ΠΡΑΚΤΙΚΑ ΤΗΣ ΑΚΑΔΗΜΙΑΣ ΑΘΗΝΩΝ

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ΜΑΘΗΜΑΤΙΚΑ.— **A sufficient condition for univalency**, by *Nicolas Samaris**, διὰ τοῦ Ἀκαδημαϊκοῦ κ. Νικολάου Ἀρτεμιάδη.

Abstract: A sufficient condition is given for an analytic function $f(z) = z + c_2 z^2 + \ldots$, to be univalent in the unit disk $|z| < 1$.

Let **A** denote the class of functions f which are analytic in the unit disk $U = \{ z : |z| < 1 \}$ and $f(0) = f'(0) - 1 = 0$. Let **B** be the class of functions in **A** which are univalent and bounded in U, and map U onto a convex domain.

Theorem 1.

Let $f \in \mathbf{A}$, $q \in \mathbf{B}$, $q_0 = sup\{ |q(z)|, z \in U \}$. If

$$\left| \frac{q^2_0}{q'} \left[\frac{z^2 f'}{f^2} - \frac{z^2 q'}{q^2} \right] \right| \leq 1, \quad z \in U \tag{1}$$

then f is univalent U.

Proof.

We have

$$\frac{1}{f(z)} = \frac{1}{z} + a_0 + \ldots, \quad \frac{1}{q(z)} = \frac{1}{z} + b_0 + \ldots .$$

Hence $\frac{1}{f} - \frac{1}{q}$ is analytic in U.

Put

$$\varphi(z) = \frac{q^2_0}{q'} \left(\frac{1}{q} - \frac{1}{f} \right)' = -\frac{q^2_0}{q'} \left(\frac{q'}{q^2} - \frac{f'}{f^2} \right). \tag{2}$$

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From (1) we get

$$|z^2 \varphi(z)| \leq 1. \qquad (3)$$

By applying Schwarz's Lemma twice to $z^2 \varphi(z)$ we get $|\varphi(z)| \leq 1$ for $z \in U$. From (2) we have

$$\left(\frac{1}{q} - \frac{1}{f}\right)' = \varphi \cdot \frac{q'}{q^2_0}.$$

or

$$\left[\frac{1}{f} - \frac{1}{q}\right]_{z_1}^{z_2} = \int_{[z_1, z_2]} \varphi \frac{q'}{q^2_0} dz.$$

Put $q(z) = \omega$. Then $dq(z) = q'(z)\, dz$, $dz = \frac{d\omega}{q'}$, and for $z_1 \in U$, $z_2 \in U$ we have

$$\left|\frac{1}{f(z_2)} - \frac{1}{q(z_2)} - \left(\frac{1}{f(z_1)} - \frac{1}{q(z_1)}\right| = \frac{1}{q^2_0}\right) \Big| = \int_{q([z, {}_1z_2])} \varphi(q^{-1}(\omega))\, d\omega \Big|. \qquad (4)$$

Sinqe $q(U)$ is convex we have

$$[q(z_1), q(z_2)] \subset q(U).$$

Using the fact that $|\varphi(z)| \leq 1$ in U we get

$$\left|\int_{q([z_1, z_2])} \varphi(q^{-1}(\omega))\, d\omega\right| = \left|\int_{[q(z_1), q(z_2)]} \varphi(q^{-1}(\omega))\, d\omega\right| \leq |q(z_2) - q(z_1)|.$$

For $z_1 \neq z_2$, if we had $f(z_1) = f(z_2)$, then from (4) and (5) it would follow that

$$\frac{|q(z_2) - q(z_1)|}{|q(z_1)\, q(z_2)|} \leq \frac{|q(z_1) - q(z_2)}{q^2_0}$$

or since q is univalent we would have

$$|q(z_1)||q(z_2)| \geq q^2_0,$$

which contradits the Maximum modulus principle applied to q and U. The theorem is proven.

Remark. Observe that for $q(z) \equiv z$ we get the Osaki and Nunokawa theorem [2].

Corollary.

Let $f \in \mathbf{A}$, $q \in \mathbf{B}$ such that

$$\left|\frac{1}{q'}\left[\frac{z}{f(z)} - \frac{z}{q(z)}\right]''\right| \leq \lambda, \; z \in U. \qquad (6)$$

where

$$\lambda = inf\{ \left| \frac{q'(z)}{q^2_0\, q(z)} \right|, \ z \in U \}.$$

Then f is univalent in U.

Proof.

Put

$$P(z) = z^2 \frac{f'(z)}{f(z)} - z^2 \frac{q'(z)}{q(z)}.$$

Then

$$P'(z) = -z \left[\frac{z}{f(z)} - \frac{z}{q(z)} \right]''.$$

From (6) we get

$$|P'(z)| \leq \lambda |z| \leq \lambda. \tag{7}$$

Put

$$\xi = q(\omega), \quad d\xi = q'(\omega)\, d\omega, \quad \omega = q^{-1}(\xi).$$

Then we have

$$P(z) = \int_{[0,\, z]} P'(\omega)\, d\omega = \int_{q\,[0,\, z]} \frac{P'(q^{-1}(\xi))}{q'(q^{-1}(\xi))} d\xi = \int_{[0,\, q(z)]} \frac{P'(q^{-1}(\xi))}{q'(q^{-1}(\xi))} d\xi \tag{8}$$

From (7) and (8) follows that

$$|P(z)| \leq \lambda |q(z)| \leq \left| \frac{q'(z)}{q^2_0} \right|$$

or

$$\left| \frac{q_0^2}{q'} \left(z^2 \frac{f'}{f^2} - z^2 \frac{q'}{q^2} \right) \right| \leq 1.$$

It follows from (1) that f is univalent in U. This proves the Corollary.

Remark. Observe that for $q(z) \equiv z$ the corollary provides the Nunokawa-Obraponic-Owa theorem [1].

We now give two examples. In Ex. 1 the function $f = \left(\frac{1}{q} + \frac{q}{q^2_0} \right)^{-1}$ is proven to be Univalent by using Th. 1. The same function cannot be proven by using Ozaki and Nunokawa Theorem in [2]., by taking $q(z) \equiv z$.
Similarily in Ex. 2 the function

$$f(z) = \frac{z}{1 - k^2 z^2 + \ldots}$$

is proven to be univalent by the Cor. of Th. The same function cannot be proven Univalent by taking $q(z) \equiv z$ in Theorem Osaki-Obradov-Owa in [1]. The above remark shows that the Th. and its Cor. of the present paper are stronger than Th. Osaki - Nunokawa and Th. Osaki - Obradovic - Owa, respectively.

Example 1.

Let $\omega(z) = \frac{z}{1 - z^2}$ and $r > 0$ its convexity radius. Then
$Re\left(1 + \frac{z\omega''(z)}{\omega'(z)}\right) > 0$ when $|z| < r$.
If $k > r$ and $q(z) = \frac{1}{k}\omega(kz)$, then it is obvious that $q \in \mathbf{B}$ and $q_0 = \frac{1}{1 - k^2}$. If

$$f = \left(\frac{1}{q} - \frac{q}{q^2_0}\right)^{-1}$$

then

$$\left(\frac{1}{f} - \frac{1}{q}\right)' = -\frac{q'}{q^2_0}$$

or

$$\left|\left(\frac{q^2_0}{q^2} z^2 \frac{f'}{f^2} - \frac{q'}{q^2}\right)\right| = |z|^2 \leq 1.$$

If $q_1(z) = z$, the condition of Theorem is $|z^2 \varphi(z)| \leq 1$, where $\varphi = -\frac{q'}{q^2_0} + \frac{1}{z^2} - \frac{q'}{q^2}$.
But

$$\varphi(z) = -\frac{1 + k^2 z^2}{(1 - k^2 z^2)^2}(1 - k^2)^2 + \frac{1}{z^2} - (1 + k^2 z^2).$$

As $z \to 1$,

$$lim\, |z^2 \varphi(z)| = |1 + 2k^2| > 1.$$

Example 2.

If

$$q(z) = \frac{z}{1 - k^2 z^2} \in \mathbf{B}$$

then

$$q'(z) = \frac{1 + k^2 z^2}{(1 - k^2 z^2)^2},\ q_0 = \frac{1}{1 - k^2},\ \left[\frac{z}{q(z)}\right]'' = -2k^2,$$

$$q_0 = \frac{1}{1-k^2}, \left| \frac{q'(z)}{q(z)} \right| = \left| \frac{1+k^2z^2}{(1-k^2z^2)^2 . z} \right|$$

and

$$\lambda = \inf_z \left| \frac{q'(z)}{q^2_0 \, q(z)} \right| = \frac{(1-k^2)^3}{1+k^2}$$

We can find k, $0 < k < 1$ such that

$$h(z) = 1 - k^2 z^2 + \frac{(1-k^2)^3}{2k^2(1+k^2)} Log(1-k^2 z^2) \neq 0, \ \forall z \in U.$$

Since

$$| Log(1-k^2 z^2), | = \left| \sum_{n=1}^{\infty} \frac{(k^2 z^2)^n}{n} \right| \leq \ log(1-k^2)$$

then

$$|h(z)| \geq (1-k^2) - \frac{(1-k^2)}{2k^2(1+k^2)} \, | log(1-k^2) |$$

for $k \to 0$ the limit of the second part of the above inequality is $\frac{3}{4}$.

If

$$f(z) = \frac{z}{h(z)}.$$

then

$$\left| \frac{1}{q'} \left[\frac{z}{f(z)} - \frac{z}{q(z)} \right]'' \right| = \lambda$$

but

$$\left| \left[\frac{z}{f(z)} \right]'' \right| - 2k^2 + \frac{(1-k^2)^3}{(1+k^2)} \frac{(1+k^2 z^2)}{(1-k^2 z^2)^2}$$

and

$$\lim_{z \to 1} \left| \left[\frac{z}{(f z)} \right]'' \right| = 1 + k^2 > 1.$$

REFERENCES

1. Nunokawa, M. Odradovic and S. Owa, «One criterion for univalency. Proc., Amer. Math. Soc. 106 (1989) 1035-1037».
2. Ozaki and M. Nucokawa, «The Scharzian derivative and univalent functions, Proc. Amer. Math. Soc. 33 (1972), 392-394».

ΠΕΡΙΛΗΨΗ

Μία ἱκανὴ συνθήκη γιὰ univalency.

Ἔστω **B** ἡ κλάση τῶν ἀναλυτικῶν συναρτήσεων στὸ μοναδιαῖο δίσκο $U = \{z : |z| < 1\}$ οἱ ὁποῖες εἶναι Univalent, φραγμένες στὸ U κυρτὲς καὶ ἐπιπλέον ἰσχύει $q(0) = q'(0) - 1 = 0$.

Στὸ Θεώρημα καὶ τὸ πόρισμα τῆς παρούσης ἐργασίας δίδονται ἱκανὲς συνθῆκες γιὰ νὰ εἶναι μία συνάρτηση Univalent μὲ τὴν χρήση κατάλληλης συνάρτησης $q \in \mathbf{B}$.

Τὰ ἀποτελέσματα τῆς ὡς ἄνω ἐργασίας, στὴν περίπτωση $q(z) \equiv q$, ταυτίζονται μὲ ἀντίστοιχα ἀποτελέσματα τῶν Ozaki-Nunokawa καὶ Ozaki-Obradovic-Owa.

Στὴ συνέχεια δίδονται δύο παραδείγματα συναρτήσεων ποὺ τὸ Θεώρημα καὶ τὸ πόρισμα τῆς ἐργασίας ἀπαντοῦν γιὰ τὶς συναρτήσεις αὐτὲς ὅτι εἶναι Univalent ἐνῶ τὰ ἀντίστοιχα τῶν παραπάνω συγγραφέων δὲν ἀπαντοῦν.

Ἔτσι ἀποδεικνύεται ὅτι τὰ ἀποτελέσματα τῆς παρούσης ἐργασίας εἶναι ἰσχυρότερα ἀπὸ τὰ προαναφερθέντα τῶν ὡς ἄνω συγγραφέων.