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ΜΗΧΑΝΙΚΗ.— **The Analytic Function Theory Applied to Elastodynamic Grack Problems** by *P. S. Theocaris*, in collaboration with *H. G. Georgiadis* *, ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Περικλῆ Θεοχάρη.

ABSTRACT

Closed-form solutions are given, regarding steady-state elastodynamic crack problems. The configuration of the problem treated consists of an infinite body containing a semi-infinite crack under plane extension. The loading was formed either by constant stresses, acting normally to the crack faces, or wedge-imposed constant displacements, both moving with the velocity of the crack tip. The state of stresses at the crack tip was taken to be, either singular, or bounded. In the last case, the Dugdale hypothesis was adopted. In the domain of elliptic wave-equations the solution of the problems was reduced to a solution of a Dirichlet problem, whereas in the case of hyperbolic wave-equations, viz. for transonic crack velocities, a Hilbert problem was formulated and solved.

1. Introduction

When a time-dependent force is applied to a body, which is assumed to be rigid, every point of the body responds instantaneously to the externally applied load, and the effect of the applied force is to produce a uniform acceleration of the rigid body, together with an angular acceleration about its mass-centre. In many physical situations the assumption of rigidity leads to results, which are sufficiently accurate. However, there are other physical

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situations, in which effects cannot be described only by means of rigid-body dynamics. An important feature of this type of problems is that, when a time-dependent force is applied to the body, all its points do not respond instantaneously to the applied load, and the disturbance takes time to propagate from its source to other positions of the body. Another typical case is the case of a moving crack, the motion of which results in a continuous change of stresses along the body.

The theory of elastodynamics was originated by Poisson, Cauchy, Lamé, Stokes and other authors towards the end of the 19th century and since then it has remained an active field of research. The major stimulus to this activity has been the field of geophysics, particularly earthquake phenomena. More recently the theory has been applied to a variety of other engineering and physical situations. Much of the early work has been summarised by Love [**1**], Kolsky [**2**] and Eringen et al. [**3**].

Two general classes of dynamic-crack propagation problems have been considered in literature. The steady-state type of crack propagation assumes constant crack-length and loading-conditions, and thus results in a field intensity, independent of time and crack velocity. Some of the earlier analytical studies, concerning steady-state crack-motion, are due to Yoffé [**4**], Radok [**5**] and Craggs [**6**]. Sih [**7**] gave a unified approach to their problems by using the Muskhelishvili approach with complex stress functions [**8**].

The transient type of crack propagation assumes a continuously increasing crack length at a constant rate and time-independent loading conditions again. The field intensity now has a square root dependence on time. Transient crack problems were treated by Baker [**9**], Broberg [**10**] and Craggs [**11**], among others. More recently, Sih and Chen [**12**] presented solutions of the latter configurations, derived in a more elegant and simple manner.

It is noticeable that the spatial distribution of the stress- and displacement-fields, resulted by the steady-state and transient solutions, is identical. This somewhat justifies the policy to assume steady-state elastodynamic crack situations. In particular, the case of a steadily moving semi-infinite crack *(Craggs' model)* is obviously closer to reality, than a constant crack-length configuration *(Yoffé's model)*.

On the other hand, although the elastodynamic solutions imply the physically non-justified existence of singularities at the crack tips, they have been proved adequate to describe the field existing at the core region [**13**].

However, an alternative approach is the introduction of the well-known Dugdale's model.

The present paper communicates closed-form solutions in problems, concerning steady-state crack propagation in infinite elastic bodies. The crack is assumed semi-infinite, and propagating in plane extension under the action of normal to the crack faces constant stresses or displacements. In Section 3 the fundamental solution of the problem refering to an infinite plate *(Craggs' model)* is derived by using conformal mapping. The crack-tip velocity is in the subsonic regime, whereas both singular and strip-yield zone behavious of stresses is assumed for the crack end. In treating the singular-elastic case of this problem, Craggs [**6**] failed to extract in a straightforward way the singular part of the field. He determined this part somewhat arbitrarily, based only on physical grounds. However, Sih [**7**] gave a direct solution of the problem by reducing it to a Hilbert problem.

In Section 4 the configuration of a semi-infinite crack, opened by a moving wedge, was considered again for subsonic crack-tip velocities. In Section 5 the problem of two semi-infinite cracks propagating collinearly was solved by the method presented in Section 3. It was assumed that Dugdale's plastic zones were developed in both crack ends.

In Sections 6 and 7 of the paper of solution of the problem considered in Section 3 is presented for crack-tip velocities in the transonic and supersonic regime. In this case the wave equations become hyperbolic and the simple Dirichlet-problem formulation is not applicable. The solution here was obtained, without using any transformation, by considering a non-homogeneous Hilbert problem [**8**, **14**, **15**]. It is worth mentioning that this part of the present study was motivated by the exploratory experiments by Winkler et al. [**16**], which have obtained crack speeds greater than the body-wave speeds in the medium. The same researchers also have presented a model, explaning physically this important phenomenon [**17**]. The requirements for hyper-velocity cracks are: energy directly delivered to the crack tip, i.e. crack-surface loading, large input energy, relatively to the fracture energy, and a weak fracture plane.

The important feature of transonic crack-propagation is the change of the strength of the singularity from the common case of $z_1^{-1/2}$ to $z_1^{\gamma-1}$, where γ is a complex number.

2. BASIC PRELIMINARIES

Assume that a body is set into motion by a crack moving with a constant velocity υ on the x'—axis. After a long time of steady motion of the crack, a steady field may be deveoloped around an observer attached to the crack tip and transient effects may be omitted.

A moving coordinate system (x, y), which is attached to the moving crack tip is introduced as $x = x' - \upsilon t$, $y = y'$ where (x', y') is a fixed coordinate system. With this transformation the wave equations in plane elastodynamics, which involve three independent variables x', y' and t, become (see for instance in [**3**]):

$$(1 - M_1^2)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0, \quad (1 - M_2^2)\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \tag{1}$$

where φ, ψ are scalar and vector potentials respectively and $M_j = \upsilon/c_j$ $(j = 1,2)$ are the Mach numbers. The c_1 and c_2 parameters are the longitudinal- and shear-wave velocities in the elastic medium. In terms of the shear modulus μ, Poisson's ratio ν, and mass density ρ, c_1 and c_2 are given by:

$$c_1 = \begin{cases} [2\mu\,(1-\nu)/\rho(1-2\nu)]^{1/2}, \text{ for plane strain} \\ [2\mu/\rho\,(1-\nu)]^{1/2}, \text{ for plane sress} \end{cases}$$

$$c_2 = (\mu/\rho)^{1/2} \tag{2}$$

Then, the stress- and displacement-fields can be found by:

$$\begin{aligned} u_x &= (\partial\varphi/\partial x) + (\partial\psi/\partial y), \quad u_y = (\partial\varphi/\partial y) - (\partial\psi/\partial x) \\ \sigma_x/\mu &= (2 - 2M_1^2 + M_2^2)\,(\partial^2\varphi/\partial x^2) + 2\,(\partial^2\psi/\partial x\partial y) \\ \sigma_y/\mu &= (M_2^2 - 2)\,(\partial^2\varphi/\partial x^2) - 2\,(\partial^2\psi/\partial x\partial y) \\ \tau_{xy}/\mu &= 2\,(\partial^2\varphi/\partial x\partial y) + (M_2^2 - 2)\,(\partial^2\psi/\partial x^2) \end{aligned} \tag{3}$$

Without going into further details, the above equations can be set in a complex-variable form, due to Sneddon [**18**] and Radok [**5**]:

$$u_x = 2\,Re\,\Omega_1\,(z_1) - 2\beta_2\,Re\,\Omega_2\,(z_2) \tag{4.1}$$

$$u_y = -2\beta_1\,Im\,\Omega_1\,(z_1) + 2\,Im\,\Omega_2\,(z_2) \tag{4.2}$$

$$\sigma_x = 2\mu\,[(2\beta^2_1 - \beta^2_2 + 1)\,Re\,\Omega'_1\,(z_1) - 2\beta_2\,Re\,\Omega'_2\,(z_2)] \tag{4.3}$$

$$\sigma_y = -2\mu\,[(1 + \beta^2_2)\,Re\,\Omega'_1\,(z_1) - 2\beta_2\,Re\,\Omega'_2\,(z_2)] \tag{4.4}$$

$$\tau_{xy} = -\,2\mu\,[2\beta_1\,Im\,\Omega'_1\,(z_1) - (1 + \beta^2_2)\,Im\,\Omega'_2\,(z_2)] \tag{4.5}$$

where:

$$\beta_j = (1 - M_j^2)^{1/2} \text{ and } z_j = x + i\beta_j y$$

The functions $\Omega_j(z_j)$ and $\Omega'_j(z_j)$ are sectionally analytic functions and therefore it is valid that $\nabla^2(Re\ \Omega'_j) = 0$, $\nabla^2(Im\ \Omega'_j) = 0$ in the domain of analyticity.

For $|z_j| \to \infty$, the complex potentials take the form [**3**]:

$$\begin{aligned} \Omega'_1(z_1) &= \frac{A_1 + iB_1}{2\mu} + 0\left(\frac{1}{z_1}\right) \\ \Omega'_2(z_2) &= \frac{A_2}{2\mu} + 0\left(\frac{1}{z_2}\right) \end{aligned} \tag{5}$$

where:

$$\begin{aligned} A_1 &= \frac{\sigma_x{}^\infty + \sigma_y{}^\infty}{2(\beta^2_1 - \beta^2_2)}, \quad B_1 = -\frac{\tau_{xy}{}^\infty}{2\beta_1} \\ A_2 &= \frac{\sigma_y{}^\infty}{2\beta_2} + \frac{1+\beta^2_2}{4\beta_2(\beta^2_1 - \beta^2_2)}(\sigma_x{}^\infty + \sigma_y{}^\infty) \end{aligned} \tag{6}$$

Finally, it may be noticed that the notation of [**3**] is adopted here, but with ω_j being substituted by Ω_j.

3. FUNDAMENTAL SOLUTION

Fig. 1a illustrates the geometrical configuration and the loading conditions of the problems considered. The crack is opened under the action of normal stress p, applied to a segment α of its lips. The crack velocity υ is assumed *subsonic*. Therefore, both equations in (**1**) are elliptic and the problem may be solved as a boundary-value problem of the potential theory. Two cases may be considered, as regards the nature of the near-tip stresses. In the first case the stresses are singular, whereas in the second case they behave according to Dugdale's hypothesis [**19**].

As it is well-known the preceding model assumes that: a) Yielding occurs in a narrow wedge-shaped zone R, b) The material in the zone is under a uniform tensile yield stress, σ_0, c) A Tresca yield criterion is obeyed and d)

The material outside the zone is elastic. This simplified model was applied to dynamic crack configurations by Goodier and Field [**20**], Kanninen [**21**] and Willis [**22**], among others.

The boundary conditions for the singular elastic case can be written as:

$$\left.\begin{array}{lll} \sigma_y\,(x,0) = -\,p & \text{for} & -\,\alpha < x < 0 \\ \sigma_y\,(x,0) = 0 & \text{»} & -\,\infty < x < -\,\alpha \\ \tau_{xy}\,(x,0) = 0 & \text{»} & -\,\infty < x < \infty \\ \sigma_x,\ \sigma_y,\ \tau_{xy} = 0 & \text{»} & |\,z_j\,| \to \infty \end{array}\right. \tag{7}$$

whereas, for the strip-yield zone case it is valid that:

$$\left.\begin{array}{lll} \sigma_y\,(x,0) = \sigma_0 & \text{for} & -\,R < x < 0 \\ \sigma_y\,(x,0) = -\,p & \text{»} & -\,\alpha < x < -\,R \\ \sigma_y\,(x,0) = 0 & \text{»} & -\,\infty < x < -\,\alpha \\ \tau_{xy}\,(x,0) = 0 & \text{»} & -\,\infty < x < \infty \\ \sigma_x,\ \sigma_y,\ \tau_{xy} = 0 & \text{»} & |\,z_j\,| \to \infty \end{array}\right. \tag{8}$$

The solution of problems (7) and (8) is derived by considering the respective Dirichlet problems and by using the conformal mapping technique. Then, Eq. (4.5) yields for the real x-axis:

$$2\beta_1\ \Omega'_2\,(z_2) - (1+\beta^2_2)\ \Omega'_2\,(z_2) = 0 \qquad \text{or}$$

$$\Omega'_2\,(z_2) = \frac{2\beta_1}{1+\beta^2_2}\ \Omega'_1\,(z_1) \tag{9}$$

Eqs. (4.4) and (9) give:

$$\sigma_y\,(x,0) = \frac{2\mu\ R\,(\upsilon)}{1+\beta^2_2}\ Re\ \Omega'_1\,(z_1) = \frac{\mu\ R\,(\upsilon)}{\beta_1}\ Re\ \Omega'_2\,(z_2) \tag{10}$$

where, the well-known Rayleigh equation $R(\upsilon) = 4\beta_1\,\beta_2 - (1+\beta^2_2)^2$ has the roots $\upsilon_1 = 0$ and $\upsilon_2 = c_R$, viz. the Rayleigh-wave velocity.

By virtue of the boundary conditions (7) and (8) and Eqs. (9), (10), the *Re*-part of the sectionally analytic functions $\Omega'_j\,(z_j)$ are given by:

$$Re\ \Omega'_j\,(z_j) = \left\{\begin{array}{lll} T_{oj}^{el.} & \text{for} & -\,\alpha < x < 0, \quad y = 0 \\ 0 & \text{»} & -\infty < x < -\,\alpha,\ y = 0 \end{array}\right. \tag{11}$$

in the singular elastic case, whereas:

$$Re\ \Omega'_j\ (z_j) = \begin{cases} T_{0j}^{pl.} & \text{for } -R < x < 0, \quad y = 0 \\ T_{0j}^{el.} & \text{»} \ -(\alpha+R) < x < -R, \quad y = 0 \\ 0 & \text{»} \ -\infty < x < -(\alpha+R), \quad y = 0 \end{cases} \tag{12}$$

in the strip-yield zone case. The boundary values of the harmonic function $Re\ \Omega'_j$ are given by:

$$T_{01}^{el.} = \frac{-p\ (1+\beta^2{}_2)}{2\mu\ R\ (\upsilon)}, \qquad T_{02}^{el.} = \frac{-p\beta_1}{\mu\ R\ (\upsilon)}$$
$$T_{01}^{pl.} = \frac{\sigma_0\ (1+\beta^2{}_2)}{2\mu\ R\ (\upsilon)}, \qquad T_{02}^{pl.} = \frac{\sigma_0\ \beta_1}{\mu\ R\ (\upsilon)} \tag{13}$$

Consider now the transformation:

$$w_j = 2iz_j^{1/2}, \qquad z_j = -w_j^2/4 \tag{14}$$

By means of (14) the infinite plane cut along the negative real axis ($-\infty < x < 0$) maps conformally onto the upper half plane of Fig. 1b. The harmonic conjugates $Re\ \Omega'_j\ (z_j)$ and $Im\ \Omega'_j\ (z_j)$ correspond to the harmonic conjugates $U_j\ (w_j)$ and $V_j\ (w_j)$, respectively. The upper and lower crack faces in the z_j -plane map onto the negative and positive real axis respectively in the w_j -plane. The point $(x, y) = (0,0)$ is a branch point of the transformation (14). Although the mapping (14) is not conformal for the branch point, it is conformal for any infinitesimally small neighbourhood of this point.

By considering the *Schwarz integral formula* (Poisson's integral formula for the half-plane) the harmonic functions $U_j\ (w_j)$ and $V_j\ (w_j)$ can be expressed by (see for instance in [**23, 24**])

$$U_j\ (u_j,\ v_j) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{v_j\ F_j\ (\xi)}{(\xi - u_j)^2 + v_j^2}\, d\xi \tag{15.1}$$

$$V_j\ (u_j,\ v_j) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{(u_j - \xi)\ F_j\ (\xi)}{(\xi - u_j)^2 + v_j^2}\, d\xi \tag{15.2}$$

where $F_j(\xi)$ are the boundary values of $U_j(w_j)$ for $v_j \to 0$ and the singular elastic case that is:

$$F_j(\xi) = \begin{cases} T_{oj}^{el.} & \text{for} \quad 2\alpha^{1/2} > |u_j|, \quad v_j = 0 \\ 0 & \text{»} \quad 2\alpha^{1/2} < |u_j|, \quad v_j = 0 \end{cases} \tag{16}$$

whereas for the strip-yield zone case we have:

$$F_j(\xi) = \begin{cases} T_{oj}^{pl.} & \text{for} \quad 2R^{1/2} > |u_j|, \quad v_j = 0 \\ T_{oj}^{el.} & \text{»} \quad 2\,[\,(\alpha + R)^{1/2} - R^{1/2}\,] > |u_j|, \quad v_j = 0 \\ 0 & \text{»} \quad 2\,(a + R)^{1/2} < |u_j|, \quad v_j = 0 \end{cases} \tag{17}$$

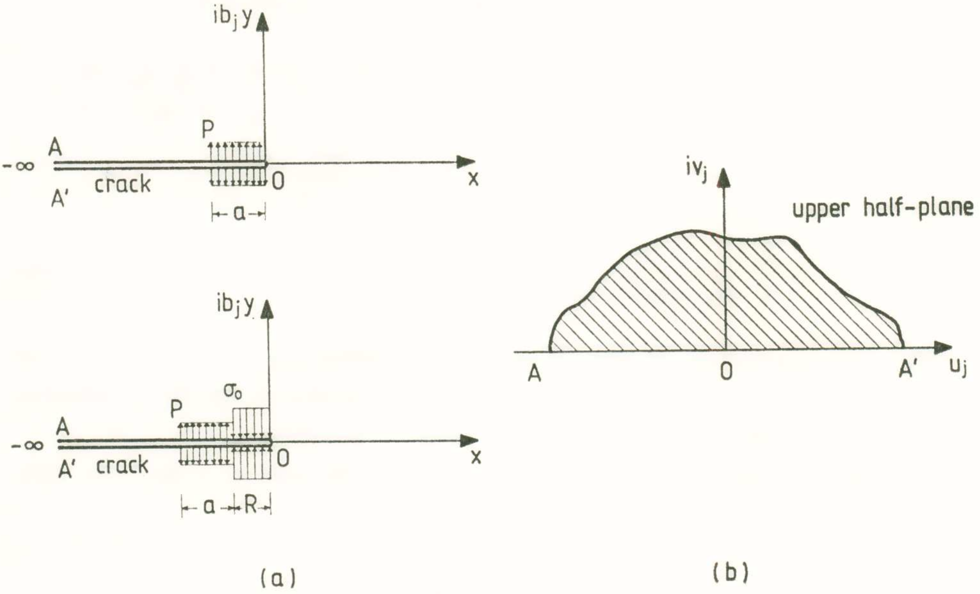


Fig. 1. a) Semi-infinite cracks possessing singularities or strip-yield zones at their ends and moving in an infinite elastic medium. b) Upper half-plane which maps conformally on to the infinite-plane cut along its negative real axis.

Particularly, in the singular elastic case the stresses become infinite, when approaching the point $z = 0$, i.e. the crack tip, coming from the internal points of the body and an additional to (11) feature of the problem must be taken into account. In this case, Eqs. (15) with the boundary conditions (16) give only the regular part of the potentials and consequently of the stress field. The boundary conditions, set by the physical problem, will determine the treatment of such a singularity. The singular part of the stress field and, consequently, of the potentials $Re\,\Omega'_j$ (see Eq. (10)) is due to the geometrical singularity at the point (0,0) in the original z-plane. This geometrical singularity has not any influence on the stresses at the crack faces, since their points are boundary points and they are subjected to appropriate boundary conditions, see Eq. (7). The geometric singularity "acts" only at the origin (0,0) and influences only the stress field of the internal points of the body. The above considerations based on physical grounds, justify the assumption of a delta-function singularity in the transformed w-plane and at the point (0,0).

Then, by virtue of (15.1), the total (regular and singular) real part of the complex potential in the transformed w_j-plane is given by:

$$U_j^{el.}(u_j, v_j) = \frac{v_j T_{0j}}{\pi} \int_{-2\alpha^{1/2}}^{2\alpha^{1/2}} \frac{d\xi}{(\xi - u_j)^2 + v_j^2} + \frac{v_j \Lambda_j}{\pi} \int_{-\infty}^{+\infty} \frac{\delta(\xi - \xi_0)\, d\xi}{(\xi - u_j)^2 + v_j^2} =$$
$$= \frac{T_{0j}}{\pi} \left[\tan^{-1}\left(\frac{2\alpha^{1/2} - u_j}{v_j} \right) + \tan^{-1}\left(\frac{2\alpha^{1/2} + u_j}{v_j} \right) \right] +$$
$$+ \frac{\Lambda_j}{\pi} \frac{v_j}{(\xi_0 - u_j)^2 + v_j^2} \tag{18}$$

where $\delta(\xi - \xi_0)$ stands for the Dirac-delta "function", $\xi_0 = 0$ and $\Lambda_j = -4\alpha^{1/2} T_{0j}$ is a factor spatially independent, which is introduced for the satisfaction of all the boundary conditions in the original problem.

Noting from (14) that $u_j = -2r_j^{1/2} \sin(\theta_j/2)$ and $v_j = 2r_j^{1/2} \cos(\theta_j/2)$, Eq. (18) gives:

$$Re\,\Omega'_j(z_j) = -\frac{T_{0j}}{\pi} \left\{ \tan^{-1}\left[\frac{2\left(\frac{\alpha}{r_j}\right)^{1/2} \cos(\theta_j/2)}{\left(\frac{\alpha}{r_j}\right) - 1} \right] + 2\left(\frac{\alpha}{r_j}\right)^{1/2} \cos(\theta_j/2) \right\} \tag{19}$$

The *Im*-part of $\Omega'_j(z_j)$ can be derived in the same manner Its singular part is of the form $r_j^{-1/2} \sin(\theta_j/2)$. The result in (19) is identical to that given by Craggs [**6**] and Sih [**7**], which as it has already mentioned, followed a different approach.

Knowing $Re\,\Omega'_j(z_j)$ and $Im\,\Omega'_j(z_j)$, stresses and displacements can be derived by the Sneddon-Radok equations (4). For instance, (4.4) and (19) give:

$$\sigma_y^{el.} = \frac{2p}{\pi R(\upsilon)} \left\{ 4\beta_1 \beta_2 \left[\frac{1}{2} \tan^{-1} \left(\frac{2\left(\frac{\alpha}{r_2}\right)^{1/2} \cos(\theta_2/2)}{\left(\frac{\alpha}{r_2}\right) - 1} \right) + \left(\frac{\alpha}{r_2}\right)^{1/2} \cos(\theta_2/2) \right] - \right.$$

$$\left. - (1+\beta_2^2)^2 \left[\frac{1}{2} \tan^{-1} \left(\frac{2\left(\frac{\alpha}{r_1}\right)^{1/2} \cos(\theta_1/2)}{\left(\frac{\alpha}{r_1}\right) - 1} \right) + \left(\frac{\alpha}{r_1}\right)^{1/2} \cos(\theta_1/2) \right] \right\} \quad (20)$$

The scaling r_j — and θ_j – coordinates are related with the r– and θ–coordinates in the physical plane as follows:

$$\tan \theta_j = \beta_j \tan\theta, \quad r_j = r(\cos^2\theta + \beta_j^2 \sin^2\theta)^{1/2} \quad (21)$$

In the strip-yield zone case it is not necessary to consider singularities and the complex potentials can be directly serived from (15) by taking into account (17):

$$Re\,\Omega'_j = -\frac{T_{oj}^{pl.}}{\pi} \tan^{-1} \left(\frac{2(R/r_j)^{1/2} \cos(\theta_j/2)}{(R/r_j) - 1} \right) -$$

$$-\frac{T_{oj}^{el.}}{\pi} \tan^{-1} \left(\frac{2\left[(\alpha+R)^{1/2} - R^{1/2}\right] r_j^{-1/2} \cos(\theta_j/2)}{\left[(\alpha+R)^{1/2} - R^{1/2}\right]^2 r_j^{-1} - 1} \right) \quad (22)$$

Introducing (22) in (4) one can evaluate stresses and displacements. Particularly, for the displacements, by integrating Ω'_j, or solving the appropriate Neumann problem, the constant of integration contributes a rigid body displacement to the system.

An interesting feature of the dynamic crack propagation is the well-known tendency of the state of stresses to diverge from the hydrostatic tension state, encountered ahead of stationary cracks [**4**]. This phenomenon is responsible for crack branching in high crack-tip velocities.

Fig. 2 shows the influence of crack-tip velocity on the ratio σ_y/σ_x ahead of the moving tip for the singular elastic case and the strip-yield zone case. It is seen that the increase of the plastic-zone size, R, causes a decrease to the tendency of reducing ratios σ_y/σ_x, for increasing crack-tip velocity.

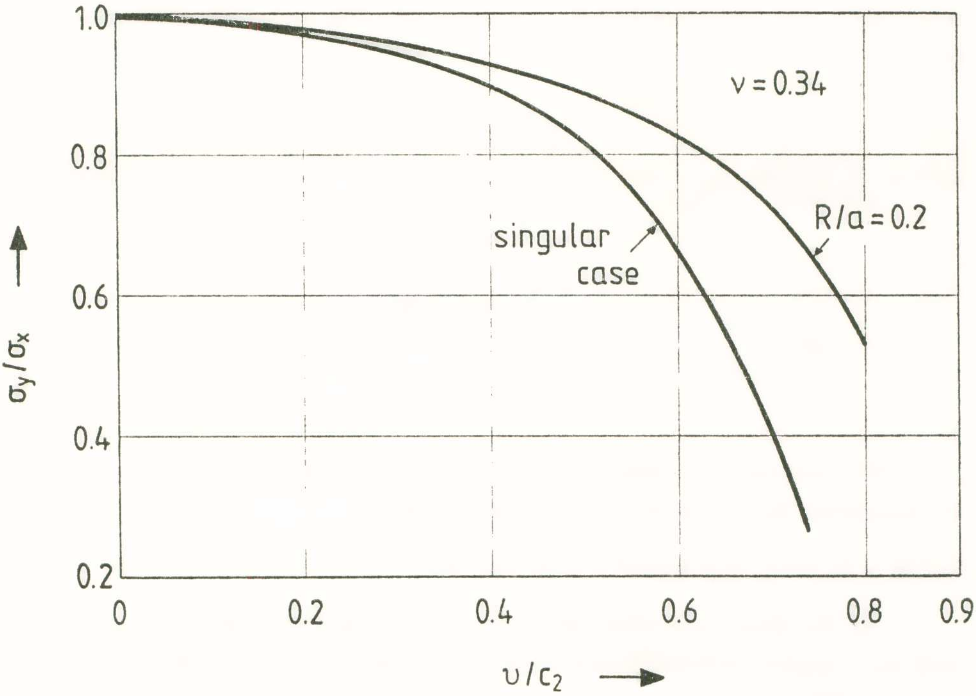


Fig. 2. The ratio σ_y/σ_x ahead of the moving crack tip plotted against crack velocity for the singular elastic case and the strip-yield zone case.

4. DYNAMIC CRACK OPENED BY A FRICTIONLESS SYMMETRIC WEDGE.

Consider a semi- infinite crack, propagating under the action of a wedge, which imposes constant displacements at the crack faces. Assume also that the wedge is moving without friction, it is smooth, and it does not contain corners, that is the usual assumptions of the theory of elasticity are made here. In particular, it is assumed that the deformed surface of the crack does not differ from the undeformed one in any marked degree, so that the loading is assumed to be applied to the boundary $y = 0$ and not to the faces of the deformed crack. The geometrical configuration and the coordinate system for this problem are shown in Fig. 3.

The boundary conditions can be written as:

$$\tau_{xy}(x, 0) = 0 \qquad \text{for} \qquad -\infty < x < \infty \tag{23.1}$$

$$u_y(x, 0) = j(x) \qquad \text{»} \qquad -\infty < x < 0 \tag{23.2}$$

where j(x) is an arbitrary, but known function, i.e. the shape of the moving wedge.

Because of the symmetry of the problem about the real axis we seek a solution of the form:

$$\Omega'_j\,(z_j) = C_j\ \Phi\,(z_j) \tag{24}$$

where C_j are constants spatially independent, to be determined. Then:

$$Re\ \Omega'_j\,(z_j) = \frac{C_j\,[\,\Phi\,(z_j) + \overline{\Phi\,(z_j)}\,]}{2},\quad Im\ \Omega'_j\,(z_j) = \frac{C_j\,[\,\Phi\,(z_j) - \overline{\Phi\,(z_j)}\,]}{2i} \tag{25}$$

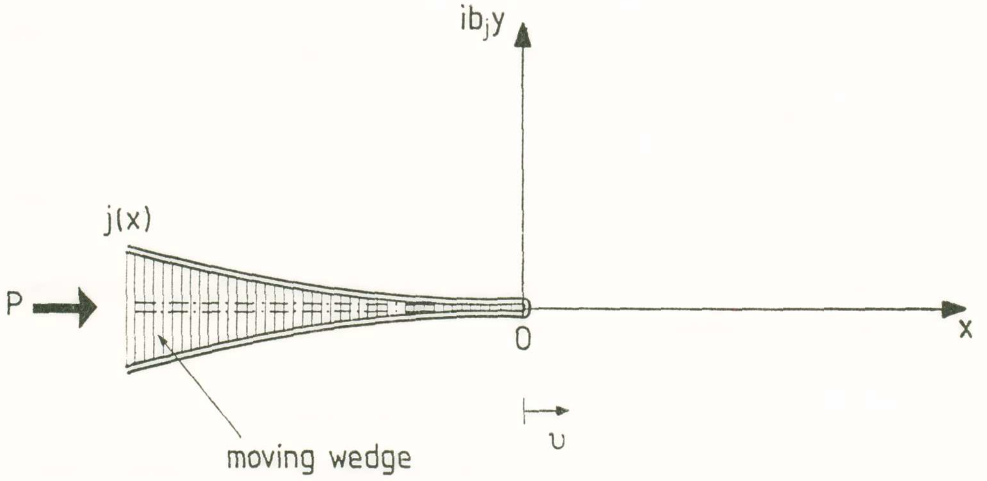


Fig. 3. A crack opened by a moving wedge in an infinite elastic medium.

Note also that:

$$\left.\begin{aligned} \Phi\,(z_1) + \overline{\Phi\,(z_1)} &= \Phi\,(z_2) + \overline{\Phi\,(z_2)} \\ \Phi\,(z_1) - \overline{\Phi\,(z_1)} &= \Phi\,(z_2) - \overline{\Phi(z_2)} \end{aligned}\right\} \quad \text{for } -\infty < x < \infty,\quad y = 0 \tag{26}$$

Differentiation of (4.2) and consideration of the boundary conditions (23) give:

$$\begin{aligned} &-\beta_1\,[\,\Omega'_1\,(x) - \overline{\Omega'_1\,(x)}\,] + [\,\Omega'_2\,(x) - \overline{\Omega'_2\,(x)}\,] = i\ j'\,(x) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for } -\infty < x < \infty \qquad (27) \\ &2\,\beta_1\,[\,\Omega'_1\,(x) - \overline{\Omega'_1\,(x)}\,] - (1 + \beta^2_2)\,[\,\Omega'_2\,(x) - \overline{\Omega'_2\,(x)}\,] = 0 \end{aligned}$$

or:

$$\beta_1 C_1 [\Phi(x) - \overline{\Phi(x)}] + C_2 [\Phi(x) - \overline{\Phi(x)}] = i j'(x) \tag{28.1}$$

$$2\beta_1 C_1 [\Phi(x) - \overline{\Phi(x)}] - (1 + \beta^2_2) C_2 [\Phi(x) - \overline{\Phi(x)}] = 0 \tag{28.2}$$

These relations yield the system:

$$\begin{aligned} -\beta_1 C_1 + C_2 &= 1 \\ 2\beta_1 C_1 - (1 + \beta^2_2) C_2 &= 0 \end{aligned} \tag{29}$$

which give for the unknown constants C_j.

$$C_1 = \frac{1 + \beta^2_2}{\beta_1 (1 - \beta^2_2)}, \qquad C_2 = \frac{2}{(1 - \beta^2_2)} \tag{30}$$

Then (25), (28.1) and (30) give:

$$Im\ \Omega'_j (z_j) = \frac{C_j\, j'(x)}{2} \tag{31}$$

With relation (31) valid, one has to solve a simple boundary-value problem, following the procedure of Section 3.

5. TWO DYNAMIC COLLINEAR CRACKS

Fig. 4a shows an infinite body, which contains two dynamic cracks, moving with the same velocity υ. Constant stresses of magnitude p are applied to a segment α of the crack faces. Moreover, strip-yield zones with length R have been created ahead of the crack tips. The crack motion is assumed to be in a steady state, so that the distance 2b between the tips of the two cracks may be taken constant.

Then, the boundary conditions can be written as:

$$\begin{aligned} &\sigma_y (x, 0) = \sigma_0 && \text{for} && b < |x| < (b + R) \\ &\sigma_y (x, 0) = -p && \text{»} && (b + R) < |x| < (b + R + \alpha) \\ &\sigma_y (x, 0) = 0 && \text{»} && (b + R + \alpha) < |x| < \infty \\ &\tau_{xy} (x, 0) = 0 && \text{»} && -\infty < x < \infty \\ &\sigma_x, \sigma_y, \tau_{xy} = 0 && \text{»} && |z_j| \to \infty \end{aligned} \tag{32}$$

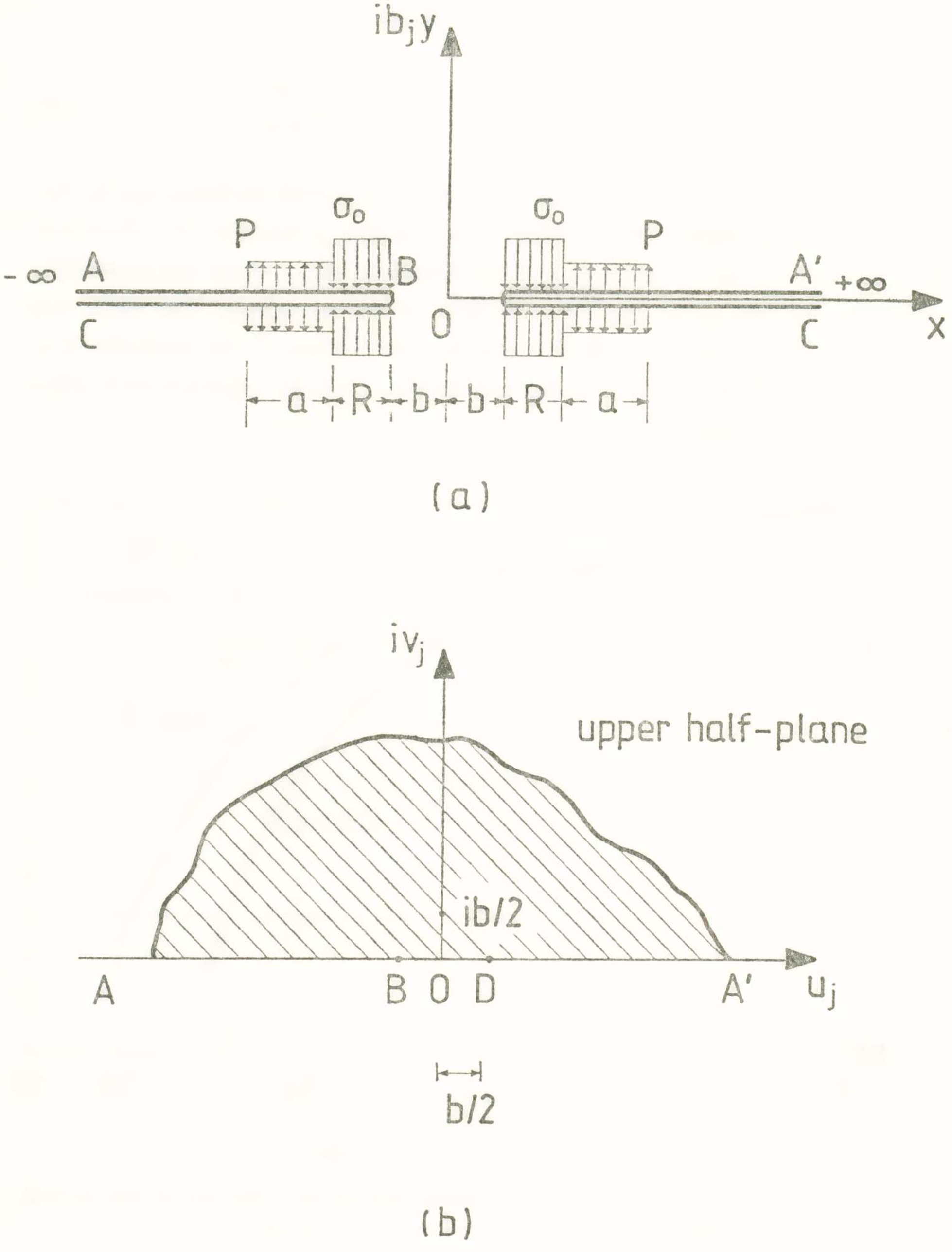


Fig. 4. a) Two semi-infinite dynamic collinear cracks possessing strip yield zones at their ends. b) Upper half-plane which maps conformally onto the infinite-plane cut along segments of its real axis.

The configuration in Fig. 4a can be conformally mapped onto the upper half plane, $v > 0$, by the transformation [**25**]:

$$w_j = \frac{z_j + (z_j^2 - b^2)^{1/2}}{2}, \qquad z_j = w_j + \frac{b^2}{4\,w_j} \tag{33}$$

Following the procedure in Section 3, a simple Dirichlet problem can be formulated, which, when solved, gives the complex potentials Ω'_j. However, the σ_y/σ_x-ratio at the point $(x, y) = (0,0)$ of the original z_j-plane can be directly evaluated for the point $(u_j, v_j) = (0, ib/2)$ of the w_j-plane. This ratio was plotted in Fig. 5 versus crack velocity for two values of the quantity b/α. The figure clearly shows that as the two dynamic cracks approach each other the σ_y/σ_x-ratio fairly decreases.

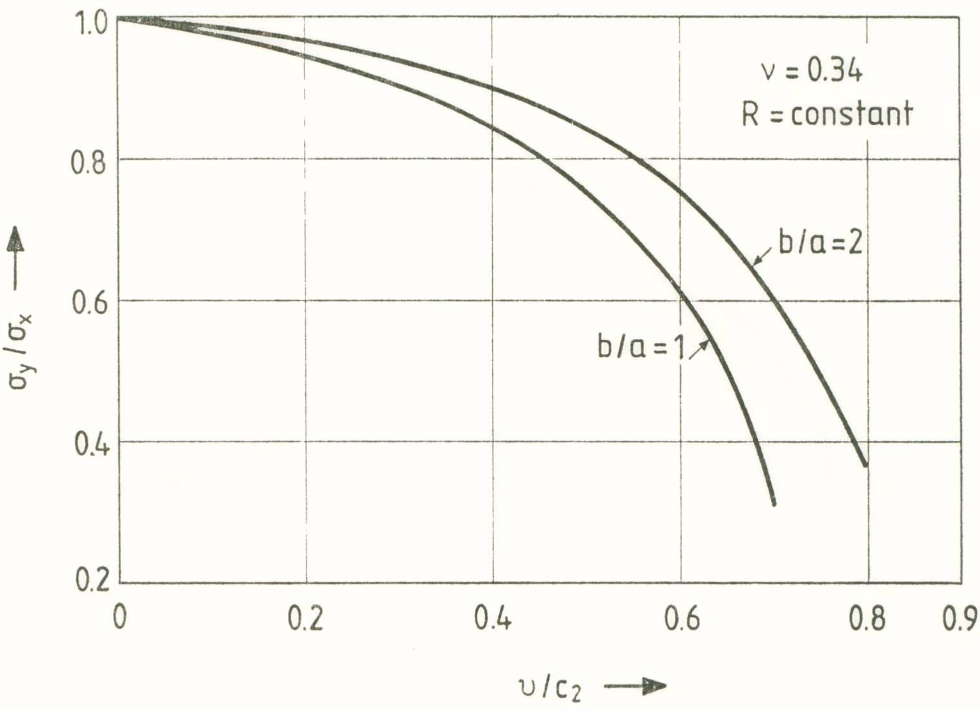


Fig. 5. The ratio σ_y/σ_x at a point which is equally distant from the tips of two moving collinear cracks plotted against crack velocity.

6. SEMI-INFINITE CRACK, PROPAGATING WITH TRANSONIC VELOCITIES

In the case of a source propagating with a velocity greater than the shear-wave velocity, but lower than the longitudinal-wave velocity, the Mach numbers are:

$$M_1 = \upsilon/c_1 < 1, \quad M_2 = \upsilon/c_2 > 1 \tag{34}$$

Consequently, the second of the wave equations (1) becomes *hyperbolic* and, then, the Sneddon-Radok equations (4) need some modifications. Accordingly, the solution obtained in Section 3, or in [**6**] and [**7**], is not valid for the transonic regime. However, the first of the wave equations (1) still remains *elliptic* and the scalar potential φ can be expressed again by:

$$\varphi = \omega_1 (z_1) + \bar{\omega}_1 (\bar{z}_1) \tag{35}$$

where:

$$\omega_1 (z_1) = \int \Omega_1 (z_1)\, dz_1 \tag{36}$$

The solution to ψ can be written as:

$$\psi = F (x + m_2 y) + G (x - m_2 y) \tag{37}$$

where $m_2 = ib_2 = (M^2_2 - 1)^{1/2}$ is a real number and F and G are arbitrary functions (see, for instance, refs. [**24**, **26**]). The straight lines $(x + m_2 y) =$ const. and $(x - m_2 y) =$ const. are the characteristics of the second differential equation in (1), i.e. curves along which partial information about the solution propagates. Fig. 6 shows the "natural" set of coordinates for the hyperbolic equation, formed by the characteristics.

It is seen in this figure that G is constant on wavefronts $x = m_2 y +$ const. that travel toward larger x, as y increases, whereas F is constant on wavefronts that travel toward decreasing x. Any solution may, therefore, be expressed as the sum of two waves, one traveling to the right in x, and the other to the left· information (shear stresses) propagates along with the waves. But, since the medium in front of the moving crack is not disturbed by shear stresses, only backward running shear waves exist, and therefore the function $G (x - m_2 y)$ is eliminated.

The displacement- and stress-fields can be written as:

$$u_x = 2\ [Re\ \Omega_1\ (z_1) + m_2\Omega_2\ (x + m_2y)] \tag{38.1}$$

$$u_y = -2\ [\beta_1\ Im\ \Omega_1\ (z_1) + \Omega_2\ (x + m_2y)] \tag{38.2}$$

$$\sigma_x = 2\mu\ [(2\beta^2_1 + m^2_2 + 1)\ Re\ \Omega'_1\ (z_1) + 2m_2\Omega'_2\ (x + m_2y)] \tag{38.3}$$

$$\sigma_y = 2\mu\ [-(1 - m^2_2)\ \ Re\ \Omega'_1\ (z_1) - 2m_2\Omega'_2\ (x + m_2y)] \tag{38.4}$$

$$\tau_{xy} = 2\mu\ [-2\beta_1\ Im\ \Omega'_1\ (z_1) - (1 - m^2_2)\ \Omega'_2\ (x + m_2y)] \tag{38.5}$$

$$\text{where } \Omega_2\ (x + m_2y) = \frac{1}{2}\ F'\ (x + m_2y).$$

Determination of the functions $\Omega_1(z_1)$ and $\Omega_2(x + m_2y)$ is now more difficult than that for subsonic crack-tip velocities. It is well-known (see,

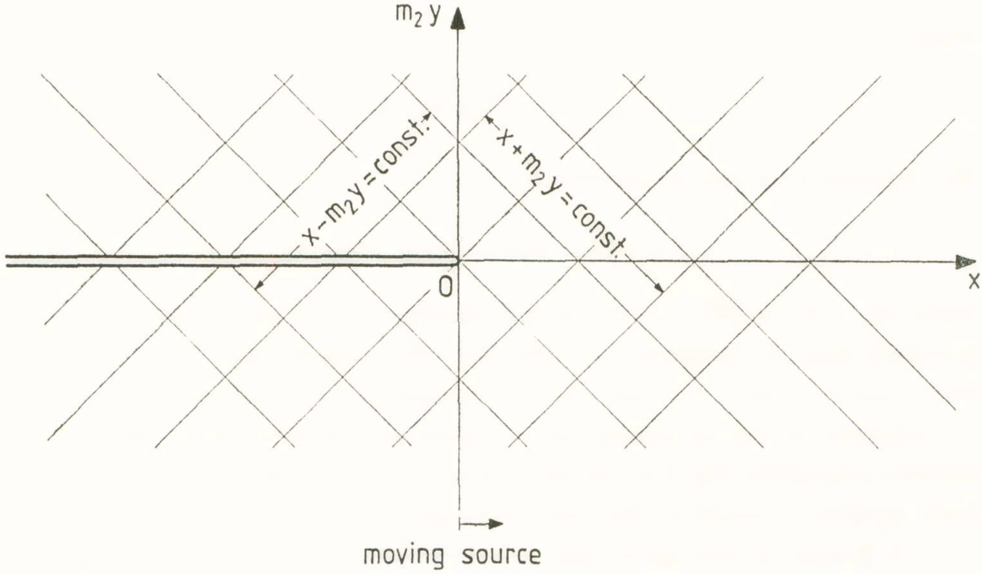


Fig. 6. The pattern of characteristics in a plane with a transonically moving crack.

for instance, refs. [**24**, **26**]) that the type of solutions of Dirichlet or Neumann problems is not suitable for hyperbolic equations. Consequently, the problem can not be treated in the same manner as those in the preceding Sections. However, the more general Hilbert problem may be proved adequate to apply.

The boundary conditions are stated in (7). The condition of vanishing of shear stress along the real x-axis, combined with (38.5), gives:

$$\Omega'_2\,(x + m_2 y) = \frac{-2\beta_1}{(1-m^2_2)}\, Im\,\Omega'_1\,(z_1), \quad \text{for } y \to 0 \tag{39}$$

Then, from (38.4):

$$\sigma_y = 2\mu\,[-\,(1-m^2_2)\, Re\,\Omega'_1\,(z_1) + \frac{4\beta_1\, m_2}{(1-m^2_2)}\, Im\,\Omega'_1\,(z_1)\,], \quad \text{for } y \to 0 \tag{40}$$

This means that there is a *linear relationship* between the *Re*- and *Im*-parts of the sectionally analytic function $\Omega'_1\,(z_1)$ along the cut L (line of discontinuity). Therefore, we can treat this boundary-value problem as a Hilbert problem.

Noting that:

$$Re\,\Omega'_1\,(z_1) = \frac{1}{2}\left[\Omega'_1(z_1) + \overline{\Omega'_1(z_1)}\right], \quad Im\,\Omega'_1\,(z_1) = \frac{1}{2i}\left[\Omega'_1\,(z_1) - \overline{\Omega'_1(z_1)}\right] \tag{41}$$

Eq. (40) becomes:

$$\sigma_y = c[\,\Omega'_1\,(z_1) + \overline{\Omega'_1\,(z_1)}\,] + d[\,\Omega'_1\,(z_1) - \overline{\Omega'_1\,(z_1)}\,], \quad \text{for } y \to 0 \tag{42}$$

where:

$$\begin{aligned} & c = -\mu\,(1 - m^2_2), \quad c + d = -\frac{4\beta_1\, m_2 - i\,(1 - m^2_2)^2}{i\,(1 - m^2_2)}\,\mu \\ & d = \frac{4\mu\beta_1 m_2}{i\,(1 - m^2_2)}, \quad c - d = -\frac{4\beta_1\, m_2 + i\,(1 - m^2_2)^2}{i\,(1 - m^2_2)}\,\mu \end{aligned} \tag{43}$$

or:

$$\sigma_y = (c + d)\,\Omega'_1\,(z_1) + (c - d)\,\overline{\Omega'_1\,(z_1)}, \quad \text{for } y \to 0 \tag{44}$$

Taking the boundary values of σ_y -stress along the line of discontinuity L we have:

$$\begin{aligned} & \sigma_y^{+} = (c + d)\,\Omega'^{+}_1\,(x) + (c - d)\,\overline{\Omega'^{+}_1\,(x)}, \quad \text{for } x \in L, \; y \to 0^{+} \\ & \sigma_y^{-} = (c + d)\,\Omega'^{-}_1\,(x) + (c - d)\,\overline{\Omega'^{-}_1\,(x)}, \quad \text{for } x \in L, \; y \to 0^{-} \end{aligned} \tag{45}$$

By using the Schwarz reflection principle, (45) yields:

$$
\begin{aligned}
\sigma_y^+ &= (c+d)\,\Omega'^+_1(x) + (c-d)\,\overline{\Omega'^+_1(x)}, \qquad \text{for } x \in L,\ y \to 0^+ \\
\sigma_y^- &= (c+d)\,\Omega'^+_1(x) + (c-d)\,\overline{\Omega'^+_1(x)}, \qquad \text{for } x \in L,\ y \to 0^-
\end{aligned} \tag{46}
$$

Now, by taking into account that:

$$\overline{\Omega'^+_1(x)} = \bar{\Omega}'^-_1(x)$$

$$\overline{\Omega'^+_1(x)} = \overline{w_1(x,-y)} - i\overline{w_2(x,-y)} = \bar{w}_1(x,-y) + i\bar{w}_2(x,-y) = \bar{\Omega}'^-_1(x)$$

(46) becomes:

$$
\left.
\begin{aligned}
\sigma_y^+ &= (c+d)\,\Omega'^+_1(x) + (c-d)\,\bar{\Omega}'^-_1(x) \\
\sigma_y^- &= (c+d)\,\bar{\Omega}'^+_1(x) + (c-d)\,\Omega'^-_1(x)
\end{aligned}
\right\},\ x \in L \tag{47}
$$

Adding and subtracting the two relations in (50) we take:

$$\sigma_y^+ + \sigma_y^- = (c+d)\,[\Omega'_1(x) + \bar{\Omega}'_1(x)]^+ + (c-d)\,[\Omega'_1(x) + \bar{\Omega}'_1(x)]^- \tag{48.1}$$

$$\sigma_y^+ - \sigma_y^- = (c+d)\,[\Omega'_1(x) - \bar{\Omega}'_1(x)]^+ + (d-c)\,[\Omega'_1(x) - \bar{\Omega}'_1(x)]^- \tag{48.2}$$

But $\sigma_y^+ - \sigma_y^- = 0$ and thus:

$$\Omega'_1(z_1) - \bar{\Omega}'_1(z_1) = 0 \text{ or } \Omega'_1(z_1) = \bar{\Omega}'_1(z_1), \text{ allover the z-plane} \tag{49}$$

Then, by virtue of (49), (48.1) becomes:

$$(c+d)\,[\Omega'_1(x)]^+ + (c-d)\,[\Omega'_1(x)]^- = \frac{\sigma_y^+ + \sigma_y^-}{2} \quad \text{or}$$

$$[\Omega'_1(x)]^+ + \frac{c-d}{c+d}\,[\Omega'_1(x)]^- = \frac{p}{c+d} \quad \text{or}$$

$$[\Omega'_1(x)]^+ - g[\Omega'_1(x)]^- = f \tag{50}$$

where:

$$g = \frac{d-c}{c+d}, \quad f = \frac{p}{c+d} \tag{51}$$

(50) is a non-homogeneous Hilbert problem with the following solution [**8**, **14**, **15**]:

$$\Omega'_1 (z_1) = \frac{X_0 (z_1)}{2\pi i} \int_L \frac{f\,dx}{X_0^+ (x) (x - z_1)} + X_0 (z_1) P (z_1) \tag{52}$$

where $X_0 (z_1) = z_1^{\gamma-1}$ is the *Plemelj function* for the semi-infinite crack L, $\gamma = \log(g)/2\pi i$ and $P (z_1) = \Gamma_n z_1^n + G_{n-1} z_1^{n-1} + \ldots + \Gamma_0$, which for a single crack becomes $P (z_1) = \Gamma_1 z_1 + \Gamma_0$. In the latter polynomial $\Gamma_1 = A_1$, where A_1 has been defined in Eq. (6) and thus for the present problem $\Gamma_1 = 0$, since $\sigma_x^{\infty} = \sigma_y^{\infty} = 0$. Further, the single-valuedness condition of the displacements requires $\Gamma_0 = 0$.

The quantity γ expresses the complex singularity of the field, and it is given by:

$$\gamma = \frac{1}{2\pi i} \log W + \frac{1}{2\pi} \tau \tag{53}$$

where:

$$W = \left\{ \frac{[16\beta^2_1 m^2_2 - (1 - m^2_2)^4]^2 + 64\beta^2_1 m^2_2 (1 - m^2_2)^4}{[16\beta^2_1 m^2_2 + (1 - m^2_2)^4]^2} \right\}^{1/2}$$

$$\tau = \tan^{-1} \left\{ \frac{8\beta_1 m_2 (1 - m^2_2)^2}{16\beta^2_1 m^2_2 - (1 - m^2_2)^4} \right\} \tag{54}$$

The integral in (52) has the form:

$$I (z_1) = \frac{1}{2zi} \int_{-\alpha}^{0} \frac{x^{1-\gamma}}{x - z_1} dx = H_0 + H_1 z_1 + H_2 z_1^2 + \ldots \tag{55}$$

Then, the sectionally analytic function $\Omega'_1 (z_1)$ is found to be:

$$\Omega'_1 (z_1) = f\,(H_0 z_1^{\gamma-1} + H_1 z_1^{\gamma} + H_2 z_1^{\gamma+1} + \ldots) \tag{56}$$

Stresses and displacements can be evaluated by Eq. (56) by inserting the real and imaginary parts of $\Omega'_1 (z_1)$. As in the case of transonically moving loads in the surface of an elastic semi-space [**27**], terms of $\delta(x + m_2 y)$ will be arise in stresses and displacements. This implies that a part of the solution (of shear character) represents a plane shock-wave, $x + m_2y = 0$, attached to the load and associated with a jump in displacements and an impulse in stresses. This wavefront called a *head wave*, or *Mach wave*, is well-known in gas dynamics. Fig. 7a shows schematically such a wavefront propagating backward to the running crack tip.

It may be emphasized that, in contrast to the case of a transonically moving load in the surface of an elastic semi-space [**27**], the singularity of stresses in the problem considered here is changed with respect to that in the case of a subsonically moving crack. This fact can be clearly observed from relation (56). This important difference between the case of a crack moving in an infinite body and a load moving on the surface of a half-plane is due to the different types of singularities in these problems.

Finally, the ratio σ_y/σ_x, showing the reduction of the stress-triaxiality, ahead of the transonically moving crack tip, can be easily evaluated directly from Eqs. (38.3) and (38.4) as:

$$\frac{\sigma_y}{\sigma_x} = \frac{m^2_2 - 1}{2\beta^2_1 + m^2_2 + 1} \qquad (57)$$

This ratio was plotted in Fig. 8 and shows again the tendency of σ_x–stress to increase with increasing velocities.

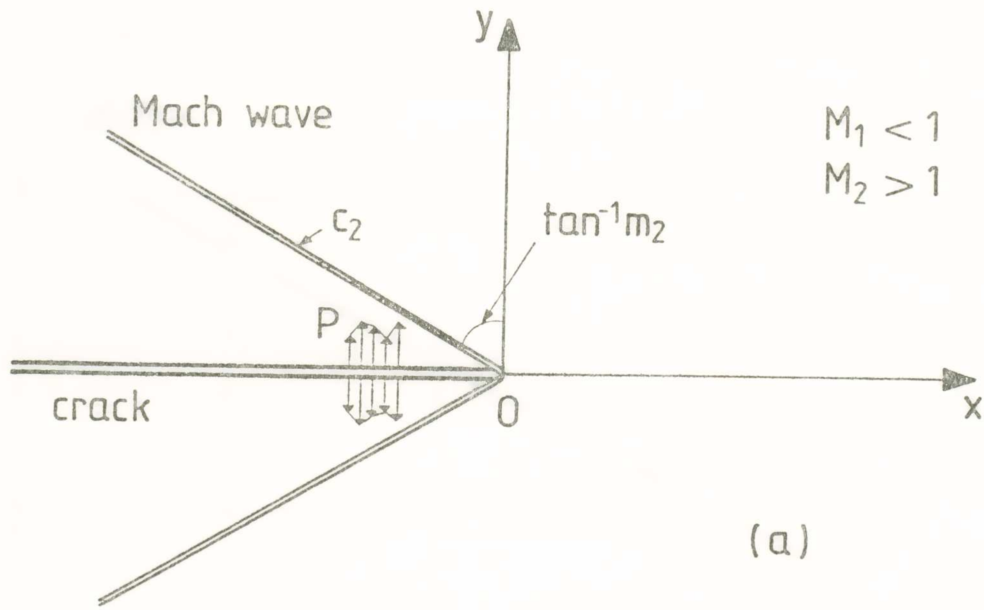
7. SEMI-INFINITE CRACK PROPAGATING WITH SUPERSONIC VELOCITIES

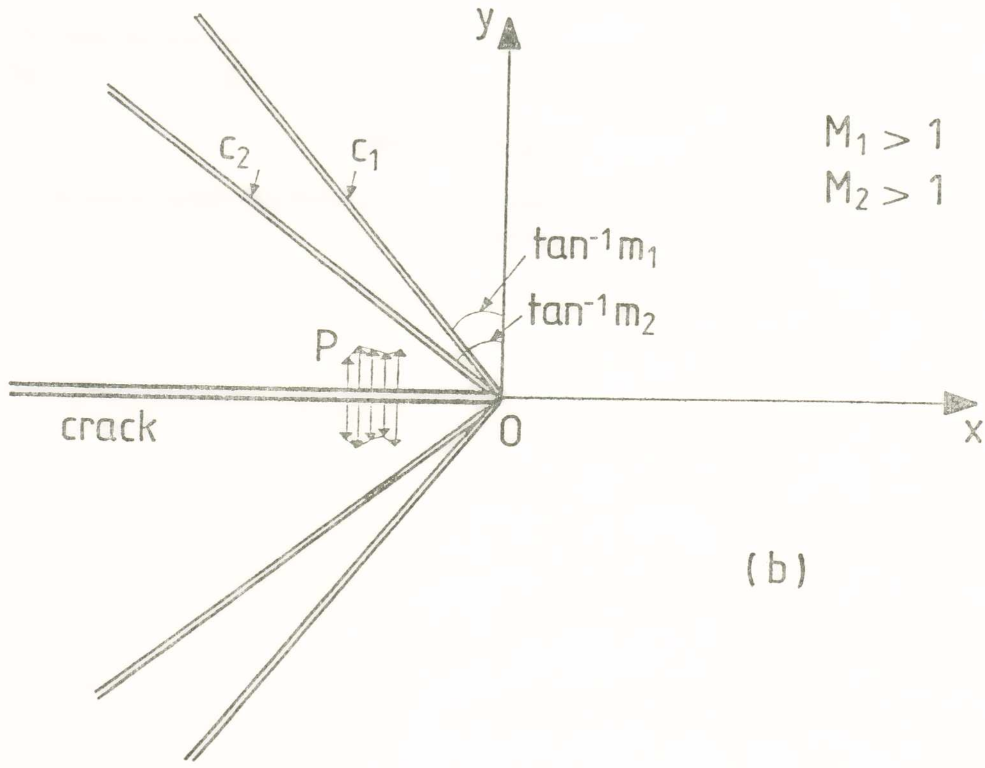
In this case both Mach numbers, M_1 and M_2, are greater than unity. Based on the analysis of Section 7 we can write the solution as:

$$\varphi = \Upsilon_1 (x + m_1y), \quad \psi = \Upsilon_2 (x + m_2y) \qquad (58)$$

This shows that the stresses and displacements are constant behind the plane-shock wavefronts, $x + m_1y - \upsilon t = 0$ and $x + m_2y - \upsilon t = 0$. Only terms of $\delta(x+m_1y)$ and $\delta(x+m_2y)$ will arise in stresses and displacements, in accordance with the case of a supersonically moving load on the surface of a semi-space [**27**]

Since the field here is spatially independent, the supersonic case presenst less interest, from the view-point of stress analysis, than the transonic case. Fig. 7b presents the longitudinal- and shear-Mach wavefronts attached to the moving crack tip.





Mach waves

Fig. 7. Mach wavefronts formed behind transonic and supersonic cracks moving in an infinite elastic medium.

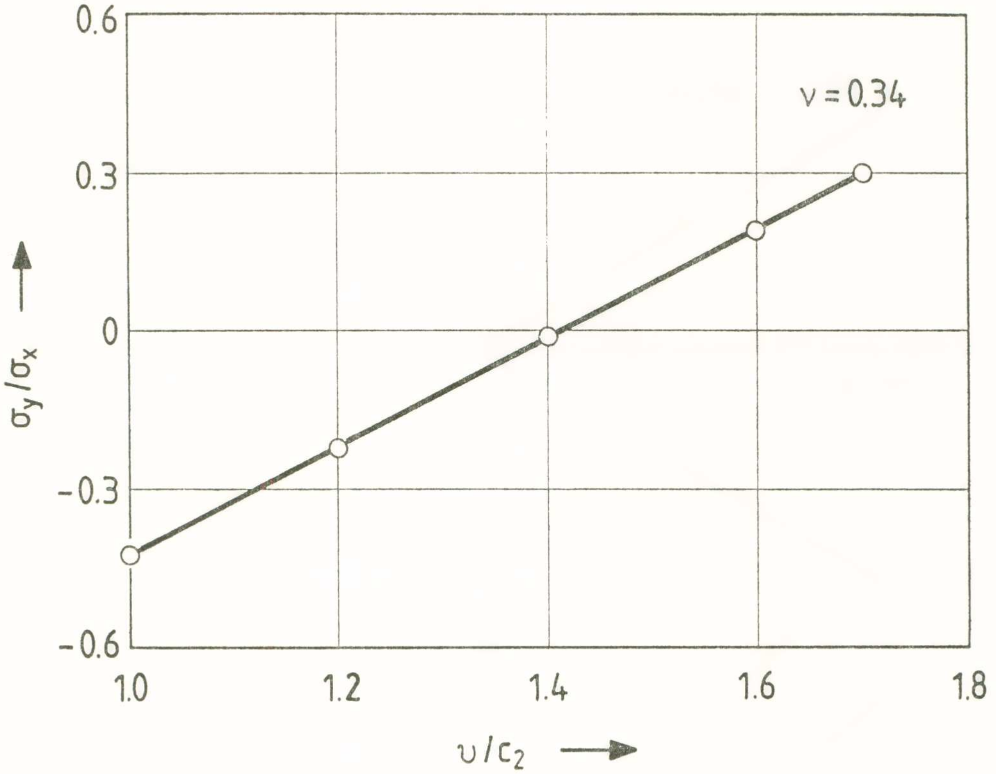


Fig. 8. The ratio σ_y/σ_x ahead of a transonically moving crack tip plotted against crack velocity.

8. CONCLUSIONS

A class of elastodynamic crack problems was solved in this paper. The method of treating these problems has a unifield character, whereas the configurations concern semi-infinite cracks in infinite bodies. Analytic-function theory was proved an adequate tool to solve such problems in closed from. However, crack problems in fiinite bodies need heavier mathematical methods, such as Integral Transforms or Integral Equations. In the latter cases, solutions in closed form are not generally obtained.

The particular case of a transonically propagating crack presents the interesting feature of the change of singularity of the stress field. On the other hand, the analysis becomes extremely simple, when the crack moves in the supersonic regime.

ΠΕΡΙΛΗΨΙΣ

Ἡ παροῦσα μελέτη ἀφορᾷ εἰς τὴν θεωρητικὴν διερεύνησιν τῶν φαινομένων τὰ ὁποῖα συνδέονται μὲ τὴν διάδοσιν ρωγμῶν εἰς ἐλαστικὰ μέσα. Ἡ ταχεῖα διάδοσις ρωγμῶν εἶναι δυναμικὸν φαινόμενον τὸ ὁποῖον προκαλεῖ τὴν συνεχῆ ἀλλαγὴν τῶν τάσεων καὶ παραμορφώσεων εἰς ὅλα τὰ σημεῖα τοῦ σώματος. Εἶναι φανερὸν ὅτι εἰς τοιαύτας καταστάσεις παύει νὰ ἰσχύει ἡ ὑπόθεσις τοῦ ἀπαραμορφώτου στερεοῦ, καθόσον ἀπαιτεῖται πεπερασμένος χρόνος διαδόσεως τῆς διαταραχῆς ἀπὸ τὴν κινουμένην πηγὴν εἰς ἄλλας θέσεις ἐντὸς τοῦ ὑλικοῦ. Ἡ ἀλλαγὴ τῆς ἐντατικῆς καταστάσεως λαμβάνει χώραν μὲ ὡρισμένας χαρακτηριστικὰς διὰ κάθε ὑλικὸν ταχύτητας, ἤτοι τῶν διαμήκων καὶ ἐγκαρσίων ἐλαστικῶν κυμάτων.

Μὲ ἄλλους λόγους τὸ ἐλαστικὸν μέσον παρουσιάζει ἀδράνειαν εἰς τὴν ἀλλαγὴν τῆς ἐντατικῆς του καταστάσεως καὶ οἱ ἀντίστοιχοι στατικαὶ λύσεις τῶν προβλημάτων τῆς Ἐλαστικότητος παύουν νὰ ἰσχύουν.

Καίτοι ὁ κλάδος τῆς Δυναμικῆς Ἐλαστικότητος ἔχει ζωὴν σχεδὸν ἑνὸς αἰῶνος καὶ εἰς τὴν βιβλιογραφίαν ὑπάρχει πλῆθος θεωρητικῶν καὶ πειραματικῶν ἐργασιῶν, ὑπάρχουν ἀκόμη ἐνδιαφέροντα προβλήματα χρήζοντα ἐπιλύσεως. Πρέπει ἐπίσης νὰ σημειωθῇ τὸ ἐκτενὲς πεδίον ἐφαρμογῶν αὐτῶν τῶν προβλημάτων εἰς διαφόρους κλάδους τῆς Ἐπιστήμης καὶ Τεχνολογίας, ὅπως ἡ Γεωφυσικὴ καὶ ἡ Σεισμολογία, καθὼς καὶ οἱ κάθε εἴδους κατασκευαὶ μὲ τὰς δυναμικὰς καταπονήσεις των.

Κατὰ τὴν θεωρητικὴν διερεύνησιν τοῦ φαινομένου τῆς δυναμικῆς θραύσεως δύο τύποι προβλημάτων ἀντιμετωπίσθησαν ἀπὸ τοὺς ἐρευνητάς. Κατὰ τὴν πρώτην κατηγορίαν, καλουμένην τῆς σταθερᾶς καταστάσεως, τὰ γεωμετρικὰ χαρακτηριστικὰ τοῦ προβλήματος λαμβάνονται ἀμετάβλητα συναρτήσει τοῦ χρόνου, μὲ ἀποτέλεσμα τὴν εἰς ἱκανὸν βαθμὸν ἁπλοποίησιν τοῦ προβλήματος. Ἀντιθέτως, κατὰ τὴν δευτέραν κατηγορίαν, καλουμένην τῆς παροδικῆς καταστάσεως, ἡ γεωμετρία τοῦ προβλήματος ὑπόκειται εἰς μεταβολὴν σταθεροῦ ρυθμοῦ, συναρτήσει τοῦ χρόνου, καὶ μόνον ἡ ἐξωτερικὴ φόρτισις θεωρεῖται ἀμετάβλητος.

Αἱ λύσεις εἰς τὰς ὁποίας καταλήγουν αἱ δύο αὗται κατηγορίαι προβλημάτων διαδόσεως ρωγμῶν ἔχουν τὸ μὲν τμῆμα τῆς ἐξαρτήσεως τῶν τάσεων ἀπὸ τὰς συντεταγμένας τῶν σημείων τοῦ ἐλαστικοῦ μέσου ἀκριβῶς τὸ αὐτό, τὸ δὲ τμῆμα τῆς ἐντάσεως, ἤτοι τῆς ἐξαρτήσεως ἀπὸ τὴν φόρτισιν, τὴν γεωμετρίαν καὶ τὴν ταχύτηταν, τελείως διάφορον.

Εἶναι ἐν γένει γνωστὸν ὅτι, μὲ τὴν βοήθειαν τῆς θεωρίας τῶν Ἀναλυτικῶν Συναρτήσεων, δύνανται νὰ ἐπιλυθοῦν ὑπὸ κλειστὴν μορφὴν μόνον προβλήματα τῆς πρώτης κατηγορίας, ἐνῷ ἐκεῖνα τῆς δευτέρας χρήζουν πλέον περιπλόκων μεθόδων τῶν Μαθηματικῶν, ὅπως οἱ Ὁλοκληρωτικοὶ Μετασχηματισμοὶ καὶ οἱ Ὁλοκληρωτικαὶ Ἐξισώσεις.

Εἰς τὴν παροῦσαν μελέτην δίδονται λύσεις, ὑπὸ κλειστὴν μορφήν, προβλημάτων τοῦ πρώτου τύπου. Θεωρεῖται ἐλαστικὸν σῶμα ἀπείρων διαστάσεων καὶ ἡμι-ἄπειρος ταχέως διαδιδομένη ρωγμή. Ἡ φόρτισις ἐπιβάλλεται εἰς τὰς παρειὰς τῆς ρωγμῆς καὶ ἀποτελεῖται εἴτε ἀπὸ σταθερὰς τάσεις, περίπτωσις ἐσωτερικῆς πιέσεως, εἴτε σταθερὰς μετατοπίσεις, περίπτωσις κινουμένου σφηνός. Ἡ ταχύτης διαδόσεως τῆς ρωγμῆς θεωρεῖται εἴτε μικροτέρα αὐτῆς τῶν ἐλαστικῶν κυμάτων ἐντὸς τοῦ μέσου, ἤτοι ὑποηχητικὴ εἴτε μεγαλυτέρα, ἤτοι παρηχητικὴ καὶ ὑπερηχητική. Τὸ πρόβλημα εἰς τὴν ὑποηχητικὴν περιοχὴν ταχυτήτων ἔχει ἐπιλυθεῖ εἰς τὸ παρελθὸν ὑπὸ τῶν J. W. Craggs καὶ G. C. Sih, διὰ χρησιμοποιήσεως τελείως διαφόρων μεθόδων ἀπὸ αὐτὴν τῆς παρούσης μελέτης. Τὸ πρόβλημα εἰς τὴν παρηχητικὴν καὶ ὑπερηχητικὴν περιοχὴν ταχυτήτων ἀντιμετωπίζεται διὰ πρώτην φορὰν καὶ δίδεται ἡ ἀκριβής του λύσις ὑπὸ κλειστὴν μορφήν.

Εἰς τὴν ὑποηχητικὴν περιοχὴν ταχυτήτων ἀμφότεραι αἱ κυματικαὶ ἐξισώσεις εἶναι ἐλλειπτικαὶ καὶ τὸ πρόβλημα δύναται νὰ ἀντιμετωπισθῇ ὡς συνοριακὸν πρόβλημα τῆς θεωρίας Δυναμικοῦ, δι' ἐφαρμογῆς τοῦ καταλλήλου συμμόρφου μετασχηματισμοῦ. Ἀντιθέτως, ὅταν ἡ ταχύτης τοῦ ἄκρου τῆς ρωγμῆς ὑπερβεῖ τὴν ταχύτητα διαδόσεως τῶν ἐγκαρσίων κυμάτων εἰς τὸ ὑλικόν, ἡ κυματικὴ ἐξίσωσις, ἡ ὁποία ἐκφράζει τὸν διανυσματικὸν χαρακτῆρα τοῦ πεδίου, γίνεται ὑπερβολικὴ καὶ ἡ ἀνάλυσις καθίσταται δύσκολος. Εἰς τὴν περίπτωσιν αὐτὴν ἡ λύσις ἐξάγεται διὰ θεωρήσεως τοῦ προβλήματος τῆς γραμμικῆς συσχετίσεως ἢ προβλήματος Hilbert.

Δι' ὑπερηχητικάς ταχύτητας ἡ κατανομή τῶν τάσεων καὶ τῶν παραμορφώσεων δὲν εἶναι πλέον συνεχής ἀπὸ σημείου εἰς σημεῖον, καὶ ὑπάρχουν ἀπότομοι ἀλλαγαὶ ἐπὶ τῶν μετώπων τῶν κυμάτων, τὰ ὁποῖα ἀκολουθοῦν τὸ κινούμενον ἄκρον τῆς ρωγμῆς. Αὐτὰ εἶναι τὰ γνωστὰ ἀπὸ τὴν Ἀεροδυναμικὴν κύματα Mach, ποὺ διαδίδονται μὲ ταχύτητας ἴσας μὲ τὰς χαρακτηριστικὰς ταχύτητας τῶν διαμήκων καὶ ἐγκαρσίων κυμάτων ἐντὸς τοῦ μέσου.

Τέλος, ἕτερον ἐνδιαφέρον χαρακτηριστικὸν εἶναι ἡ ἀλλαγὴ τοῦ τύπου τῆς ἰδιομορφίας, ἤτοι τοῦ νόμου βάσει τοῦ ὁποίου ἀπειρίζονται οἱ τάσεις εἰς περιοχὰς προσεγγιζούσας τὸ ἄκρον τῆς ρωγμῆς, ἀπὸ $z^{1/2}$ εἰς τὴν ὑποηχητικὴν περιοχὴν ταχυτήτων ρωγμῆς εἰς $z^{\gamma-1}$, ἔνθα γ μιγαδικός, εἰς τὴν παρηχητικὴν περιοχήν.

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