Products and lengths in halfgroupoids (first part), *by S. P. Zervos**.

Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Κ. Π. Παπαϊωάννου.

I. REFERENCES AND CONTENTS

Our basic reference text is R. H. Bruck's *«A Survey of Binary Systems»* (Springer, second printing, 1966, p. 1 - 8). Abbreviated reference to it: R.H.B. The terms «lemma» and «theorem» always refer to results stated there, while we preserved the term «proposition» for results stated here.

In section III, we define the natural notion of product in a halfgroupoid (in multiplicative notation). In the special case of a groupoid, such products reduce to the classical «words» [see, for words, A. G. Kurosh, «Lectures on General Algebra» (Chelsea, 1963), p. 138]. In spite of its simplicity, this notion of product is not at all used in R. H. B. However, its usefulness is attested by, for instance, propositions 1 - 3, which immediately give for some theorems proofs simpler and more natural than the ones in R. H. B. Section IV has, merely, a preliminary character to section V. This last section constitutes the main part of the present paper; here, the introduction of the new, as far as I know, notion of the *length* of an element in a halfgroupoid (in two versions) permits to us the beginning of a close investigation of the structure of the general halfgroupoid. This will be continued in forthcoming papers.

II. TERMINOLOGY AND NOTATIONS

The Bourbaki set-theoretic terminology and notations (in particular, we use «families» instead of R H.B.'s «collections»), with $\subseteq$ in place of $\subset$, this last symbol meaning here $\subseteq$ and $\neq$. M = either $\mathbb{N}$ (i.e. the set of all natural numbers), or the set of all elements of $\mathbb{N}$ $\leqslant$ than a given one. μ, ν and ϱ will denote elements of $\mathbb{N}$ (possibly, of M; this will be explicitly indicated). Abbreviations: Iff = if and only if. Resp. = respectively. | denotes the end of a proof.

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The algebraic terminology is, in general, that of R.H.B.. However, in the notation for a binary operation (called, also, for reasons of suggestiveness, «multiplication») we write a . b or a * b in place of his ab; and we denote by (A,·) or (A,*) the halfgroupoid defined resp. by or * on the set A, to avoid confusion between the halfgroupoid and its carrier.

Abbreviations and notations. hgr = halfgroupoid. subhgr = subhalfgroupoid. $(A,\cdot) \subseteq (B,\cdot)$ means: (A,·) is a subhgr of the hgr (B,·). $(A,\cdot) \underset{e}{\subseteq} (B,\cdot)$ means: (B,·) is an extension of its subhgr (A,·). (Further abbreviations of this sort will be introduced later.) mec (A_o, A)= the maximal extension chain $((A_\nu,\cdot))_{\nu\in N}$ of $(A_o,\cdot)$ in (A,·). Gen $(A_o, A) = (\bigcup_{\nu\in N} A_\nu,\cdot)$. (This will be justified later.)

Further terminology and notations will be introduced in the sequel.

III. PRODUCTS, AND THEIR SYSTEMATIC USE

Definition 1. Given a hgr (A,·), any finite sequence of elements of A,· and parentheses (it is possible that no parenthesis or even no parenthesis and no · appear in this sequence) which is defined in (A,·) and equal to an element of A will be called *product of elements of* A or, more simply, *product in* (A,·). The elements of A appearing in this sequence will be called the *factors* of the product and, also, of the element of A represented by it. Notation: a = prod (A,·). Abridged notation: a = prod A. Every $a \in A$ is, trivially, prod A.

If $(A,\cdot) \subseteq (B,\cdot)$ and b = prod B with all its factors belonging to A, we shall write: b = prod (A, B). The following «transitivity» of prod holds, then, obviously: *If* $(A,\cdot) \subseteq (B,\cdot) \subseteq (C,\cdot)$ *and* c = prod (B, C) *with all its factors being* prod (A, B), *then* c = prod (A, C). Notation: Prod (A, B) = the set of all the elements of B which are prod (A, B). The following «transitivity» of Prod holds, then, obviously: *If* $(A,\cdot) \subseteq (B,\cdot) \subseteq (C,\cdot)$, $B \subseteq$ Prod (A, B) *and* $C \subseteq$ Prod (B, C), *then* $C \subseteq$ Prod (A, C).

Proposition 1. *Given mec* (A_o, A), $\bigcup_{\nu\in N} A_\nu =$ Prod (A_o, A).

Proof. 1) Every element of $A_1 - A_o$ is, by the definition of A_1, a product of two elements of A_o, hence, prod (A_o, A). Suppose that, for some μ, $A_\mu \subseteq$ Prod (A_o, A). Then, $A_{\mu+1} - A_\mu$, which, by its definition, consists of the products, defined in (A,·) and not already belonging to A_μ, of two elements of A_μ, is also contained in Prod (A_o, A). Hence, it is inductively proved that $\bigcup_{\nu \in N} A_\nu \subseteq$ Prod (A_o, A).

2) Every product, defined in (A,·), of two elements of A_o is also defined in (A_1,·). Suppose that, for some μ, every product, defined in (A,·), of at most μ+1 elements of A_o is also defined in (A_μ,·) and consider a product a, defined in (A,·), of at most μ + 2 elements of A_o. a is also a product of two factors, each one of them being a product of at most μ+1 elements of A_o, hence belonging to A_μ; hence, $a \epsilon A_{\mu+1}$, while the prod (A_o, A) = a is defined in ($A_{\mu+1}$,·). It is therefore inductively proved that all prod (A_o, A) are defined in Gen (A_o, A) and that, consequently, Prod (A_o, A) $\subseteq \bigcup_{\nu \in N} A_\nu$. | Corollary of the proof : *All prod* (A_o, A) *are defined in Gen* (A_o, A).

Proposition 2. *If* (H,·) $\subseteq$ (Θ,·) $\subseteq$ (K,·), *a necessary condition for* (Θ,·) *to be closed in* (K,·) *is* Prod (H, K) $\subseteq$ Θ.

Proof. Suppose (Θ,·) is closed in (K,·) and let $u \epsilon K$ be a product of ν elements of H. For ν = 1, $u \epsilon H$, hence, trivially, $u \epsilon \Theta$. For ν = 2, u = $n_1 . n_2$ with $(n_1, n_2) \epsilon H^2$, hence [since (Θ,·) is closed in (K,·)], $u \epsilon \Theta$. Suppose, now, that for ν = μ, every element of Prod (H, K) with at most μ factors belongs to Θ and consider $u \epsilon$ Prod (H, K) with at most μ + 1 factors. Then, u is also a product of two factors, each one of them being a product of at most μ elements of H, hence, an element of Θ. Hence, also $u \epsilon \Theta$. This completes the induction. |

Propositions 1 and 2 give immediate proofs of the following results mentioned in p. 2 of R.H.B. *Given mec* (A_o, A), 1° *Gen* (A_o, A) *is closed in* (A,·), *and* 2° *every subhgr closed in* (A,·) *and containing* (A_o,·) *also contains Gen* (A_o, A). *Proof.* 1°. Every product, defined in (A,·), of two elements of $\bigcup_{\nu \in N} A_\nu$ is, by proposition 1, prod (A_o, A) and hence, according to its proof, is defined in Gen (A_o, A). (*Note.* The direct proof of 1° is equally simple.) 2°. Apply proposition 2, with (H,·) = (A_o,·), (K,·) = (A,·) and (Θ,·) a subhgr of (A,·) closed in (A,·) and containing (A_o,·). Then, according to it, Prod (A_o, A) $\subseteq$ Θ. This and proposition 1 imply $\bigcup_{\nu \in N} A_\nu \subseteq \Theta$. Then, since we are referring to a closed subhgr of (A,·), ($\bigcup_{\nu \in N} A_\nu$,·) $\subseteq$ (Θ,·). |

If a and b are prod (A_0, A), set $a.b=c$ iff $a.b=c$ in $(A,\cdot)$. With . so defined, consider the hgr (Prod $(A_0, A),\cdot)$; it obviously coincides with Gen (A_0, A).

Hence, we have the following three definitions of Gen (A_0, A), which are entirely equivalent: 1. Gen $(A_0, A) = (\bigcup_{\nu\in N} A_\nu, \cdot)$, where $((A_\nu, \cdot))_{\nu\in N}$ is mec (A_0, A). 2. Gen (A_0, A) is the intersection of the family of all closed in $(A,\cdot)$ subhgr of $(A,\cdot)$ containing $(A_0,\cdot)$. (This is an immediate consequence of the above proved known results 1° and 2°.) 3. Gen $(A_0, A) =$ (Prod $(A_0, A),\cdot)$ (our definition).

We recall that they say: «$(A_0,\cdot)$ generates Gen (A_0, A) in $(A,\cdot)$.» In the special case where Gen $(A_0, A) = (A,\cdot)$, they also say: «$(A_0,\cdot)$ generates $(A,\cdot)$». [These justify the notation Gen $(A_0, A) = (\bigcup_{\nu\in N} A_\nu, \cdot)$, used here.] Notation: We shall, sometimes, write, $(B,\cdot) \underset{g}{\subseteq} (C,\cdot)$ for $(B,\cdot)$ generates $(C,\cdot)$.

The following lemma 1.1 is proved in R.H.B.: «*If* $(G,.) \underset{g}{\subseteq} (H,\cdot)$ *and* $(H,\cdot) \underset{g}{\subseteq} (K)$, *then* $(G,\cdot) \underset{g}{\subseteq} (K,\cdot)$.» Let us rephrase it, in terms of Prod. It becomes: $(H,\cdot) =$ (Prod $(G, H),\cdot)$ and $(K,\cdot) =$ (Prod $(H, K),\cdot)$ imply $(K,\cdot) =$ (Prod $(G, K),\cdot)$. But, this is an obvious consequence of the transitivity of Prod. ❙

Proposition 3. $(A_0,\cdot) \underset{=}{\subset} (A,\cdot)$, $(B, *)$ *is a hgr and* f *is a homomorphism of* $(A,\cdot)$ *into* $(B,\cdot)$. *Then, if* $a\in A$ *is a product of elements* $a_\iota \in A_0$, $f(a)$ *is obtained if we leave unchanged the, eventually existing, parentheses and replace* . *by* * *and each* a_ι *by* $f(a_\iota)$.

Proof. For $a = a_1 . a_2$, the assertion is the very definition of homomorphism. Suppose that the assertion is also valid for every prod (A_0, A) containing at most μ parentheses and consider a prod (A_0, A) with $\mu + 1$ parentheses. This will, necessarily, be of one of the forms: $a'.(\quad)$, $(\quad).(\quad)$, $(\quad).a'$, where every indicated exterior parenthesis contains a prod (A_0, A) having at most μ parentheses, hence satisfying the hypothesis of the induction. This and the validity of the assertion for $a = a_1 . a_2$ prove it for any of the three forms considered above, so completing the induction. (This sort of demonstrative argument, with induction on the number of parentheses, seems useful in some questions concerning hgr.) |

Notation. When a = prod A and we are interested in the factors of a, we can, to indicate it, write a = prod a_ι. Then, f (a) = prod f (a_ι) will have the meaning conferred to it by proposition 3.

The following lemma 1.2 is proved in R.H.B.: «*If* $(G,\cdot) \underset{g}{\subset} (H,\cdot)$ *and if* ϑ *is a homomorphism of* $(G,\cdot)$ *into a hgr* $(K,\cdot)$, *then* ϑ *can be extended in at most one way (and, possibly, in none) to a homomorphism of* $(H,\cdot)$ *into* $(K,\cdot)$.» Proposition 3 and the definitions give the following alternative proof: Suppose that there exists a homomorphism f of $(H,\cdot)$ into $(K,\cdot)$. Since $(H,\cdot) = (\text{Prod}\,(G, H),\cdot)$, every $a \in H$ is prod (G, H) = prod a_ι. Hence, f (a) = prod f (a_ι). Now, if f extends ϑ, f (a_ι) = ϑ (a_ι) (since $a_\iota \in G$) and, therefore, f (a) is completely defined by ϑ. |

Abbreviations. Let j take the values f, g, and f, g. Then, $(A,\cdot) \underset{j}{\subset} (B,\cdot)$ means, resp.: $(A,\cdot) \subset (B,\cdot)$ and $(B,\cdot)$ is free over $(A,\cdot)$ resp. $(A,\cdot)$ generates $(B,\cdot)$ resp. $(A,\cdot)$ freely generates $(B,\cdot)$.

We recall, also, lemma 1.3: «*If* $(G,\cdot) \underset{e}{\subset} (H,\cdot)$, $(G,\cdot) \underset{f}{\subset} (H,\cdot)$ iff $(H,\cdot)$ *is an open extension of* $(G,\cdot)$.»

IV. PRELIMINARY REMARKS TO V

We expose here in an explicit way remarks either written in R. H. B. in an indirectly condensed form, or not mentioned at all but useful. Since their proofs do not require original arguments, we generally omit them. We also introduce some new terminology. Abbreviation. «P. R.» for «preliminary remark».

P. R. 1. For all hgr $(A,\cdot)$, $(A,\cdot) \underset{j}{\subset} (A,\cdot)$ (Reflexivity).

P. R. 2. $(A,\cdot) \underset{j}{\subset} (A',\cdot)$ and $(A',\cdot) \underset{j}{\subset} (A'',\cdot)$ imply $(A,\cdot) \underset{j}{\subset} (A'',\cdot)$ (Transitivity).

If in a compatible sequence $((A_\nu,\cdot))_{\nu \in M}$ of hgr, for all $\nu + 1 \in M$, $(A_\nu,\cdot) \subset (A_{\nu+1},\cdot)$, the sequence will be called *increasing*. Given such an increasing sequence, a hgr $(B,\cdot)$ and a sequence $(\varphi_\nu)_{\nu \in M}$ of homomorphisms $\varphi_\nu : (A_\nu,\cdot) \rightarrow (B,\cdot)$, this last will be called *increasing* iff, for every $\nu + 1 \in M$, $\varphi_{\nu+1}$ extends φ_ν. Then, the ordered triple $(((A_\nu,\cdot))_{\nu \in M}, (B,\cdot), (\varphi_\nu)_{\nu \in M})$ will be called an *extension system.*

Every φ_ν being a mapping $A_\nu \rightarrow B$, hence, a subset R_ν of $A_\nu \times B$, $\bigcup_{\nu\in M} R_\nu$ is a mapping $\varphi : \bigcup_{\nu\in M} A_\nu \rightarrow B$, extending all φ_ν. This justifies setting $\varphi = \bigcup_{\nu\in M} \varphi_\nu$ and leads to

P. R. 3. *Given an extension system* $(((A_\nu, \cdot))_{\nu\in M}, (B, \cdot), (\varphi_\nu)_{\nu\in M})$, *there is one and only one homomorphism* $\varphi : (\bigcup_{\nu\in M} A_\nu, \cdot) \rightarrow (B, \cdot)$, *extending all* φ_ν.

P. R. 4. *Let* $((A_\nu, \cdot))_{\nu\in M}$ *be an increasing sequence. Then, if, for all* $\nu + 1 \in M$, $(A_\nu, \cdot) \underset{j}{\subseteq} (A_{\nu+1}, \cdot)$, *also, for all* ϱ *such that* $\nu + \varrho \in M$, $(A_\nu, \cdot) \underset{j}{\subseteq} (A_{\nu+\varrho}, \cdot)$; *and, then,* $(A_\nu, \cdot) \underset{j}{\subseteq} (\bigcup_{\mu\in M} A_\mu, \cdot)$.

P. R. 5. *If* $((A_\nu))_{\nu\in N}$ *is a complete extension chain,* $(\bigcup_{\nu\in N} A_\nu, \cdot)$ *is a groupoid.* (This is contained in the proof of theorem 1.1 in R. H. B.).

P. R. 6. *If* $((A_\nu, \cdot))_{\nu\in N}$ *is an open extension chain,* $(A_0, \cdot) \underset{f}{\subseteq} (\bigcup_{\nu\in N} A_\nu, \cdot)$.

(This is, essentially, contained in the above - mentioned proof in R.H.B., but it is more explicit and natural to consider it as a consequence of his lemma 1.3 and of P. R. 4.)

We recall theorem 1.1 in R. H. B.: «*Every hgr* $(G, \cdot)$ *freely generates at least one groupoid* $(H, \cdot)$. *If* $(G, \cdot)$ *freely generates two groupoids* $(H, \cdot)$ *and* $(H', \cdot)$ *there exists an isomorphism* ϑ *of* $(H, \cdot)$ *onto* $(H', \cdot)$, *which induces the identity mapping on* G.»

Proposition 4. (Characterization of all hgr freely generated by a given hgr.) *Suppose* $(A_0, \cdot) \underline{\subseteq} (A, \cdot)$ *and let* j *take the values* f *and* f, g. *Then,* $(A_0, \cdot) \underset{j}{\subseteq} (A, \cdot)$ *iff there exists a groupoid* $(B, \cdot)$, *with* $(A, \cdot) \underline{\subseteq} (B, \cdot)$ *and* $(A_0, \cdot) \underset{j}{\subseteq} (B, \cdot)$.

Proof. a) *Necessity.* Suppose $(A_0, \cdot) \underset{j}{\subseteq} (A, \cdot)$. By theorem 1.1, there exists a groupoid $(B, \cdot)$, with $(A, \cdot) \underset{j}{\subseteq} (B, \cdot)$. Hence, by P. R. 2, $(A_0, \cdot) \underset{j}{\subseteq} (B, \cdot)$.

Sufficiency. Suppose there is a groupoid $(B, \cdot)$, with $(A, \cdot) \underline{\subseteq} (B, \cdot)$ and $(A_0, \cdot) \underset{j}{\subseteq} (B, \cdot)$. Then, for every groupoid $(C, \cdot)$ and for every homomorphism $\varphi : (A_0, \cdot) \rightarrow (C, \cdot)$, there is a homomorphism $f : (B, \cdot) \rightarrow (C, \cdot)$, extending φ. The restriction of f to (A,·) is a homomorphism (A,·) → (C,·), extending φ. Hence, $(A_o,\cdot) \underset{f}{\subseteq} (A,\cdot)$; hence, $(A_o,\cdot) \underset{j}{\subseteq} (A,\cdot)$. ∎

V. LENGTH OF AN ELEMENT

Changes in R.H.B.'s terminology: 1) Given a hgr (H,·) he calls a ∈ H *prime in* (H,·) iff a has no divisors in (H,·). This seems to us inconvenient for the following reason: In the multiplicative hgr (ℕ,·) prime numbers are not «prime» elements, since, for all a ∈ ℕ, a = a . 1. We therefore shall replace his term «prime» by «strictly prime». 2) We say «divisor chain in (H,·)» instead of his «divisor chain of (H,·)».

Additional terminology. Given a divisor chain in (H,·), its «length» is 1) if the chain is finite, the number of its terms, 2) otherwise, $+\infty$. *Notation.* $\mathbb{N}_\infty = \mathbb{N} \cup \{+\infty\}$, ordered as usual.

Definition 2. a. Let (G,·) ⊆ (H,·). Suppose that, for some γ ∈ H, there exists a divisor chain in (H,·) finite over (G,·). Then, denote by Σ_γ the set of all such chains for γ, let correspond to every element of Σ_γ its length over (G,·), considered (this length) as an element of $\mathbb{N}_\infty$ and call g_γ the so defined mapping $\Sigma_\gamma \to \mathbb{N}_\infty$. Sup. $g_\gamma(\Sigma_\gamma)$ (this supremum is taken in $\mathbb{N}_\infty$) will be called *the length of* γ *over* (G,·).

Definition 2. b. Let (H,·) be a hgr and γ ∈ H. Denote by Σ'_γ the set of all the divisor chains for γ in (H,·) and let correspond to every element of Σ'_γ its length, considered as an element of $\mathbb{N}_\infty$, consider the so defined mapping $g'_\gamma : \Sigma'_\gamma \to \mathbb{N}_\infty$. Sup. $g_\gamma(\Sigma'_\gamma)$ will be called *the length of* γ.

Notations [with respect to a hgr $(A_o,\cdot) \subseteq (A,\cdot)$]. 1) $\gamma_1|\gamma$ for γ_1 divides γ in (A,·). 2) $l(\gamma)$ [resp. $l_{A_o}(\gamma)$] = length of γ [resp. over $(A_o,\cdot)$] in (A,·).

Obviously, a) for all γ ∈ A, $l(\gamma)$ [resp. $l_{A_o}(\gamma)] \geqslant 1$; b) γ is strictly prime in (A,·) iff $l(\gamma) = 1$; and c) if $g'(\Sigma'_\gamma)$ is an infinite set, $l(\gamma) = +\infty$.

Proposition 5. *Let* $(A_o,\cdot) \subseteq (A,\cdot)$, $\gamma_i \in A$ $(i = 0,1)$ *and* $l_{A_o}(\gamma_o) = \nu + 1$. *Then,* a) *if* $\gamma_1|\gamma_o$ *in* (A,·), $l_{A_o}(\gamma_1) \leqslant \nu$; *and* b) *there exists at least one divisor chain* $\gamma_o, \ldots, \gamma_\nu$, *for every term* γ_λ $(0 \leqslant \lambda \leqslant \nu)$ *of which* $l_{A_o}(\gamma_\lambda) = \nu + 1 - \lambda$.

Abbreviation (here). Divisor chain = divisor chain finite over $(A_o,\cdot)$.

Proof. a) Suppose there existed a $\gamma_1|\gamma_o$ in (A,·) [then, $l_{A_o}(\gamma_o) \geqslant 2$, so that $\nu \geqslant 1$] with $l_{A_o}(\gamma_1) > \nu$. There would exist, then, a divisor chain $\gamma_1, \ldots, \gamma_{\nu+\varrho}$, in (A,·), with $\varrho \geqslant 1$, $\gamma_{\nu+\varrho-1} \notin A_o$ and $\gamma_{\nu+\varrho} \in A_o$, hence, a divisor chain $\gamma_o, \gamma_1, \ldots, \gamma_{\nu+\varrho}$ of length $> \nu + 1$ over $(A_o, \cdot)$; this would imply $l_{Ao}(\gamma_o) > \nu + 1$, contrary to our supposition. b) Suppose all (possibly existing) divisors γ_1 of γ_o in $(A,\cdot)$ have length at most $\nu - 1$ over $(A_o, \cdot)$. Then, every (possibly existing) divisor chain $\gamma_o, \gamma_1, \ldots$ in $(A,\cdot)$ would have length at most $\nu - 1$ over $(A_o, \cdot)$ hence, every (possibly existing) divisor chain $\gamma_o, \gamma_1, \ldots$ in $(A,\cdot)$ would have length at most ν over $(A_o, \cdot)$, so that $l_{Ao}(\gamma_o) \leqslant \nu$, contrary to our supposition. When $\nu > 0$, there exists at least one $\gamma_1 | \gamma_o$ in $(A,\cdot)$, hence (by the above arguments) $l_{Ao}(\gamma_1) = \nu$; in this case, the proof is completed by an obvious induction. The case $\nu = 0$ is trivial. ❙

Notations. 1) $L^{(\nu)}_{Ao}(A) =$ the set of all $\gamma \in A$ with, in $(A,\cdot)$, $l_{Ao}(\gamma) \leqq \nu + 1$. 2) $L^{f}_{Ao}(A) = \bigcup_{\nu \in N} L^{(\nu)}_{Ao}(A)$, i.e. the set of all elements of A having, in $(A,\cdot)$, finite length over $(A_o, \cdot)$. 3) Finally, A_ν is defined in mec (A_o, A).

Proposition 6. *Let* $(A_o, \cdot) \subseteq (A, \cdot)$. *Then,* $L^{(\nu)}_{Ao}(A) \subseteq A_\nu$.

Proof. By induction. The assertion is obvious for $\nu = 0$. Suppose it is true for all $\nu < \mu$ and consider a $\gamma_o \in A$ with $l_A(\gamma_o) = \mu + 1$ (hence, $\geqslant 2$). According to proposition 5, there is, in $(A,\cdot)$, at least one $\gamma_1 | \gamma_o$ with $l_{Ao}(\gamma_1) = \mu$, while all the divisors of γ_o have length $\leqslant \mu$ over $(A_o, \cdot)$. In $(A,\cdot)$, either $\gamma_1 . \gamma_1' = \gamma_o$ or $\gamma_1' . \gamma_1 = \gamma_o$. In both cases, $(\gamma_1, \gamma_1') \in A^2_{\mu-1}$, by the hypothesis of the induction; hence [by the definition of A_μ in mec (A_o, A)], the (conveniently ordered) product of γ_1 and γ_1' belongs to A_μ; hence, $\gamma_o \in A_\mu$. ❙

C o r o l l a r y. $(A_{\nu+1} - A_\nu) \cap L^{(\nu)}_{Ao}(A) = \varnothing$.

We so come to the following (rather interesting for the subject, it seems to us)

Proposition 7. $(A_o, \cdot) \subseteq (A, \cdot)$ *and* $A = L^{f}_{Ao}(A)$ *imply* $(A_o, \cdot) \underset{g}{\subseteq} (A, \cdot)$. [In words: *If* $(A_o, \cdot)$ *is a subhgr of the hgr* $(A,\cdot)$ *and if all the elements of* A *have finite length over* $(A_o, \cdot)$, *then* $(A_o, \cdot)$ *generates* $(A,\cdot)$.]

First proof. It suffices to show that $A \subseteq \bigcup_{\nu \in N} A_\nu$ [then, necessarily, $(A,\cdot) = (\bigcup_{\nu \in N} A_\nu, \cdot)$]. But, $\gamma \in A$ i.e. $\gamma \in L^{f}_{Ao}(A)$ implies $\gamma \in L^{(\nu)}_{Ao}(A)$, for some ν, hence, according to proposition 6, $\gamma \in A_\nu$. Hence, $A \subseteq \bigcup_{\nu \in N} A_\nu$. ❙

Second proof (without using proposition 6). It suffices to show that all elements of A are prod (A_o, A). This we now prove by induction. For $\gamma_o \in A$ with $l_{Ao}(\gamma_o) = 1$, it is trivially true. Suppose it is also true for the elements of $L^{(\nu)}_{Ao}(A)$ and consider $\gamma_o \in A$ with $l_{Ao}(\gamma_o) = \nu + 2$. By the definition of length, there exists, then, in $(A,\cdot)$, a factorization $\gamma_o = \gamma_1 . \gamma_1'$. Then, by proposition 5, γ_1 and γ_1' belong to $L^{(\nu)}_{Ao}(A)$, hence, by the hypothesis of the induction, they are prod (A_o, A). Hence, $\gamma_o = \gamma_1 . \gamma_1'$ is also prod (A_o, A).

Note. $A = L^{f}_{Ao}(A)$ is *not,* however, a *necessary* condition in order that, when $(A_o,\cdot) \subseteq (A;)$, $(A_o,\cdot) \underset{g}{\subseteq} (A,\cdot)$. *Example.* Take $A_o = \{a_o\}$ and $A = \{a_o, a_1\}$, with $a_o . a_o = a_1$ and $a_1 . a_o = a_o$. Then, $(A_2,\cdot) = (A,\cdot)$, hence, $(A_o,\cdot) \underset{g}{\subseteq} (A,\cdot)$, but the divisor chain $a_o, a_1, a_o, a_1, \ldots$ (with infinitely many repetitions of a_o and a_1) shows that a_o and a_1 have not finite length over $(A_o,\cdot)$. Hence, *it is possible that* $L^{f}_{Ao}(A) = \varnothing$ *and* $(A_o,\cdot) \underset{g}{\subseteq} (A,\cdot)$.

Notation. In the sequel M is always finite; $m = \text{Max}\, M$. When, in $\gamma = \text{prod}\, a_i$, we must indicate the number of the factors a_i, we write $\underset{i \in M}{\text{prod}}\, a_i$. *Convention.* As usual, $\nu + \infty = +\infty + \nu = +\infty$.

Proposition 8. *Let* $(A_o,\cdot) \subseteq (A,\cdot)$ *and* $\gamma = \underset{i \in M}{\text{prod}}\, a_i$. *Then,* [resp. *if, for all* $i \in M$, $l_{Ao}(a_i)$ *is defined*], $l(\gamma) \geqq 1 + \text{Max}\{l(a_i) \mid \iota \in M\}$ [resp. *also* $l_{Ao}(\gamma)$ *is defined and* $l_{Ao}(\gamma) \geqq 1 + \text{Max}\{l_{Ao}(a_i) \mid \iota \in M\}$].

Proof. Inductive proof of the assertions.

Note. The above proposition suggests the study of the general functional inequality, for positive functions, $f(\underset{i \in M}{\text{prod}}\, a_i) \geqq 1 + \text{Max}\{f(a_i) \mid \iota \in M\}$, where 1 may be replaced by a constant; also, when M and prod may take some «infinite» meanings.

ΠΕΡΙΛΗΨΙΣ

Εἰς τὴν παροῦσαν ἀνακοίνωσιν σπουδάζονται αἱ δομαὶ μιᾶς μερικῶς ὡρισμένης ἐσωτερικῆς πράξεως διὰ τῆς εἰσαγομένης ἀνωτέρω συστηματικῆς χρήσεως τῆς ἐννοίας τοῦ «γινομένου» καὶ διὰ τῆς εἰσαγομένης νέας ἐννοίας τοῦ «μήκους» στοιχείου ἐντὸς τοιαύτης δομῆς.

★

Ὁ Ἀκαδημαϊκὸς κ. **Κ. Π. Παπαϊωάννου** ἀνακοινῶν τὴν ὡς ἄνω ἐργασίαν εἶπε τὰ ἑξῆς:

Διὰ τῆς παρούσης ἀνακοινώσεως ἄρχεται ἡ παρουσίασις τῶν ἀποτελεσμάτων τῆς ἐρευνητικῆς ἐργασίας τοῦ καθηγητοῦ κ. Σπυρίδωνος Π. Ζερβοῦ, κατὰ τὸ ἔτος 1968, ἐπὶ τῶν δομῶν μιᾶς μερικῶς ὡρισμένης ἐσωτερικῆς πράξεως.

Εἰσάγει ὁ κ. Ζερβὸς τὴν συστηματικὴν χρῆσιν τοῦ «γινομένου» στοιχείων ἐντὸς μιᾶς τοιαύτης δομῆς, ἡ ὁποία ἁπλοποιεῖ τὰς ἀποδείξεις γνωστῶν ἀποτελεσμάτων καὶ ἐπιτρέπει τὴν ἐξαγωγὴν νέων. Τὸ κυριώτερον, ὅμως, στοιχεῖον τῆς παρούσης ἀνακοινώσεως εἶναι ἡ εἰσαγωγὴ τῆς νέας ἐννοίας τοῦ «μήκους», στοιχείου ἐντὸς μιᾶς τοιαύτης δομῆς. Δι' αὐτῆς ἐξάγονται πλεῖστα ἐνδιαφέροντα νέα ἀποτελέσματα, ἑρμηνεύονται δὲ βαθύτερον αἱ αἰτίαι κλασσικῶν ἀποτελεσμάτων. Τὸ πρῶτον μέρος τῶν νέων αὐτῶν ἐξαγομένων ἐκτίθεται εἰς τὴν παροῦσαν ἀνακοίνωσιν.