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ΑΝΑΚΟΙΝΩΣΕΙΣ ΜΗ ΜΕΛΩΝ

ΜΑΘΗΜΑΤΙΚΑ.— **On generalized spectra of topological tensor algebras,** *by Anastasios Mallios* *. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Φ. Βασιλείου.

1. Introduction.

The purpose of the present note is to give an extension of previous results referring to spectra of topological tensor product algebras [4] to the case one replaces the complex numbers by an infinite - dimensional topological algebra as the range of the algebra homomorphisms involved. The motivation to the present study was a paper by H. Porta and J. T. Schwartz [9] dealing with representations of the Banach algebra of bounded operators on a Hilbert space to another such space, the results reported in the following being obtained in an attempt to put certain of their results into a more general setting as an application of the technique developed in [4], [5]. Applications along the lines of this note to certain topological analytic function algebras will be given elsewhere [8], as well as proofs and more details of the results presented herein.

2. Terminology and preliminary results.

The category of topological algebras we are dealing with in the sequel is that of (complex) locally convex topological ones with (ring) multiplication jointly (: in both variables) continuous. Thus, if E, F are

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objects in this category, we denote by Hom (E, F) the respective set of morphisms, i. e. the set of continuous (algebra) homomorphisms between the corresponding topological algebras. We topologize the set hom (E, F) with the relative topology induced on it by L_s (E, F), i. e. the space of continuous linear maps between the respective topological vector spaces equipped with the topology of simple convergence in their domain of definition.

Referring to the topological space Hom (E, F) as above, we say that a set $A \subseteq \mathit{Hom}$ (E, F) is *locally equicontinuous* if for every $h \in A$ there exists a neighborhood U of h in Hom (E, F), which is an equicontinuous subset of Hom (E, F) $\subseteq L_s$ (E, F).

In all that follows we denote by **A** the category of topological algebras into which we are working, as indicated above.

Thus, let $E, F \in \mathbf{A}$ and let $E \otimes F$ be the corresponding tensor product algebra. We say that a Hausdorff topology τ on $E \otimes F$ (we denote the respective topological space by $E \underset{\tau}{\otimes} F$) is *admissible* (with respect to the tensor product structure of $E \otimes F$ and the given category **A**) if $E \underset{\tau}{\otimes} F \in \mathbf{A}$.

Furthermore, we shall usually require for admissible topologies as above to satisfy one or both of the following two conditions:

(2. 1) The canonical bilinear map of $E \times F$ into $E \underset{\tau}{\otimes} F$ is separately continuous.

(2. 2) For every $G \in \mathbf{A}$ and for any $f \in \mathit{Hom}(E, G)$ and $g \in \mathit{Hom}(F, G)$, one has $f \otimes g \in L(E \underset{\tau}{\otimes} F, G)$.

We remark that the projective tensorial topology π on $E \otimes F$ (A. Grothendieck) is an admissible one in the preceding sense, which also satisfies the preceding two conditions (cf. [4], p. 176, Prop. 3. 2 and [2], Chap. I, p. 30, Prop. 2).

A stronger version of condition (2. 2) is the following one, which we shall also use in the sequel:

(2. 3) For any equicontinuous subsets $A \subseteq \mathit{Hom}(E, G)$ and $B \subseteq \mathit{Hom}$ (F, G), one has that $A \otimes B$ is an equicontinuous subset of $L(E \underset{\tau}{\otimes} F, G)$.

We remark that the preceding condition is also satisfied in case $\tau = \pi$.

Moreover, if $G \in \mathbf{A}$ is a commutative algebra, then condition (2.3) implies that $A \otimes B$ is actually a subset of $\mathit{Hom}(E \underset{\tau}{\otimes} F, G)$, and a similar remark holds also true for condition (2.2).

Now, for $E, F \in \mathbf{A}$, we call the subspace $M(E, F)$ of the non-zero elements of $\mathit{Hom}(E, F)$ endowed with the relative topology (of the simple convergence in E) the *generalized spectrum* (with respect to F) of the algebra E.

For E, F given as above, the *generalized Gel'fand map* is the (algebra) homomorphism g of E into $C(\mathit{Hom}(E, F), F)$ the algebra of continuous F-valued maps on $\mathit{Hom}(E, F)$, given by

$$(2.4) \qquad g : x \mapsto g(x) = \hat{x} : h \mapsto \hat{x}(h) := h(x),$$

for every $x \in E$ and $h \in \mathit{Hom}(E, F)$. On the other hand, for every $x \in E$, the map $\hat{x} = g(x) : \mathit{Hom}(E, F) \to F$, defined by (2.4), is called the *generalized Gel'fand transform* of the element $x \in E$.

We shall usually consider the range of g equipped with the topology of compact convergence, so that concerning the continuity of the map g, one has the following.

Theorem 2.1. For any $E, F \in \mathbf{A}$ the corresponding generalized Gel'fand map $g : E \to C(\mathit{Hom}(E, F), F)$ is continuous, when the range space is endowed with the topology of compact convergence, if and only if every compact subset of $\mathit{Hom}(E, F)$ is also equicontinuous.

The preceding theorem specializes to a familiar situation for $G = C$ (: the algebra of complex numbers) and E an m-barreled locally convex (topological) algebra (cf. [7], p. 305, Th. 3.1 as well as p. 306, Corol. 3.1).

Now, as a first auxiliary result to the main theorem of the next section, one has the following theorem, which combines, into the context of the present note, results in [9] and [4]. That is, we have

Theorem 2.2. Let $E, F, G \in \mathbf{A}$ such that E, F have identity elements and let τ be an admissible topology on $E \otimes F$, which satisfies the conditions (2.1), (2.2). Then, there exists a homeomorphism

$$(2.5) \qquad \varphi : \mathit{Hom}(E \underset{\tau}{\otimes} F, G) \mapsto \mathit{Hom}(E, G) \times \mathit{Hom}(F, G),$$

whose range are those (f, g) for which $f(x) g(y) = g(y) f(x)$ in G, for every $(x, y) \in E \times F$. Moreover, by restricting (2.5) to the generalized spectrum of $E \underset{\tau}{\otimes} F$, we get the following injection, which is also a homeomorphism:

$$(2.6) \qquad \psi : M(E \underset{\tau}{\otimes} F, G) \mapsto M(E, G) \times M(F, G).$$

In particular, for G a commutative algebra, the preceding homeomorphisms are, moreover, surjections.

In connection with the preceding theorem, we remark that for any $E, F, G \in \mathbf{A}$ with G complete, if τ is an admissible topology on $E \otimes F$, then the completion $E \widehat{\underset{\tau}{\otimes}} F$ of $E \underset{\tau}{\otimes} F$ is again an object of the category $\mathbf{A}$ under consideration, and moreover there exists a (canonical) continuous bijection of $\mathit{Hom}\,(E \widehat{\underset{\tau}{\otimes}} F, G)$ into $\mathit{Hom}\,(E \underset{\tau}{\otimes} F, G)$, so that, by the preceding theorem, one obtains a continuous bijection of Hom $(E \widehat{\underset{\tau}{\otimes}} F, G)$ onto the range space of the map φ defined by the relation (2.5) above. That is, we have the following continuous injection

$$(2.7) \qquad \chi : \mathit{Hom}\,(E \widehat{\underset{\tau}{\otimes}} F, G) \rightarrow \mathit{Hom}\,(E, G) \times \mathit{Hom}\,(F, G),$$

which is also a (continuous) bijection for G commutative, and analogous considerations hold for the generalized spectra of the respective topological algebras as indicated in Th. 2.2 above.

Now, *the map χ is not, in general, a homeomorphism:* This has been conjectured in [4], even for the particular case $G = C$ and the subcategory of $\mathbf{A}$ consisting of the locally m-convex topological algebras, and positive answers thereof have recently been given in [1], [3].

The next section is mainly concerned with conditions ensurring that the preceding map is actually a homeomorphism.

3. The main results.

Our primary objective in this section is Theorem 3.1 below. We start with the following:

Lemma 3.1. Let $E, F, G \in \mathbf{A}$ such that E, F have identity elements, and let τ be an admissible topology on $E \otimes F$, satisfying the conditions (2.1), (2.3) above. Moreover, let $\mathit{Hom}\,(E, G) \subseteq L_s\,(E, G)$ and $\mathit{Hom}\,(F, G) \subseteq L_s\,(F, G)$ be locally equicontinuous sets. Then, $\mathit{Hom}\,(E \underset{\tau}{\otimes} F, G) \subseteq L_s\,(E \underset{\tau}{\otimes} F, G)$ is also a locally equicontinuous set.

The interference of local equicontinuity in the preceding lemma is explained by the following lemma and Prop. 3.1 below. Thus, we have:

Lemma 3.2. Let $E, F \in \mathbf{A}$ such that F is complete and let $\hat{E}$ be the completion of E. Moreover, let S be a set of generators of E. Finally, suppose that the following condition is satisfied:

(3.1) For every $h \in \mathcal{H}om\,(E, F)$ there exists a neighborhood U of h in $\mathcal{H}om\,(E, F) \subseteq L_s(E, F)$ such that for every $z \in \hat{E}$ and for every net (x_i) in E converging to z, the net $(\widehat{x_i})$ (: generalized Gel'fand transform of the net considered) converges to $\hat{z}$ uniformly on U.

Then, the topology of $\mathcal{H}om\,(\hat{E}, F) \subseteq L_s(\hat{E}, F)$ is the «weak topology» defined on it by the maps $\hat{x}$, $x \in S$.

The preceding lemma has been motivated by the following result, which is fundamental for the next theorem. That is, we have

Proposition 3.1. Let $E, F \in \mathbf{A}$ such that F is complete and let $\hat{E}$ be the completion of E. Moreover, suppose that $\mathcal{H}om\,(E, F) \subseteq L_s(E, F)$ is a locally equicontinuous set. Then, $\mathcal{H}om\,(E, F)$ satisfies the preceding condition (3.1), so that one has $\mathcal{H}om\,(E, F) = \mathcal{H}om\,(\hat{E}, F)$ within a homeomorphism, with respect to the topology of simple convergence of the respective spaces.

In connection with Prop. 3.1 above, we remark that it is in the last relation of this statement, that one really draws profit from the completeness of the algebra F.

We are now in a position to state our basic result. That is, we have the following

Theorem 3.1. Let E, F, G be objects of the category $\mathbf{A}$ with identity elements and with G complete. Moreover, let $\mathcal{H}om\,(E, G) \subseteq L_s(E, G)$ and $\mathcal{H}om\,(F, G) \subseteq L_s(F, G)$ be locally equicontinuous sets, and let τ be an admissible topology on $E \otimes F$, which satisfies the conditions (2.1), (2.3) above. Then, there exists a bicontinuous injection

$$(3.2) \quad \varphi : \mathcal{H}om\,(E \mathbin{\widehat{\otimes}_{\tau}} F, G) \rightarrowtail \mathcal{H}om\,(E, G) \times \mathcal{H}om\,(F, G),$$

whose range space is determined as in Theorem 2.2 (cf. the relation (2.5)). On the other hand, by restricting (3.2) to the corresponding generalized spectra of the algebras involved, one obtains a bicontinuous injection

$$(3.3)\qquad \psi : M(E \hat{\otimes}_{\tau} F, G) \mapsto M(E, G) \times M(F, G).$$

Moreover, when G is a commutative algebra, the two preceding maps are homeomorphisms (bijections) between the respective spaces.

The preceding theorem specializes to previous results in [4], [5] for the case $G = C$, as well as to results in [9] concerning representations of operator-valued analytic function algebras by operators in a Hilbert space. A brief report of the same theorem has been given in [6]. On the other hand, applications, along the lines of results in [9], concerning vector-valued analytic function algebras are considered in [8] within the context of the present more general setting.

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Π Ε Ρ Ι Λ Η Ψ Ι Σ

Εἰς τὴν παροῦσαν ἐργασίαν ἐπιτυγχάνεται ἐπέκτασις προηγουμένων ἀποτελεσμάτων τοῦ συγγραφέως [4 - 7], ἀναφερομένων εἰς τοπολογικὰ τανυστικὰ γινόμενα τοπολογικῶν ἀλγεβρῶν καὶ τὴν σχέσιν μεταξὺ τῶν ἀντιστοίχων φασμάτων τῶν θεωρουμένων ἀλγεβρῶν, εἰς τὴν περίπτωσιν κατὰ τὴν ὁποίαν οἱ μιγαδικοὶ ἀριθμοὶ ἀντικαθίστανται ὑπὸ μιᾶς ἀλγέβρας (μὴ πεπερασμένης διαστάσεως), ἀνηκούσης εἰς μίαν ἀρκούντως γενικὴν κατηγορίαν τοπολογικῶν ἀλγεβρῶν.

Ἀφορμὴν εἰς τὴν παροῦσαν μελέτην ἔδωσε πρόσφατος ἐργασία τῶν H. Porta καὶ J. T. Schwartz [9], ἀναφερομένη εἰς παραστάσεις ἀλγεβρῶν ἀπὸ τελεστὰς ἑνὸς χώρου τοῦ Hilbert καὶ ἐφαρμογὰς εἰς ἀντιστοίχους παραστάσεις ἀλγεβρῶν ἀναλυτικῶν συναρτήσεων περισσοτέρων μιγαδικῶν μεταβλητῶν. Ἡ ἐφαρμογὴ τῶν μεθόδων τῶν ἀναπτυσσομένων εἰς τὰς ἐργασίας [4 - 7], καθὼς καὶ ἡ ἐπέκτασις τούτων διὰ τῆς παρούσης μελέτης, ἐπιτρέπει μεταξὺ ἄλλων τὴν διατύπωσιν καὶ ἐπέκτασιν τῶν ἀποτελεσμάτων τῶν ἀνωτέρω συγγραφέων, τῶν ἀναφερομένων εἰς ἀλγέβρας ἀναλυτικῶν συναρτήσεων, εἰς τὰ πλαίσια τῆς γενικῆς θεωρίας τῶν τοπολογικῶν τανυστικῶν ἀλγεβρῶν [6, 7].

Λεπτομερεστέρα ἔκθεσις ὡς καὶ ἐφαρμογαὶ εἰς τὴν περίπτωσιν τοπολογικῶν ἀλγεβρῶν ἀναλυτικῶν συναρτήσεων ἐπὶ γενικῶν μιγαδικῶν χώρων, εἰς τὰ πλαίσια τῆς θεωρίας τῶν τοπολογικῶν δεσμῶν, περιέχονται εἰς προσεχεῖς δημοσιεύσεις (πρβλ. [8]).

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