.

Due to the inequality

$$R_e\, F(z) \geqslant 0, \qquad z \in D$$

We have

$$(16) \qquad |c_n| \leqslant 2\, c_0 = \frac{2}{1 + b_1\lambda}$$

From (15) we get

$$zf'(z) = \sum_{n=1}^{\infty} n\alpha_n z^n = \sum_{n=1}^{\infty} [\alpha_n - b_n(-\lambda)^n] z^n \cdot \sum_{n=0}^{\infty} c_n z^n$$

The last equation gives the following relationship between the coefficients α_n, b_n, c_n:

$$n\alpha_n = \sum_{k=1}^{n} [\alpha_k - (-\lambda)^k b_k]\, c_{n-k}, \quad n = 1, 2, \ldots$$

or

$$(17) \qquad (n - c_0)\, \alpha_n = \sum_{k=1}^{n-1} \alpha_k\, c_{n-k} - \sum_{k=1}^{n} (-\lambda)^k b_k\, c_{n-k}$$

If we set $\lambda = 0$ then (17) and (16) provide the well known inequality $|\alpha_n| \leqslant n$, $n = 2, 3, \ldots$.

From (17) we obtain, on account of (13) and (16),

$$|\alpha_n| \leqslant \frac{2c_0}{n - c_0} \sum_{k=1}^{n-1} |\alpha_k| + \frac{1}{n - c_0} \sum_{k=1}^{n} \lambda^k |c_{n-k}|\, |b_k|$$

$$\leqslant \frac{2c_0}{n - c_0} \sum_{k=1}^{n-1} |\alpha_n| + \frac{\lambda_n b_1 c_0}{n - c_0} + \frac{1}{n - c_0} \sum_{k=1}^{n-1} 2b_1\, c_0 \lambda^k$$

Now if we let $\lambda \to 1$ we get, since $b_1\sigma = 1 - \sigma$

$$(18) \qquad |\alpha_n| \leqslant \frac{2\sigma}{n - \sigma} \sum_{k=1}^{n-1} |\alpha_k| + \frac{(1 - \sigma)(2n - 1)}{n - \sigma}, \qquad n \geqslant 2$$

From (18) we deduce the following

$$(19) \qquad |\alpha_n| \leqslant A_n(f, \alpha) + R_{n-1}(\sigma) = M_n(f, \alpha) \leqslant A_n(f, \alpha)$$

The last part of (19) follows immediately from Lemma 3, because $R_n(\sigma)$ is nonpositive for $n \geqslant 1$ and $(1/2) \leqslant \sigma \leqslant 1$.

To prove the first part of (19) we proceed by induction on n. It is easily seen that for $n = 2, 3$, (18) provides

$$|\alpha_2| \leqslant 1 + \frac{1}{2-\sigma} = A_2(f, \alpha) + R_1(\sigma) = A_2(f, \alpha)$$

$$|\alpha_3| \leqslant 1 + \frac{2!2}{(2-\sigma)(3-\sigma)} + A_3(f, \alpha) + R_2(\sigma) = A_3(f, \alpha)$$

because $R_1(\sigma) = R_2(\sigma) = 0$, which proves that (19) holds for $n = 2, 3$. Assume that (19) holds for n. We get from (18), after some calculations:

$$|\alpha_{n+1}| \leqslant \frac{1}{n+1-\sigma} \sum_{k=1}^{n} |\alpha_k| + \frac{(1-\sigma)(2n+1)}{n+1-\sigma}$$

$$\leqslant A_{n+1}(f, \alpha) + R_n(\sigma) = M_{n+1}(f, \alpha)$$

It follows that (19) holds for $n + 1$.

This proves assertion (iv) of Th. 1, while assertion (v) is obvious. The theorem ls proved.

Remark. If in (13) and (16) equality holds fon $n = 2, 3, 4$ then for

$$c_1 = c_2 = c_3 = 2\sigma, \quad b_2 = b_4 = -b_1, \quad b_3 = b_1, \quad \lambda = 1$$

it is easily checked that (22) is sharp for $n \leqslant 4$. Indeed we find

$$\alpha_2 = 1 + \frac{1}{2-\sigma}, \quad \alpha_3 = 1 + \frac{4}{(2-\sigma)(3-\sigma)},$$

$$\alpha_4 = 1 + \frac{18}{(2-\sigma)(3-\sigma)(4-\sigma)} + \frac{\sigma^2 - \sigma}{(2-\sigma)(3-\sigma)(4-\sigma)}$$

However the sharpness of (19) for all n remains open.

We make the following conjecture which we believe it is true.

Conjecture. Let $f \in \mathscr{S}_0$, $\alpha \in \mathring{S}_f$. Then

$$(*) \qquad |\alpha_n| \leqslant A_n(f, \alpha) + R_{n-1}(\sigma) + H_n(\sigma), \qquad n \geqslant 2$$

where

$$H_n(\sigma) = \sum_{k=3}^{n-2} \left[R_k(\sigma)\,(2\sigma)^{n-k-1} \Big/ \prod_{p=2}^{n+3-k} (p-\sigma) \right]$$

for $n \geqslant 5$ and $H_n(\sigma) = 0$ for $n < 5$.

Furtheremore if equality holds in (13) and (16) and if

$$c_n = 2\sigma, \quad b_{2q} = -b_1, \quad b_{2q-1} = b_1, \quad n = 1, 2, ..., q = 1, 2, ...,$$

then for the α_n obtained from (17), (*) is sharp.

Theorem 2.

Let f_1, f_2 be functions in $\mathscr{S}_0$. Let $B(f_1, n)$, $B(f_2, n)$ be the corresponding bounds to the Taylor coefficients of f_1 and f_2 respectively, as these are defined in Theorem 1(v). Suppose $\mathring{S}_{f_1} \subset \mathring{S}_{f_2}$. Then

$$B(f_2, n) \leqslant B(f_1, n) \tag{20}$$

Proof. Let $\alpha \in \mathring{S}_{f_1}$. Let G_1 be the function obtained from f_1 exactly the same way as G was obtained from f in (10). Similarly, since α also belongs to $\mathring{S}_{f_2}$, let G_2 be the function obtained from f_2. We have

$$G_1(D) = \mathring{S}_{f_1} \subset \mathring{S}_{f_2} = G_2(D), \ G_1(0) = G_2(0) = 0$$

and both G_1 and G_2 are regular and univalent in D. It follows that G_1 is subordinate to G_2, so that $G_1(z) = G_2(\varphi(z))$, where φ is analytic in D and $|\varphi(z)| \leqslant |z|$. We have $G'_1(z) = G'_2(z)\,(\varphi(z)).\ \varphi'(z)$, or

$$G'_1(0) = \mu(f_1, \alpha) = G'_2(0)\,\varphi'(0) = \mu(f_2, \alpha)\,\varphi'(0)$$

since $|\varphi'(0)| \leqslant 1$ we have

$$\mu(f_1, \alpha) \leqslant \mu(f_2, \alpha) \tag{21}$$

Put

$$\sigma_1 = \frac{1}{1 + \mu(f_1, \alpha)}, \quad \sigma_2 = \frac{1}{1 + \mu(f_2, \alpha)}$$

We have from (21)

$$\sigma_1 \geqslant \sigma_2 \tag{22}$$

Now the function $M_n(f, \alpha) = A_n(f, \alpha) + R_{n-1}(s)$ defined in the statement of Th. 1 can be written as follows

$$M_n(f, \alpha) = 1 + \frac{n-1}{n-\sigma} + \frac{2\sigma}{n-\sigma} \cdot$$

$$\cdot \left[\frac{1}{2-\sigma} + \frac{2!2}{(2-\sigma)(3-\sigma)} + \ldots + \frac{(n-2)!\,(n-2)}{(2-\sigma)\ldots(n-1-\sigma)} \right]$$

It is easily seen that the derivative of $M_n(f, \alpha)$ with respect to σ is nonnegative, which implies that $M_n(f, \alpha)$ is an increasing function of σ. It follows that

$$M_n(f, \alpha) \geqslant M_n(f_2, \alpha) \tag{23}$$

By taking the infimum of the left side of (23) for $\alpha \in \mathring{S}_{f_1}$ and of the right side for $\alpha \in \mathring{S}_{f_2}$, we get (20) because $\mathring{S}_{f_1} \subset \mathring{S}_{f_2}$. This proves the theorem.

4. EXAMPLES AND COMMENTS

The function

$$f(z) = \frac{1}{2\varepsilon}\left[\left(\frac{1-z}{1-z}\right)^{\varepsilon} - 1\right], \quad z \in D, \quad 1 < \varepsilon < 2$$

belongs to the class $\mathcal{S}_0$. This is easily seen if we graf f. More precisely let $\mathcal{L}_1$, $\mathcal{L}_2$ be the rays which start from the point $(-1/2\varepsilon, 0)$ and make with the positive x-axis the angles

$$\left(2-\varepsilon\right)\frac{\pi}{2}, \qquad (\varepsilon - 2)\frac{\pi}{2}$$

respectively. Then $\mathring{S}_f$ is the open set which contains the origin and is bounded by the rays $\mathcal{L}_1$, $\mathcal{L}_2$. Let T be the symmetric set of $\mathring{S}_f$ with respect to the line

$$x = -\frac{1}{2\varepsilon}. \quad \text{Then} \quad f(D) = \mathbb{C} - \overline{T}.$$

If we choose $\alpha = 0 \in \mathring{S}_f$, then the function G considered in (10), which maps D onte $\mathring{S}_f$, is

$$G(z) = \frac{1}{2\varepsilon}\left[\left(\frac{1-z}{1-z}\right)^{2-\varepsilon} - 1\right], \quad z \in D,$$

and we have $\mu(f, 0) = G'(0) = \dfrac{2-\varepsilon}{\varepsilon}$ and $\sigma = \dfrac{\varepsilon}{2}$.

Other examples can be found in [2] (p.p. 196, 197).

We close with the following comment.

In [1] the authors present a different approach to the subject:

Given $f \in \mathcal{S}$ the index δ of starlikness of f is defined to be

$$\delta = \sup \{r : f(z) \text{ is a s.c.p of } f(D), \text{ whenever } |z| < r\}$$

Let Δ_δ be the class of all starlike functions whose index is equal to δ, $0 \leqslant \delta \leqslant 1$. For $f \in \Delta_\delta$ the following inequality holds:

$$|\alpha_n| \leqslant \prod_{k=1}^{n-1} \frac{k(1+\delta)+1-(-\delta)^k}{k(1+\delta)+\delta+(-\delta)^k} \tag{24}$$

The estimates given by (24) depend on δ, or equivalently on the size of $f(D_\delta)$ which (in the cases of interest, i.e when $0 < \delta < 1$) is *always* a bounded subset of $\mathring{S}_f$.

On the other hand the estimates in Theorem 1, above, depend on the entire set $\mathring{S}_f$. If $\mathring{S}_f$ is unbounded (see example given above) then $f(D_\delta)$ is a proper subset of $\mathring{S}_f$. Now it is possible in this case (when $\mathring{S}_f$ is unbounded) that the "unused" part of $\mathring{S}_f$ to "hide" some additional information on the α_n, including some concerning the sharpness of (24).

REFERENCES

[1] L. Raymon and D.E. Tepper: "Star Center Points of Starlike Functions" Australian Math, So., 19, Series A. 1975.

[2] Z. Nehari: "Conformal Mapping" Mc Graw-Hill, New York, 1952

ΠΕΡΙΛΗΨΗ

Παρατηρήσεις ἐπὶ τῶν Univalent Starlike συναρτήσεων.

Ἔστω $\mathscr{S}$ ἡ κλάση τῶν συναρτήσεων τῆς μορφῆς $f(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n$ οἱ ὁποῖες εἶναι ἀναλυτικὲς καὶ univalent (ἤτοι δὲν παίρνουν καμμιὰ τιμὴ παραπάνω ἀπὸ μιὰ φορὰ) στὸν μοναδιαῖο δίσκο $D = \{z \in \mathbb{C} : |z| < 1\}$; Ἐὰν $f \in \mathscr{S}$ τότε τὸ σύνολο $f(D)$ (ἤτοι ἡ εἰκόνα τοῦ D διὰ τῆς f) εἶναι ἕνα μὴ κενό, ἀνοικτὸ συνεκτικό, γνήσιο ὑποσύνολο τοῦ μιγαδικοῦ ἐπιπέδου $\mathbb{C}$.

Ἕνα σημεῖο $w \in f(D)$ καλεῖται κέντρο ἀστερότητας (star center point) τοῦ συνόλου $f(D)$, τότε καὶ μόνο τότε, ὅταν:

$$tf(z) + (1 + t)\, w \in f(D), \quad z \in D, \quad 0 \leqslant t \leqslant 1$$

Ἐὰν $f \in \mathscr{S}$, τότε τὸ σύνολο τῶν κέντρων ἀστερότητας τοῦ $f(D)$ συμβολίζεται μὲ S_f.

Ἔστω $\mathscr{S}_0$ ἡ ὑπόκλαση τῆς $\mathscr{S}$ γιὰ τὴν ὁποία ἔχομε:

$$\text{Ἐάν} \quad f \in \mathscr{S}_0 \quad \text{τότε} \quad 0 \in \mathring{S}_f,$$

ὅπου $\mathring{S}_f$ εἶναι τὸ ἐσωτερικὸ τοῦ S_f.

Στὴν παροῦσα ἐργασία ἐξετάζομε τὴν ἐπίδραση ποὺ ἀσκεῖ τὸ μέγεθος τοῦ συνόλου $\mathring{S}_f$ ἐπὶ τῶν συντελεστῶν, α_n, τοῦ Taylor μιᾶς συναρτήσεως $f \in \mathscr{S}_0$. Ἀποδεικνύομε τρία λήμματα καὶ δύο θεωρήματα.

Τὸ Θεώρημα I, ἀποτελεῖ τὸν κύριο κορμὸ τῆς ὅλης μελέτης. Ἀποδεικνύεται σ᾽ αὐτὸ ἡ βασικὴ ἀνισότητα: $|\alpha_n| \leqslant M(f, n)$, $n \geqslant 2$, ὅπου $M(f, n)$ εἶναι σταθερὲς τῶν ὁποίων τὸ μέγεθος ἐξαρτᾶται μόνο ἀπὸ τὸ σύνολο $\mathring{S}_f$. Ἡ τελευταία αὐτὴ ἀνισότητα δείχνει τὴν ἐπίδραση ποὺ ἔχει τὸ σύνολο $\mathring{S}_f$ ἐπὶ τοῦ μεγέθους τῶν συντελεστῶν α_n, Παραθέτομε σχόλια ἀναφερόμενα στὴν ἀκρίβεια (sharpness) τῆς ὡς ἄνω βασικῆς ἀνισότητας.

Τὸ Θεώρημα 2, ἔχει ὡς ἑξῆς: Ἔστωσαν f_1, f_2 συναρτήσεις τῆς κλάσεως $\mathscr{S}_0$ καὶ S_{f_1}, S_{f_2} τὰ ἀντίστοιχα σύνολα κέντρων ἀστερότητας τῶν συναρτήσεων αὐτῶν. Ἔστωσαν $M(f_1, n)$, $M(f_2, n)$ τὰ ἀντίστοιχα φράγματα τῶν συντελεστῶν τῶν f_1 καὶ f_2, ὅπως αὐτὰ ὁρίσθηκαν στὸ Θεώρημα I. Ὑποθέτομε ὅτι $\mathring{S}_{f_1} \subset \mathring{S}_{f_2}$. Τότε $M(f_2, n) \leqslant M(f_1, n)$.

Τὸ Θεώρημα 2 βεβαιώνει ὅτι ὅταν τὸ σύνολο $\mathring{S}_f$, ποὺ ἀντιστοιχεῖ σὲ μία

συνάρτηση $f \in \mathcal{S}_0$, μεγαλώνει, τότε τὰ ἀντίστοιχα φράγματα M(f, n) τῶν συντελεστῶν τῆς f μικραίνουν.

Τέλος κλείνομε τὴν μελέτη αὐτὴ παραθέτοντας μερικὰ γενικὰ σχόλια ποὺ ἀφοροῦν τὰ κτηθέντα σὲ αὐτὴν ἀποτελέσματα.