

ΜΑΘΗΜΑΤΙΚΑ.—On the Problem of Connection between Geometrical Optics and Wave Optics for Anisotropic Media, by *Nicholas Chako*\*. Ἀνεκρινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ἰωάνν. Ξανθάκη.

#### 1. INTRODUCTION.

In several papers<sup>[1,2]</sup> we have shown the procedure of affecting the transition from the equations of classical mechanics and geometrical optics to the respective equations of quantum mechanics and wave optics, and vice-versa. This relationship comes about through the equations of variations of Poincaré for classical dynamical system, or geometrical optical system, via Einstein's famous relation  $E=h\nu$  and the similar role played by  $\lambda$  (wave length of light) in optics. On the other hand, the transition from an equation of the wave type to the corresponding dynamical, or geometrical optics equation, is carried out through the asymptotic solution of the wave equation in terms of a large (small) parameter  $k = \frac{2\pi}{\lambda}$ , or the Planck constant  $\frac{2\pi}{h}$ . Both procedures lead to a certain condition to be satisfied, namely the Birkhoff-Chetaev criterion<sup>[1]</sup>. It is in this sense that the asymptotic character of classical dynamics and geometrical optics is revealed, and not as is often stated through the well known Moll-Debye-Sommerfeld transformation<sup>[1,6]</sup>. Furthermore we have also shown (l.c.) the close association of the equations of variations and the so-called transport equations which determine the various amplitudes entering in the asymptotic solution of the wave equation. Indeed, the leading transport equation satisfied by the *principal amplitude* is of the same form as the equation of variation of the corresponding classical system satisfying first order stability. Moreover, the equations of variation for higher order stability correspond to the higher order transport equations in the so-called higher order amplitudes entering in the asymptotic expansion of the wave function. This leads to the important result that the solution of a classical dynamical system, or geometrical optical system, obtained by means of perturbation theory, is closely connected to the full asymptotic solution of the associated wave equation of quantum mechanics, or wave op-

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tics, the transport equations playing the same role as the equations of variations for the classical system. For full details see ref. [1,7]. In fact, our results are derived for more general equations of wave type and embrace all the wave equations of mathematical physics.

Here, we shall be concerned only with the problem of the geometrical optics equations for anisotropic media and their associated wave equations. The case of isotropic media has been fully discussed elsewhere [1]. Our treatment will be based on Maxwell equations rather than the corresponding wave equation for anisotropic media.

## 2 THE TRANSPORT EQUATIONS OF THE ELECTROMAGNETIC FIELDS IN AN ANISOTROPIC MEDIUM.

The propagation of electromagnetic waves in an anisotropic medium is governed by Maxwell equations

$$(2.1) \quad \nabla \wedge \mathbf{H} = \frac{1}{c} \mathbf{D}_t, \quad \nabla \wedge \mathbf{E} = - \frac{1}{c} \mathbf{B}_t; \quad \left( \frac{\partial \mathbf{F}}{\partial t} = \mathbf{F}_t \right)$$

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \cdot \mathbf{B} = 0.$$

Here, we have assumed no sources in the medium. The relation between  $\mathbf{D}$  and  $\mathbf{E}$  and  $\mathbf{B}$  and  $\mathbf{H}$  are given by the linear relation

$$(2.2) \quad \mathbf{D} = \varepsilon(\mathbf{x}_i, t) \mathbf{E}, \quad \mathbf{B} = \mu(\mathbf{x}_i, t) \mathbf{H}$$

where the dielectric constant  $\varepsilon$  and the permeability  $\mu$  are functions of coordinates and time. Here, we shall assume  $\varepsilon$  to be a vector function and  $\mu$  a scalar. In general  $\varepsilon$  and  $\mu$  are tensors. We shall not discuss this case on account of the complexity of the problem<sup>(+)</sup>.

The fields  $\varepsilon$  and  $\mathbf{H}$  are assumed to be continuous functions for all  $\mathbf{x}_i$  and  $t$ , except for their first partial derivatives which we assume to be discontinuous for those values of the coordinates and for all values of  $t$  on a certain surface  $\Sigma$  given by the equation

$$(2.3) \quad \varphi(\mathbf{x}_i, t) = 0, \text{ or a constant } C.$$

As  $t$  varies, the surface  $\Sigma$  moves and changes form in the physical medium determined by  $\varepsilon$  and  $\mu$ . The surface  $\Sigma$  is called a *wave-surface* or wave front representing the electromagnetic wave. For any fixed value of

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(+) In sect. 4 the transport equations are given for the general case,  $\varepsilon$  and  $\mu$  are tensors.

$t$ ,  $\Sigma$  separates the space in two regions A and B. In each of these regions, the derivatives of the field E and H are continuous except where the coordinates are on  $\Sigma$ , in which case they suffer discontinuities. If we denote the values of the derivatives of E and H on A by a subscript 1 and on B by subscript 2, we write

$$(2.4) \quad \left[ \frac{\partial E_i}{\partial x_j} \right] = \left( \frac{\partial E_i}{\partial x_j} \right)_2 - \left( \frac{\partial E_i}{\partial x_j} \right)_1 = u_i \varphi_{x_j}, \quad \left[ \frac{\partial H_i}{\partial x_j} \right] = v_i \varphi_{x_j},$$

$$\left[ \frac{\partial E_i}{\partial t} \right] = u_i \varphi_t, \quad \left[ \frac{\partial H_i}{\partial t} \right] = v_i \varphi_t, \quad \left[ \frac{\partial D_i}{\partial x_j} \right] = (\varepsilon_i u_i) \varphi_{x_i}, \text{ etc.}$$

where  $u_i$ ,  $v_i$  may be considered as components of the vectors  $u$  and  $v$ . Relations (2.4) are known as the geometrical and kinematical compatibility conditions of E and H on  $\Sigma$ . Furthermore, since  $\varepsilon$  and  $\mu$  and their derivatives with respect to  $x_i$ ,  $t$  are assumed continuous on  $\Sigma$ , we have according to (2.2) the relations

$$(2.5) \quad \left[ \frac{\partial D_i}{\partial x_j} \right] = \varepsilon_i \left[ \frac{\partial E_i}{\partial x_j} \right], \quad \left[ \frac{\partial B_i}{\partial x_j} \right] = \mu_i \left[ \frac{\partial H_i}{\partial x_j} \right], \quad \left[ \frac{\partial D_i}{\partial t} \right] = \varepsilon_i \left[ \frac{\partial E_i}{\partial t} \right],$$

$$\left[ \frac{\partial B_i}{\partial t} \right] = \mu_i \left[ \frac{\partial H_i}{\partial t} \right]$$

$$(2.6) \quad [D_{ix_j}] = (\varepsilon_i u_i) \varphi_{x_j}, \quad [B_{ix_j}] = (\mu_i v_i) \varphi_{x_j}, \quad [D_{it}] = (\varepsilon_i u_i) \varphi_t,$$

$$[B_{it}] = (\mu_i v_i) \varphi_t$$

Inserting these expressions in (2.1)-(2.2), we get the dynamical compatibility conditions, namely [4]

$$(2.7) \quad \Sigma (\varepsilon_i u_i) \varphi_{x_i} = 0, \quad \Sigma (\mu_i v_i) \varphi_{x_i} = 0$$

$$(2.8) \quad \sum_{i=1}^3 \left[ u_{i+2} \varphi_{x_{i+1}} - u_{i+1} \varphi_{x_{i+2}} \right] + (\mu_i v_i) \frac{\varphi_t}{c} = 0,$$

$$\sum_{i=1}^3 \left[ v_{i+2} \varphi_{x_{i+1}} - v_{i+1} \varphi_{x_{i+2}} \right] - (\varepsilon_i u_i) \frac{\varphi_t}{c} = 0$$

The functions  $u_i$ ,  $v_i$  are determined by these equations once we know  $\varphi$ . To determine  $\varphi$  let  $n$  be the normal to the wave surface  $\varphi$ . It is the velocity of propagation of  $\Sigma$ , then  $n$  and  $w$  are given by

$$(2.9) \quad n_i = n^i = \frac{\varphi_{x_i}}{\Delta}, \quad w = - \frac{\varphi_t}{\Delta} \quad (\Delta = \sqrt{\sum_{i=1}^3 (\varphi_{x_i})^2})$$

and equations (2.7) and (2.8) are briefly written as follows:



$$(2.9) \quad (\epsilon \cdot u) \cdot n = 0, (\mu \cdot v) \cdot n = 0, (2.10) \quad (n \wedge u) = \frac{w}{c} (\mu \cdot v), (n \wedge v) = -\frac{w}{c} (\epsilon \cdot u)$$

where  $n = (n_1, n_2, n_3)$  is the normal vector. Eliminating  $v$ , we obtain the equation satisfied by  $u$  [4]

$$(2.10) \quad n(n \cdot u) - u = -\mu \left( \frac{w^2}{c^2} \right) (\epsilon \cdot u)$$

### 3. HAMILTON'S CHARACTERISTIC EQUATION, OR THE EICONAL EQUATION.

In order to determine  $\varphi$  we proceed as follows. Let us write (2.10) in the form

$$(3.1) \quad (n \cdot u) n_i - u_i = -\mu_i \left( \frac{w}{c} \right)^2 (\epsilon_i u_i)$$

This is a system of three linear homogenous equations in the unknowns  $u_1, u_2$  and  $u_3$ . Since  $u_i$  are independent it can be satisfied if the determinant of the coefficients vanish. The result of elimination of  $u_i$  yield the following equation in  $\varphi_{x_i}, \varphi_t$  [4]

$$(3.1) \quad H = \sum_{i=1}^3 \frac{\varphi_{x_i}^2}{\varphi_t^2 - \gamma_i^2 \Delta^2} = 0, \quad \gamma_i^2 = \frac{c^2}{(\epsilon \cdot \mu)_i}$$

This is a first order partial differential equation in  $\varphi_{x_i}, \varphi_t$ . It is Hamilton's characteristic function for the determination of  $\varphi$ , or the *multiplier equation* of Birkhoff [3], associated with Maxwell equations (2.1). At any point of the field there correspond to any direction of space given by  $n_i$  two velocities of propagation  $w$  determined by the equation

$$(3.2) \quad \sum_{i=1}^3 \frac{n_i^2}{w^2 - \gamma_i^2} = 0.$$

On the other hand the equations of the rays are obtained from the solution of the first canonical equation

$$(3.3) \quad \frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial \varphi_{x_i}} \bigg/ \frac{\partial H}{\partial \varphi_t}.$$

Performing the differentiation one arrives after some reductions to the following result

$$(3.3) \quad \frac{dx_i}{dt} = -\frac{\varphi_{x_i}}{\varphi_t} \left[ \frac{\varphi_t^2}{\Delta^2} + \frac{1}{(\varphi_t^2 - \gamma_i^2 \Delta^2)} \frac{1}{M} \right] = n_i w \left( w + \frac{N^2}{w(w^2 - \gamma_i^2)} \right),$$

where

$$(3.4) \quad M = \sum_{j=1}^3 \left( \frac{\varphi_{xj}}{\varphi_t^2 - \gamma_j^2 \Delta^2} \right)^2, \quad N^2 = \sum_{i=1}^3 \left( \frac{1}{\frac{n_i}{w^2 - \gamma_i^2}} \right)^2$$

Thus there are two rays associated with the two velocities  $w = (w_1, w_2)$  in any direction of space. If we multiply (3.3) by  $n_i$  and sum over  $i$ , we get

$$(3.5) \quad (\mathbf{n} \cdot \dot{\mathbf{x}}) = w,$$

that is the normal component to the wave front of the vector  $\dot{\mathbf{x}}$  is equal to the transport velocity  $w$  of the wave front. On the other hand it is easy to show that  $\mathbf{u}, \mathbf{v}$  are orthogonal to  $\dot{\mathbf{x}}$ , the velocity of electromagnetic energy carried along the rays.

#### 4. TRANSPORT EQUATIONS OF THE ELECTROMAGNETIC FIELD ALONG THE RAYS.

The transport equations of the field in the medium defined  $\epsilon$  and  $\mu$  can be derived in the same way as in sec. 3 of [7]. To do so we express the field functions  $\mathbf{E}$  and  $\mathbf{H}$  and also  $\mathbf{D}$  and  $\mathbf{B}$  at any time  $t$  in the form

$$(4.1) \quad \mathbf{E}_B = \mathbf{E}_A + \mathbf{u} \varphi + \frac{1}{2!} \mathbf{u}' \varphi^2 + \dots, \quad \mathbf{H}_B = \mathbf{H}_A + \mathbf{v} \varphi + \frac{1}{2!} \mathbf{v}' \varphi^2 + \dots,$$

$$(4.2) \quad \mathbf{D}_B = \mathbf{D}_A + (\epsilon \cdot \mathbf{u}) \varphi + \frac{1}{2!} (\epsilon \cdot \mathbf{u}') \varphi^2 + \dots, \quad \mathbf{B}_B = \mathbf{B}_A + (\mu \cdot \mathbf{v}) \varphi + (\mu \cdot \mathbf{v}') \varphi^2 + \dots,$$

where  $\mathbf{u}', \mathbf{v}'$ , etc., are determined from  $\mathbf{u}, \mathbf{v}$  by application of compatibility conditions as shown in [7]. In sec. 3 we have seen that  $\mathbf{u}, \mathbf{v}$  satisfy eqs. (2.7 - 2.8). Since the system is linear and homogeneous in  $u_i, v_i$ , the determinant of the coefficients must vanish. Let us denote it by  $A^{\alpha\beta}(x_i, t, \varphi) = 0$ . The matrix  $A^{\alpha\beta}$  and the operator  $T^{\alpha\beta}$  are:

$$A^{\alpha\beta} = \begin{vmatrix} \epsilon_{xx}\varphi_t + \epsilon_{xx}t & \epsilon_{xy}\varphi_t + \epsilon_{xy}t & \epsilon_{xz}\varphi_t + \epsilon_{xz}t & 0 & -\varphi_z & \varphi_y \\ \epsilon_{yx}\varphi_t + \epsilon_{yx}t & \epsilon_{yy}\varphi_t + \epsilon_{yy}t & \epsilon_{yz}\varphi_t + \epsilon_{yz}t & \varphi_z & 0 & -\varphi_x \\ \epsilon_{zx}\varphi_t + \epsilon_{zx}t & \epsilon_{zy}\varphi_t + \epsilon_{zy}t & \epsilon_{zz}\varphi_t + \epsilon_{zz}t & -\varphi_y & \varphi_x & 0 \\ 0 & \varphi_z & -\varphi_y & \mu_{xx}\varphi_t + \mu_{xx}t & \mu_{xy}\varphi_t + \mu_{xy}t & \mu_{xz}\varphi_t + \mu_{xz}t \\ -\varphi_z & 0 & \varphi_x & \mu_{yx}\varphi_t + \mu_{yx}t & \mu_{yy}\varphi_t + \mu_{yy}t & \mu_{yz}\varphi_t + \mu_{yz}t \\ \varphi_y & -\varphi_x & 0 & \mu_{zx}\varphi_t + \mu_{zx}t & \mu_{zy}\varphi_t + \mu_{zy}t & \mu_{zz}\varphi_t + \mu_{zz}t \end{vmatrix} = 0.$$

The operator  $T^{\alpha\beta}$  has the same structure as the matrix  $A^{\alpha\beta}$  if we replace  $\varphi_t, \varphi_x, \varphi_y, \varphi_z$  by  $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}$ , etc. If  $U^\alpha$  is the matrix (vector), the elements of the first row being  $E_x, E_y, E_z, H_x, H_y, H_z$  and the rest zero, the Maxwell equations (2.1) take the form  $T^{\alpha\beta}U^\alpha = 0$ . and eqs. (2.7 - 2.8) are given by  $A^{\alpha\beta}u^\alpha = 0$ ,  $u^\alpha$  stand for the row matrix with elements of the first row being  $u_1, u_2, \dots, u_6$  and the rest are zero.

If we substitute (4.1) - (4.2) in eq. (2.1) and take account of the compatibility conditions, we arrive at the following system of equations for the determination of  $u^\alpha$  :

$$(4.3) \sum_{\alpha, \beta} \frac{\partial A^{\alpha\beta}}{\partial \varphi_{x\beta}} \frac{\partial u^\alpha}{\partial x_\beta} + \sum_{\alpha} A^{\alpha\beta} u^\alpha = 0, \sum_{\alpha, \beta} \frac{\partial A^{\alpha\beta}}{\partial \varphi_{x\beta}} \frac{\partial u'^\alpha}{\partial x_\beta} + \sum_{\alpha} A^{\alpha\beta} u'^\alpha + F(u^\alpha) = 0, \text{ etc.}$$

$$(\alpha = 1 \dots 6, \beta = 1 \dots 4)$$

where  $F(u^\alpha) = T^{\alpha\beta}u^\alpha$ . These equations are the transport equations. The leading equation is called the *principal equation* and  $u^\alpha$  are the *principal amplitudes*. The other equations are linear non-homogeneous in the higher order amplitudes. For a detailed discussion and their relationship with the equations of variations of the corresponding geometrical (ray) equations see ref. [7].

The system of equations (4.3) are generalizations of the transport equations for inhomogeneous isotropic media derived by a number of authors (see ref. [5], [6]). For homogeneous anisotropic media  $\epsilon$  and  $\mu$  are constants and (4.3) are considerably simplified. A more general problem is treated in ref. [7], where different kind of expansions from (4.1) - (4.2) have been considered.

The equations of the rays is given by the system of equations

$$(4.4) \frac{dx_i}{\frac{\partial A^{\alpha\beta}}{\partial \varphi_{x_i}}} = \frac{dt}{\frac{\partial A^{\alpha\beta}}{\partial \varphi_t}} = \frac{d\varphi}{0} = \frac{-d\varphi_{x_i}}{\frac{\partial A^{\alpha\beta}}{\partial x_i}} = \frac{-d\varphi_t}{\frac{\partial A^{\alpha\beta}}{\partial t}} = d\sigma$$

where  $x_i, t, \varphi_{x_i}, \varphi_t$  are expressed in terms of the parameter  $\sigma$  and initial values of  $x_i, t, \varphi_{x_i}, \varphi_t$ . In terms of  $\sigma$  (4.3) take the simpler form

$$(4.5) \sum_{\alpha} \frac{du^\alpha}{d\sigma} + A^{\alpha\beta} u^\alpha = 0, \sum_{\alpha} \frac{du_k^\alpha}{d\sigma} + \sum_{\alpha} A^{\alpha\beta} u_k^\alpha + F(u_{k-1}^\alpha) = 0, (k=1, 2, \dots; u_0^\alpha = u^\alpha).$$

where  $\frac{d}{d\sigma}$  means differentiation along the rays.



Finally, the method outline above can be applied to Dirac equation of quantum mechanics. The transition from dynamical equations to obtain Dirac's equation has been derived by the author [2]. A similar procedure will carry over the Hamilton equation (4.3) or (3.1) to Maxwell equations (2.1).

#### SUMMARY

In this paper the *transport* equations of propagation of the electromagnetic field have been derived for the general case of an anisotropic and in homogeneous medium. They include as special cases, the problem of propagation in crystalline media, as well as Dirac's equation in quantum mechanics. Our treatment has important applications in obtaining asymptotic solutions of the propagation of electromagnetic waves in the ionosphere as well as in acoustical problems, where the *refractive* index depends on time as well as on the coordinates. We have also indicated the procedure of obtaining the transition from the equation of the rays (geometrical optics) to Maxwell equations.

#### ΠΕΡΙΛΗΨΙΣ

Εἰς τὴν παροῦσαν ἐργασίαν δίδονται αἱ ἐξισώσεις μεταφορᾶς (*transport*) ἡλεκτρομαγνητικοῦ πεδίου εἰς τὴν γενικὴν περίπτωσιν ἀνισοτρόπου καὶ ἀνομοιογενοῦς μέσου. Περιλαμβάνονται ὡς μερικαὶ περιπτώσεις τὸ πρόβλημα τῆς διαδόσεως εἰς κρυσταλλικὰ πεδία καὶ ἡ ἐξίσωσις τοῦ Dirac εἰς τὴν κβαντομηχανικὴν. Περιέχονται σημαντικαὶ ἐφαρμογαὶ εἰς τὴν ἐπίτευξιν ἀσυμπτωτικῶν λύσεων τῆς διαδόσεως ἡλεκτρομαγνητικῶν κυμάτων εἰς τὴν ἰονόσφαιραν καθὼς καὶ εἰς ἀκουστικὰ προβλήματα, ὅπου ὁ δείκτης διαθλάσεως ἐξαρτᾶται ἀπὸ τὸν χρόνον καὶ τὰς συντεταγμένας. Ἐπίσης ὑποδεικνύεται πῶς δύναται νὰ ἐπιτευχθοῦν αἱ ἐξισώσεις τοῦ Maxwell ἀπὸ τὴν ἐξίσωσιν τῶν ἀκτίνων τῆς Γεωμετρικῆς Ὀπτικῆς.

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