ΜΑΘΗΜΑΤΙΚΑ.— **Handle decompositions of differentiable manifolds,** *by George M. Rassias**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φίλωνος Βασιλείου.

The present paper is concerned with the covering of a differentiable manifold with the interiors of closed disks (not just coordinate systems). Morse-theoretic techniques are used throughout this paper.

Theorem (1). Any closed two-dimensional manifold M can be covered with the interiors of three closed disks.

Proof. A closed two-dimensional manifold is either homeomorphic to S^2 or to a connected sum of tori T^2, or a connected sum of projective planes $\mathbf{R}P^2$. If M is homeomorphic to S^2, then it can be covered with the interiors of two closed disks. We first prove the theorem in the case of the torus T^2, and the connected sum of a finite number of tori. There exists a handle-decomposition of any closed two-manifold consisting of one 0-handle, one 2-handle and its 1-handles. The complement of the 2-handle in the connected sum of tori, is a 2-disk by attaching 2g 1-handles on it, where g is the number of tori. Now, we join the 1-handles (in the way indicated in Figures 1 and 2 for the cases T^2, $T^2 \sharp T^2$, respectively), and we obtain a closed 2-disk. Notice, that the boundary of the join of the 1-handles is homeomorphic to the circle S^1. Now, we prove the theorem in the case of the projective plane $\mathbf{R}P^2$, and the connected sum of any finite number of projective planes. There exists a handle-decomposition consisting of one 0-handle, one 2-handle and its (non-orientable) 1-handles. The complement of the 2-handle in the connected sum of projective planes is a 2-dimensional disk by attaching h (non-orientable) 1-handles on it (i. e., each one of them with a half-twist), where h is the number of projective planes, as indicated in the Figures 5 and 6 for the cases of $\mathbf{R}P^2$, $\mathbf{R}P^2 \sharp \mathbf{R}P^2$, respectively. If we join the 1-handles in the way indicated in Figures 5 and 6, we obtain a closed 2-disk. Again, notice that the boundary of the join of the 1-handles is homeomorphic to the circle S^1. Thus, any closed

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two-dimensional manifold can be covered with the interiors of three closed disks. There is also a combinatorial way to see that (for example) the connected sum of tori T^2 can be covered with the interiors of three closed disks. Consider the regular $(4g+2)$-sided polygon G, where g is the number of tori and identify the sides $a_1, \ldots, a_{2g-1}$ such that $a_1 \cdot \ldots \cdot a_{2g-1} \cdot a_1^{-1} \cdot \ldots \cdot a_{2g-1}^{-1} = 1$. After, the identification this regular polygon G has only two vertices m and M. Consider, now, an open neighborhood N of m in G. Then, N is identified with corresponding open neighborhoods in the remaining positions of the vertex m, and the union of these neighborhoods gives an open 2-disk C_1. Similarly, we obtain an open 2-disk C_2 for the other vertex M. See, Figures 3 and 4 for the cases of T^2, $T^2 \# T^2$. So, the connected sum of any finite number of tori can be covered with the disks C_1, C_2 and the disk C_3 which is a sufficiently large open neighborhood of the point of intersection of radii whose vertices are m and M, i. e., there exists a covering with the interiors of three closed disks. In fact, the intersection of the two disks C_1, C_2 consists of rank $(\pi_1(M)) + 1$ disks in M as it is clearly described.

Since the fundamental group of a closed orientable two-dimensional manifold M, admits a presentation having 2g generators and a single relation, where g is the genus of M, we have that $C_1 \cap C_2$ consists of $2g+1$ disks in M. On the other hand, since the fundamental group of a non-orientable closed two-dimensional manifold admits a presentation having h generators and a single relation, where h is the genus of M, we have that $C_1 \cap C_2$ consists of $h+1$ disks in M.

Corollary. Any connected open (i. e., noncompact without boundary) two-dimensional manifold M can be covered with the interiors of two closed disks D_1, D_2 such that $(\text{int } D_1) \cap (\text{int } D_2)$ consists of rank $(\pi_1(M)) + 1$ open disks in M. Note that $\pi_1(M)$ is a free group.

Now, we present, using Morse theory, the following generalization of the previous theorem.

Theorem (2). Any closed differentiable manifold M of dimension n, can be covered with the interiors of $(n+1)$ closed disks (not just coordinate systems).

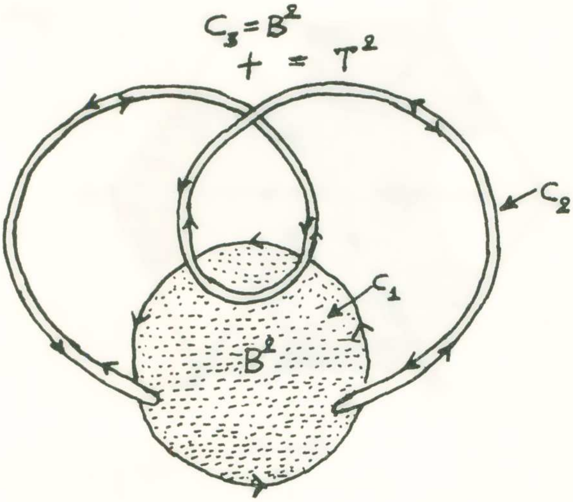

Fig. 1.

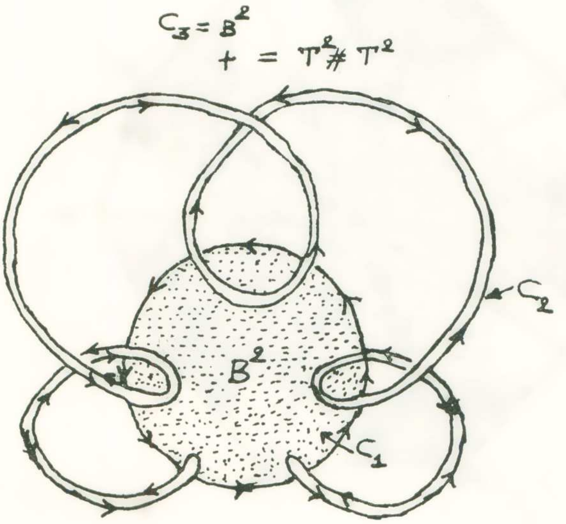

Fig. 2.

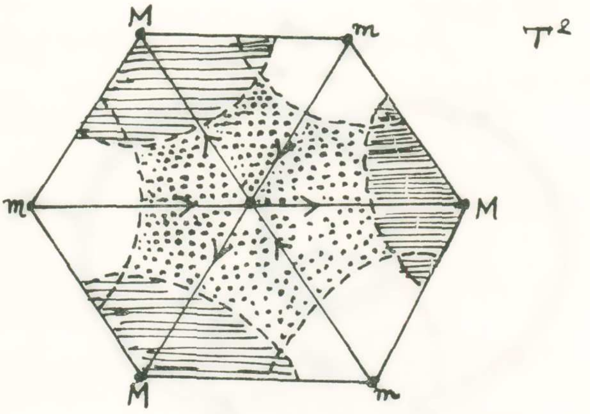

Fig. 3.

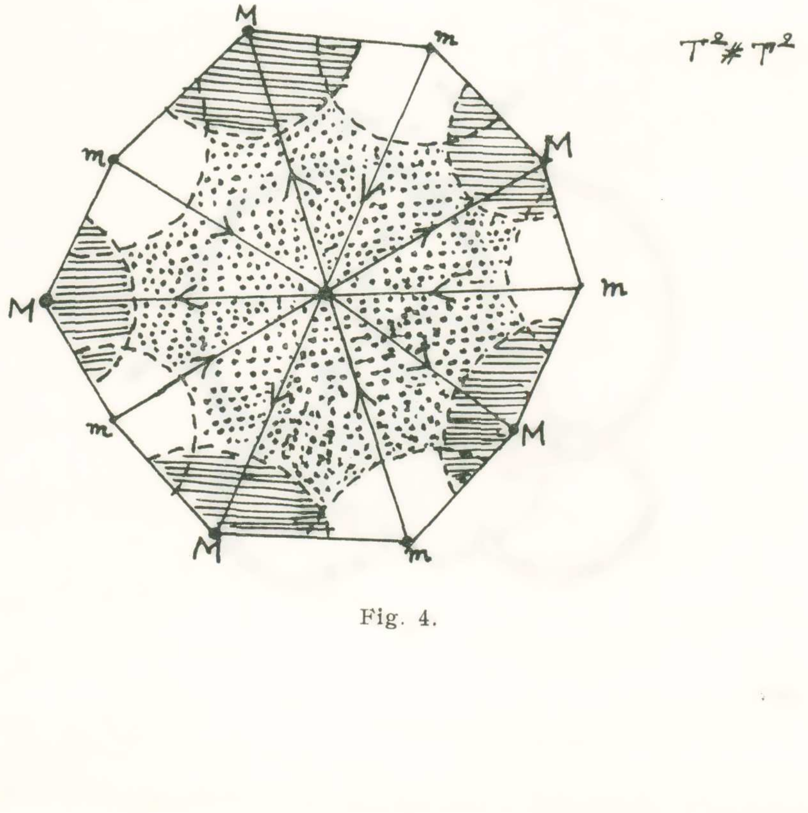

Fig. 4.

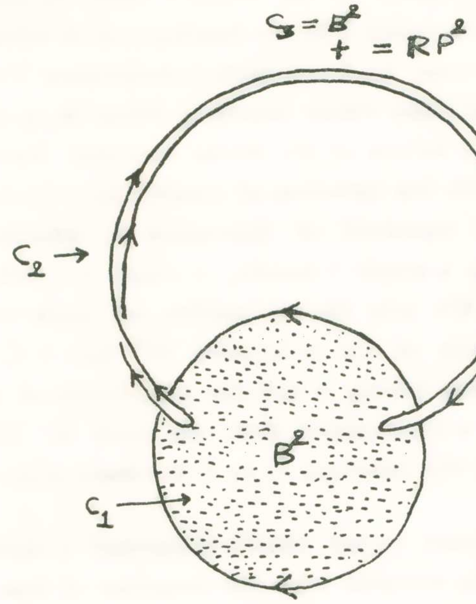

Fig. 5.

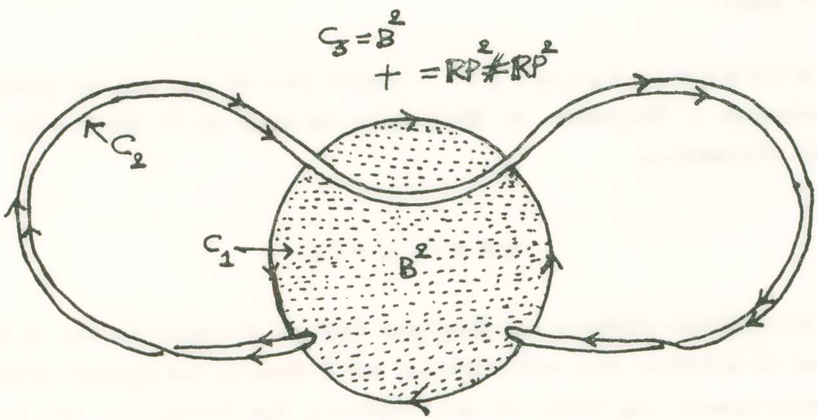

Fig. 6.

Proof. At first we show that there exist *at most* $(n+1)$ closed disks whose interiors cover M. By Smale's rearrangement lemma (see Milnor [**2**], p. 44) there exists a Morse function on M with at most $(n+1)$ critical values. However, by Lusternick-Schnirelman Theory, the minimal number of closed disks whose interiors cover M is at most equal to the number of critical values of the Morse function. Now, we show that M can be covered with the interiors of exactly $(n+1)$ closed disks. Any closed differentiable manifold of dimension n, admits a handlebody decomposition having a single 0-handle, a single n-handle, and m-handles, $1 \leqslant m \leqslant n-1$. We join the m-handles for each m separately by joining a point in each of the m-handles H_i^m (say $1 \leqslant i \leqslant k$) with D^n using a path and then taking a tubular neighborhood of this path. In this way we get an n-dimensional disk, for each m. Thus, M can be covered with exactly the interiors of $(n+1)$ closed disks. Q. E. D.

Remark. Every closed simply-connected n-dimensional manifold M, $n \geqslant 5$, can be covered with the interiors of less than n disks. This follows from Theorem 1.8 of Smale [**5**] by taking $m = 2$, $n \geqslant 5$. So, there exist critical points of index only $0, 2, 3, \ldots, n-3, n-2, n$ and not of index $1, (n-1)$. However, the Morse function can be chosen so that the critical values $f(p)$ of the critical points p, equal to the corresponding indices of the critical points. Thus, the number of critical values is less than n. Hence, the manifold M can be covered with less than n disks.

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ΠΕΡΙΛΗΨΙΣ

Τὸ πρόβλημα εὑρέσεως τοῦ ἐλαχίστου ἀριθμοῦ ἀνοικτῶν δίσκων, οἱ ὁποῖοι δύνανται νὰ καλύψουν μίαν πολλαπλότητα, καὶ ἡ σημασία τοῦ ἀριθμοῦ αὐτοῦ διὰ τὸν χαρακτηρισμὸν τῆς δομῆς τῆς πολλαπλότητος ἔχει ἀπασχολήσει τοὺς ἐρευνητὰς μαθηματικοὺς ἐπὶ πολλὰ ἔτη, καθ᾽ ὅσον ἡ λύσις τοῦ προβλήματος αὐτοῦ δίδει ἀπάντησιν εἰς τὸ θεμελιῶδες πρόβλημα τῆς Γεωμετρίας - Τοπολογίας, τὸ πρόβλημα τῆς ταξινομήσεως τῶν πολλαπλοτήτων ὅσον ἀφορᾷ εἰς τὴν τοπολογικὴν δομὴν αὐτῶν.

Ἡ παροῦσα ἐργασία περιέχει συμπεράσματα χρήσιμα διὰ τὴν ἐπίλυσιν τοῦ ἀνωτέρω προβλήματος.

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