ΑΝΑΚΟΙΝΩΣΕΙΣ ΜΗ ΜΕΛΩΝ

ΑΝΑΛΥΤΙΚΗ ΓΕΩΜΕΤΡΙΑ.— **A Conics' Treatment by the Method of Analytical and Geometrical Equivalence,** *by C. B. Glavas**.

Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Ἰω. Ξανθάκη.

The ordinary treatment of a conic $f(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + Z = 0$ involves in general two successive analytical transformations. One is the «translation» of the rectangular axes to a new origin; the other being their «rotation» by a certain angle. By the former transformation one obtains the elimination of the terms in x and y in $f(x, y) = 0$ while by the second the term in xy. The equation of the conic (ellipse or hyperbola) is thus finally reduced to its standard form $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$. With certain modifications one arrives at parabola's equation $y^2 = 2px$.

The equation $f(x, y) = 0$ refers to the rectangular system Oxy, while the standard form to another rectangular system with origin O' and with its axis of the abscissae making an angle ω with Ox. The standard form equation as the by-product of two analytical transformations is algebraically different from the original $f(x, y) = 0$ but represents geometrically the same curve with it. For this reason the conic $f(x, y) = 0$ is geometrically obtained by the construction of the corresponding one to the standard form equation.

In this paper it is proposed to study the conics by the application simultaneously of analytical and geometrical equivalence. The procedure which is going to be followed consists (1) by the statement of the new method as applied for the study of conics through (1.1) rectangular systems and (1.2) oblique ones (2) by the examination of the transformation of conics simultaneously in rectangular and oblique axes (3) by the establishment of certain theorems on sets of conics and (4) by the examination of a particular problem related to special sets of conics.

1. Given the equation $f(x, y) = 0$ of a conic we first apply the transformation $x = x' + a$, $y = y' + b$, where (a, b) are the coordinates of the new origin O' (Fig. 1). Then the equation $f(x, y) = 0$ becomes

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$f(x' + a, y' + b) = 0$. The latter is analytically equivalent to the original $f(x, y) = 0$. Now we take the geometrically equivalent of $f(x' + a, y' + b) = 0$ in the system (x, y). This is $f(x + a, y + b) = 0$. The next step is to

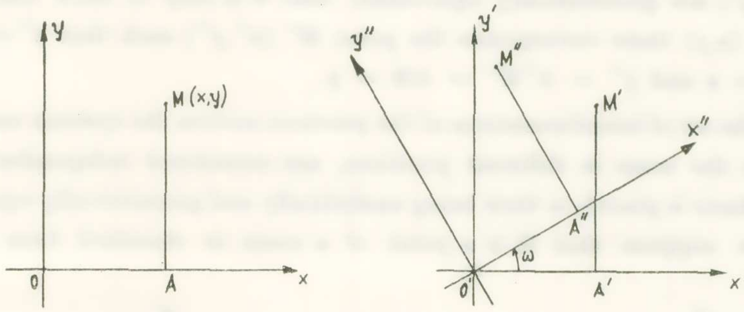

Fig. 1

apply the analytical transformation $x = x'' \cos\omega - y'' \sin\omega$, $y = x'' \sin\omega + y'' \cos\omega$ in the equation $f(x + a, y + b) = 0$. This gives the equation $f(x''\cos\omega - y''\sin\omega + a, x'' \sin\omega + y''\cos\omega + b) = 0$ in the system (x'', y''). Finally we transform geometrically the latter equation to the system (x,y) and get $f(x\cos\omega - y\sin\omega + a, x\sin\omega + y\cos\omega + b) = 0$. The above set of transformations may be represented as follows:

$$f(x,y) = 0 \overset{a}{\sim} f(x'+a, y'+b) = 0 \overset{g}{\sim} f(x+a, y+b) = 0 \overset{a}{\sim} f(x''\cos\omega - y''\sin\omega + a, x''\sin\omega + y''\cos\omega + b) = 0 \overset{g}{\sim} f(x\cos\omega - y\sin\omega + a, x\sin\omega + y\cos\omega + b) = 0.$$

We determine a, b and ω as usually so that the above last to the right equation represents the standard form of the conic $f(x, y) = 0$ in the same system (x, y). Note that in the traditional treatment the standard form equation refers to the system (x'', y'') (Fig.1). We have therefore to construct the conic corresponding to the standard form equation in the system (x, y) and then trace the original $f(x, y) = 0$.

1.1. Suppose that the final conic (standard form) was constructed. In order to obtain the original, it is necessary first to show that the systems (x,y), $(x'y')$, $(x''y'')$ are analytically and geometrically equivalent as the meaning of these terms is defined [1]. The existence of formulae of transformation

1. C. B. Glavas, «The Principle of Geometrical Equivalence and Some of its Consequences to the Theory of Curves», *Proceedings of the Academy of Athens*, 32 (1957).

among these systems establishes their analytical equivalence. On the other hand, given the point M in the system (x,y) (Fig. 1) one can take $x'=O'A'= = OA = x$ and $y' = A'M' = AM = y$, which shows that the systems (x,y) and (x',y') are geometrically equivalent. Also it is easy to show that to the point M (x,y) there corresponds the point M″ (x″,y″) such that $x''=O'A''= = OA = x$ and $y'' = A''M'' = AM = y$.

In the set of transformations of the previous section the systems employed, although the same in different positions, are considered independent. Then the emphasis is placed on their being analytically and geometrically equivalent.

Now suppose that M is a point of a conic in standard form (Fig. 2).

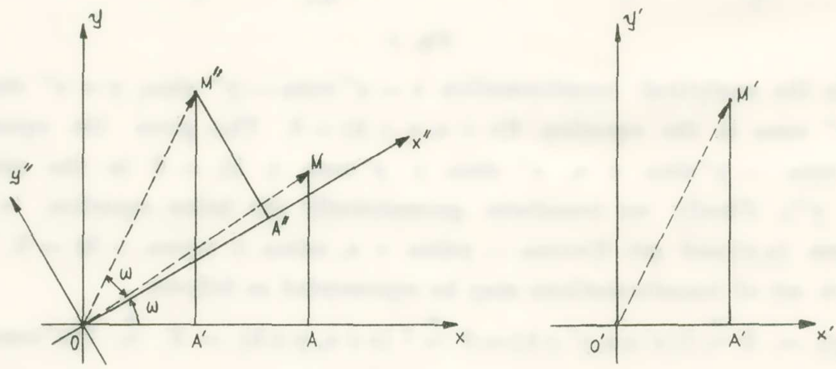

Fig. 2

The corresponding point of the geometrically equivalent to the latter conic is M″, which is found by taking $OA'' = x'' = OA = x$ and $A''M'' = y'' = =AM=y$. Then M″ is taken as a point of the conic $f(x+a, y+b) = 0$ in the system (x,y). For this reason we drop the perpendicular M″A′ on Ox. The next step is to determine the geometrically equivalent of M″ in the system (x′, y′) with origin O′(a,b). We take $O'A' = x' = OA$, and $A'M' = y' = = A'M''$. Therefore the point M′ is the required point of the given conic $f(x,y) = 0$.

From the equal triangles OAM, OA″M″, we conclude that $|\overrightarrow{OM}| = |\overrightarrow{OM''}|$ and that the angle of OM, OM″ is equal to ω. Also from the equal triangles OA′M″, O′A′M′, we see that $\overrightarrow{OM''} = \overrightarrow{O'M'}$. If we denote the conics corresponding to M, M′, M″ by C(M), C(M′), C(M″) respectively, we conclude that C(M′) can be constructed if from O′ (a, b) we draw a vector $\overrightarrow{O'M'}$ of equal magnitude to $\overrightarrow{OM}$ and making with it an angle equal to ω. This may be obtained by drawing from O′ first a vector equal to $\overrightarrow{OM}$ and then turning it counterclockwise by ω.

A first remark is that a one-to-one correspondence may be established between the points of C(M) and C(M′) and that the two conics are equal. This was to be expected because rotations and translations as isometries preserve distance, but the above result is obtained by the application of the method of analytical and geometrical equivalence.

Conversely, given the conic C(M′), one may construct its corresponding «central» C(M″) or C(M) by drawing vectors $\overrightarrow{OM''}$ or $\overrightarrow{OM}$ equal to, or making an angle -ω with, $\overrightarrow{OM'}$ respectively.

Example 1.1. Let the conic $f(x,y) \equiv x^2 + xy + y^2 - 3x - 1 = 0$ be given. If we apply the transformation $x = x' + 2$, $y = y' - 1$ we get $f_1(x', y') \equiv x'^2 + x'y' + y'^2 - 4 = 0$. Its geometrically equivalent in the system (x, y) is $f_1(x, y) \equiv x^2 + xy + y^2 - 4 = 0$.

Then we apply the transformation $x = \frac{\sqrt{2}}{2}(x''-y'')$, $y = \frac{\sqrt{2}}{2}(x''+y'')$,

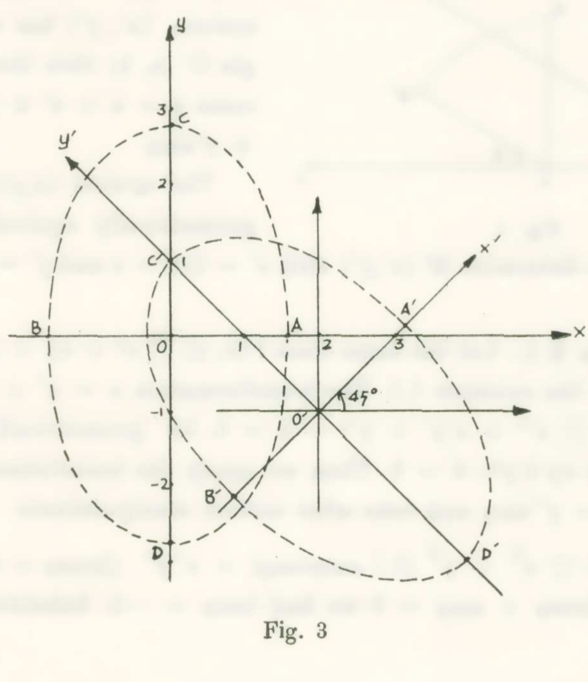

Fig. 3

which gives $f_2(x'', y'') \equiv \frac{x''^2}{8/3} + \frac{y''^2}{8} - 1 = 0$. Its geometrically equivalent in the system (x,y) is $f_2(x,y) \equiv \frac{x^2}{8/3} + \frac{y^2}{8} - 1 = 0$. The above set of transformations may be written as follows:

$$f(x,y) = 0 \overset{a}{\sim} f_1(x',y') = 0 \overset{g}{\sim} f_1(x,y) = 0 \overset{a}{\sim} f_2(x'',y'') = 0 \overset{g}{\sim} f_2(x,y) = 0$$

We first construct the ellipse $f_2(x, y) = 0$ which is represented in Figure 3 by ACBD. We take the point $O'(2, -1)$ and we draw $\overrightarrow{O'A'}$ equal in length to $\overrightarrow{OA}$ and making an angle with it equal to $\omega = 45^o$. We do the same thing for the points C, B, D, etc. constructing finally the required curve $f(x, y) = 0$ which is represented in the figure by A′C′B′D′.

2. We now come to the case of oblique systems. Let the rectangular system (x, y) be given and the oblique (x′, y′) with Ox′ coinciding with Ox and Oy′ making an angle φ with Ox (Fig. 4). The formulae of an analytical transformation between the two systems are well known to be $x = x' + y'\cos\varphi$, $y = y'\sin\varphi$. If the oblique system (x′, y′) has a different origin O′ (a, b) then the formulae become $x = a + x' + y'\cos\varphi$, $y = b + y'\sin\varphi$.

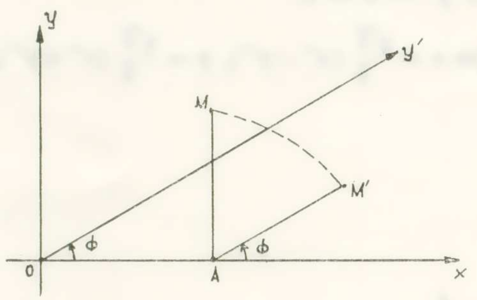

Fig. 4

The system (x,y) and (x′,y′) are geometrically equivalent. For given M(x,y) one can determine M′ (x′,y′) with x′ = OA = x and y′ = AM′=AM=y.

Example 2.1. Let the same conic $f(x, y) \equiv x^2 + xy + y^2 - 3x - 1 = 0$ be given as in the example 1.1. The transformation $x = x' + a$, $y = y' + b$ gives $f_1(x',y') \equiv x'^2 + x'y' + y'^2 - 4 = 0$. Its geometrically equivalent is $f_1(x,y) \equiv x^2 + xy + y^2 - 4 = 0$. Then we apply the transformation $x = x'' + y''\cos\varphi$, $y = y''\sin\varphi$ and take after certain manipulations:

$$f_2(x'',y'') \equiv x''^2 + y''^2 (1 + \sin\varphi\cos\varphi) + x''y'' (2\cos\varphi + \sin\varphi) - 4 = 0$$

Putting $2\cos\varphi + \sin\varphi = 0$ we find $\tan\varphi = -2$. Substituting this value in the equation $f_2(x'',y'') = 0$ we get $x''^2 + \frac{3}{5}y''^2 - 4 = 0$. The geometrically equivalent of the latter equation is $x^2 + \frac{3}{5}y^2 - 4 = 0$. The above set of transformations may be put as follows:

$$f(x,y) = 0 \overset{a}{\sim} f_1(x',y') = 0 \overset{g}{\sim} f_1(x,y) = 0 \overset{a}{\sim} f_2(x'',y'') = 0 \overset{g}{\sim} f_2(x,y) = 0$$

The conic $f_2(x, y) = 0$, i.e. $x^2 + \frac{3}{5}y^2 - 4 = 0$ is in standard form in the original rectangular system (x, y) and may be easily constructed. Let the conic $f_2(x, y) = 0$ be represented by the curve ABA'B' (Fig. 5). Let M be a point of this conic. Its geometrically equivalent is M'', which is found by drawing PM'' parallel to Oy'' and taking PM'' = PM. The point M'' has coordinates x'' = OP'' and P''M'' = y'' in the system (x, y). The geometrically

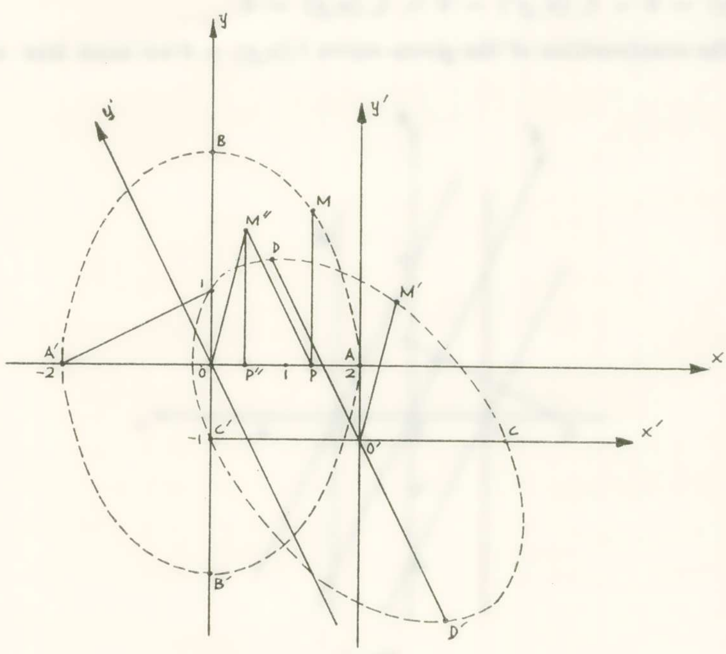

Fig. 5

equivalent of M'' in the system O'x'y', where O' has coordinates (2, — 1), is the point M', which is found by drawing the vector $\overrightarrow{O'M'}$ equal to $\overrightarrow{OM''}$.

It is not difficult to trace the conic f (x, y) = 0 which is represented by

CDC′D′. Note that the latter is the same with the one represented in Figure 3 by A′C′B′D′. Therefore by the two different methods of examples 1.1 and 2.1 one may check the correctness of one and the same curve's tracing. Also we observe that to one and the same conic there correspond two different central ones if the applied two types of transformations are different.

Example 2.2. This example refers to parabola. Let the equation $f(x,y) \equiv 4x^2 + 4xy + y^2 - 4 = 0$ be given. We apply the transformation $x = x' + y'\cos\varphi$, $y = y'\sin\varphi$ and get after the necessary manipulations the equation $f(x',y') \equiv 4x'^2 + (4\cos^2\varphi + \sin^2\varphi + 4\cos\varphi\sin\varphi)\, y'^2 + 4\,(2\cos\varphi + \sin\varphi)\, x'y' - 4 = 0$. We put $2\cos\varphi + \sin\varphi = 0$ from which we get as before $\tan\varphi = -2$. Substituting this value in the equation $f(x',y') = 0$ we take $f(x',y') \equiv 4x'^2 - 4 = 0$ or $x'^2 - 1 = 0$. Its geometrically equivalent curve is $f(x,y) \equiv x^2 - 1 = 0$. Hence we have the set of transformations:

$$f(x,y) = 0 \overset{a}{\sim} f_1(x',y') = 0 \overset{g}{\sim} f_1(x,y) = 0$$

For the construction of the given curve $f(x,y) = 0$ we must first construct

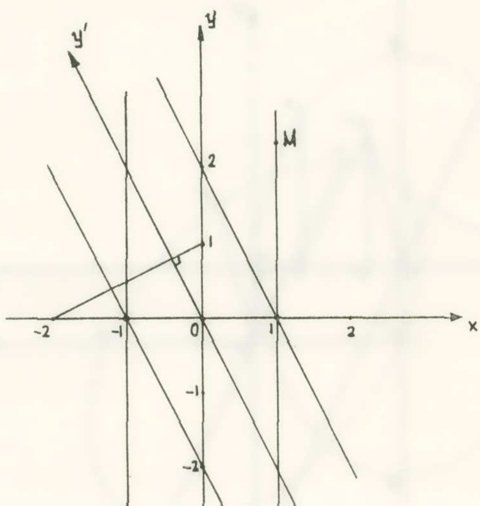

Fig. 6

$f_1(x,y) = 0$ or $x^2 = 1$. This equation represents the pair of lines $x = \pm 1$ (Fig. 6). Its geometrically equivalent curve is $f_1(x'y') = 0$. To construct the latter we must draw Oy' making with Ox an angle φ such that $\tan\varphi = -2$. Now take a point M on the line $x = 1$. The point $(1,0)$ corresponds to itself.

For M we must draw a line through (1,0) parallel to Oy' and take a length on it equal to the ordinate of M. It is clear that the line $x = 1$ is transformed to the line through (1,0) parallel to Oy'. Also the line $x = -1$ is transformed to the line through (—1,0) parallel to Oy'.

Hence the curve $f_1(x',y') = 0$ is a pair of lines through the points (1, 0) and (—1,0) parallel to Oy'. And of course these lines represent the curve $f(x, y) = 0$ which is geometrically the same with $f_1(x',y') = 0$. In this case the conic $f(x,y) = 0$ is a degenerate one.

Note that, since the lines parallel to Oy' through (1,0) and (—1,0) must have an inclination equal to that of Oy', they must pass through the points (0,2) and (0, —2) which is the case if the drawing is very accurate.

3. The above study of conics by the application of analytical and geometrical equivalence has certain further consequences for the theory of these curves. To facilitate the discussion we recall that «central» is a conic whose equation is in standard form in a rectangular system Oxy taken as «basic». By the term «set of transformations» is understood a certain sequence of analytical and geometrical transformations by which a certain conic $f(x,y) = 0$ is transformed to another one $f'(x,y) = 0$ in the same basic system Oxy. By the notation (ω, O') and (φ, O') we mean the type of transformations described respectively in sections 1 and 2 of this paper. With these remarks we proceed to the establishment of certain statements about conics.

3.1. In Section 1 we saw that to the non-central conic $f(x,y) = 0$ there corresponds the central $f_2(x,y) = 0$ under the transformation (ω, O'). The conic $f_2(x,y) = 0$ is unique for the angle ω and the point O' (a,b) are determined uniquely. By the transformation (φ, O') to the same non-central $f(x, y) = 0$ there corresponds a different central $f'_2(x, y) = 0$. Of course if we can find other types of transformations we may determine another central $f''_2(x,y) = 0$ corresponding to the same non-central $f(x,y) = 0$, etc.

Hence we establish the proposition that to a given non-central conic there may correspond more than one (possibly infinite) central conics. The conics which correspond under the transformation (ω, O') are equal, those corresponding under (φ, O') being unequal.

Of course, given a central conic one can geometrically construct the corresponding in each case non-central one. Thus a central conic may be considered as «generator» while the non-centrals as «derived» ones.

3.2. If we reverse the above process, to one and the same central conic there may correspond two non-centrals, one under the transformation (ω, O') and the other under (φ, O'). More clearly, given a central conic $f_2(x,y) = 0$, if we apply the set of transformations of examples 1.1 and 1.2, two non-central conics will be found for each point $O'(a,b)$.

Now let a point M of a central conic be given. In order to find its corresponding points under transformations (ω, O') and (φ, O') we draw MB perpendicular on Ox (Fig. 7). From B we draw $BM' = BM$ making an angle φ with Ox. From O′ we draw $\overrightarrow{O'M'} = \overrightarrow{OM}$ and turn $\overrightarrow{O'M'}$ by an angle ω, thus determining the point M_1 of the first non-central conic. From M′ we draw M′B′ parallel to MB and from $B' \overrightarrow{B'M_2} = \overrightarrow{BM'}$. The point M_2 belongs to the second non-central conic.

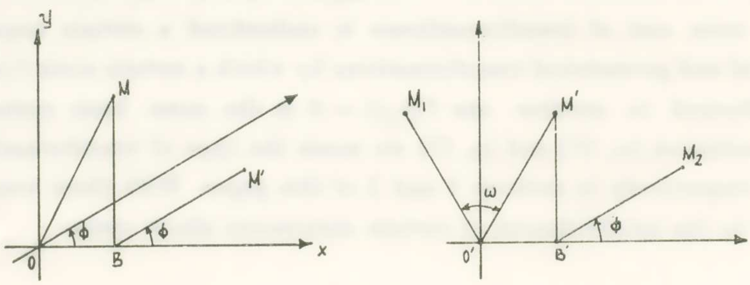

Fig. 7

Under the above conditions, given any one of the three conics, the central and the two corresponding non-centrals, one may determine geometrically the other two. Really, let M_2 be a point of the second non-central conic. We draw M_2B' making an angle φ with Ox axis. From B′ we draw $B'M' = B'M_2$. We turn O′M′ by ω counterclockwise. The point M_1 belongs to the first non-central. If from O we draw $\overrightarrow{OM} = \overrightarrow{O'M'}$, the point M belongs to the central conic, etc.

3.3. Let us consider now the case of transformations of the type (ω, O'). Let a basic system Oxy be given and a central conic C_{oo} (Fig. 8). If from O and all the other points of the plane $O', O'', \ldots$ we draw a set of vectors equal respectively to the set of vectors corresponding to the points of C_{oo}, then, according to the conclusions of section 1, there results a set of conics $C_o = [C_{oo}, C'_{oo}, C''_{oo}, \ldots]$. The conics of C_o are equal to C_{oo}. If similarly to the set of vectors of C_{oo} we draw vectors making correspondingly an angle ω_1 with the vectors of C_{oo} then another set of conics is produced $C_1 = [C_{o1}, C'_{o1}, C''_{o1}, \ldots]$. Thus we produce other sets $C_2, C_3, \ldots$. We call C the set of the sets $C_o, C_1, C_2, \ldots$, i.e. $C = [C_o, C_1, C_2, \ldots]$.

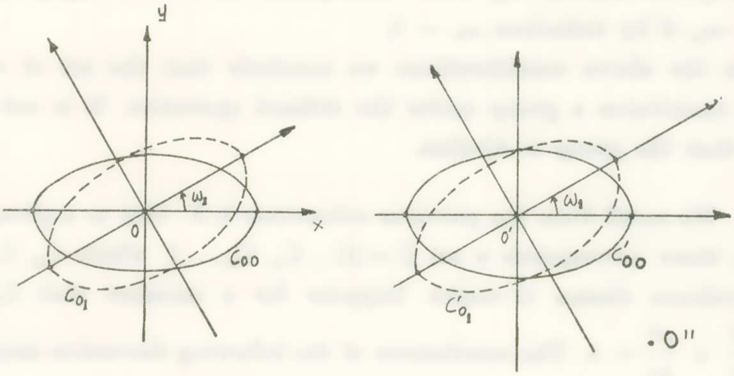

Fig. 8

It is clear now that all conics of the sets of C are equal, but not «similarly placed» in the plane. By definition two conics of the sets of C are equal, if and only if $\omega_\lambda = \omega_\mu$. For example C_{o1} is equal to C'_{o1}, but not to C'_{oo}. This definition of equality of conics leads to an equivalence relation. Really, it is obvious that $C_{o\lambda} = C_{o\mu}$. Also if $C_{o\lambda} = C_{o\mu}$ then $C_{o\mu} = C_{o\lambda}$. For the first equality implies $\omega_\lambda = \omega_\mu$, which can be written $\omega_\mu = \omega_\lambda$. Hence $C_{o\mu} = C_{o\lambda}$. Finally, if $C_{o\lambda} = C_{o\mu}$ and $C_{o\mu} = C_{o\nu}$, then $C_{o\lambda} = C_{o\nu}$. This is easily proved and since the three properties of reflexivity, symmetry, and transitivity hold, we assert the existence of an equivalence relation in the set C. Hence there exists a partition of the conics considered into equivalence classes, which are evidently the sets $C_o, C_1, C_2, \ldots$ of C. Each of these classes defines a conic of the ones considered here. As representative elements for each class we may take the conics $C_{oo}, C_{o1}, C_{o2}, \ldots$.

We define an operation « $*$ » on the set of conics considered by putting:

$$C_{o\lambda} * C_{o\mu} = C_{o(\lambda+\mu)}, \text{ where } \omega_{\lambda+\mu} = \omega_\lambda + \omega_\mu$$

It is evident that our set is closed with respect to the operation defined.

For $C_{o(\lambda+\mu)}$ is a conic of the same set. The operation in question is associative. Take $(C_{o\lambda} * C_{o\mu}) * C_{o\nu}$ and $C_{o\lambda} * (C_{o\mu} * C_{o\nu})$. The conics represented by the two expressions are equal since $(\omega_\lambda + \omega_\mu) + \omega_\nu = \omega_\lambda + (\omega_\mu + \omega_\nu)$. There is an identity element for this operation namely C_{oo} (or any element of the set C_o). Really $C_{o\lambda} * C_{oo} = C_{o\lambda}$. Of course this element is unique since all elements of C_o are equal. Finally, if we designate by $-\omega_\lambda$ the angle $2\pi - \omega_\lambda$ it is clear that to a given element $C_{o\lambda}$ there corresponds an inverse $C_{o(-\lambda)}$ for which $\omega_{(-\lambda)} = -\omega_\lambda$, if by definition $\omega_o = 0$.

From the above considerations we conclude that the set of conics in question constitutes a group under the defined operation. It is not difficult to show that the group is Abelian.

3.4. We recall from the previous subsection 3.3. that to a given central conic C_{oo} there corresponds a set $C = [C\ , C_1, C_2, \ldots]$ where $C_o, C_1, C_2, \ldots$ are equivalence classes of conics. Suppose for a moment that C_{oo} is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The conclusions of the following discussion may be easily adjusted with minor modifications to the other conics.

If we take new values a′, b′ let the new conic be represented by $\overset{(1)}{C}_{oo}$. To this conic there corresponds a set $\overset{(1)}{C} = [\overset{(1)}{C}_o, \overset{(1)}{C}_1, \overset{(1)}{C}_2, \ldots]$, where $\overset{(1)}{C}_o, \overset{(1)}{C}_1, \overset{(1)}{C}_2, \ldots$ are equivalence classes of conics, etc. Take the set $K = [C_{oo}, \overset{(1)}{C}_{oo}, \overset{(2)}{C}_{oo}, \ldots]$. Then it is clear that to each pair (a, b) of real numbers there corresponds a unique element of K and conversely. We have in other words a one to-one correspondence between the set $R \times R$ and the set K. Let us take the set $L = [C, C^{(1)}, C^{(2)}, \ldots]$. We observe that there exists a one-to-one correspondence between the sets K and L. The set K is the set of central-generator conics while the sets of L contain the derived non-central conics. It is clear that the product of the one-to-one mappings $R \times R \longleftrightarrow K$ and $K \longleftrightarrow L$ is the one-to-one mapping $R \times R \longleftrightarrow L$.

3.5. From the previous discussion we see that given a central conic C_{oo}, the ellipse for example $\frac{x^2}{\lambda^2} + \frac{y^2}{\mu^2} = 1$, one can determine analytically and geometrically a family of sets of conics $C = [C_o, C_1, C_2, \ldots]$. The problem now is: (a) to establish the conditions which a conic given by its equation

$f(x,y) \equiv Ax^2 + Bxy + Cy^2 + Dx + Ey + Z = 0$ must satisfy in order to have C_{oo} as its corresponding central and (b) to determine C_{oo}.

We shall examine first the case of a conic (ellipse) whose equation is of the form $f_1(x,y) \equiv Ax^2 + Cy^2 + Dx + Ey + Z = 0$ and which of course is an element of one of the subsets denoted previously by C_o, $\overset{(1)}{C_o}$, $\overset{(2)}{C_o}$, ...

We observe that if $f_1(x,y) = 0$ has C_{oo} or $\frac{x^2}{\lambda^2} + \frac{y^2}{\mu^2} = 1$ as its corresponding central then the type of transformations in this particular case must be:

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\mu^2} = 1 \overset{a}{\sim} \frac{(x'+a)^2}{\lambda^2} + \frac{(y'+b)^2}{\mu^2} = 1 \overset{g}{\sim} \frac{(x+a)^2}{\lambda^2} + \frac{(y+b)^2}{\mu^2} = 1$$

We want the last equation on the right in the above set to be the same with the given conic $f_1(x,y) = 0$. Hence equating the coefficients of equal powers in both equations we get:

$$\frac{1}{\lambda^2} = A, \ \frac{1}{\mu^2} = C, \ \frac{2a}{\lambda^2} = D, \ \frac{2b}{\mu^2} = E, \ \frac{a^2}{\lambda^2} + \frac{b^2}{\mu^2} - 1 = Z$$

Eliminating λ, μ, a, b in the latter five relations we take:

$$\boxed{\frac{D^2}{4A} + \frac{E^2}{4C} - 1 = Z}$$

We have thus established the required relation which must be satisfied by the coefficients of equations $f_1(x,y) = 0$ corresponding to conics which are elements of one of the subsets C_o, $\overset{(1)}{C_o}$, $\overset{(2)}{C_o}$, ... This gives the answer to problem (a). The equation of Coo is evidently $Ax^2 + Cy^2 = 1$. Also it is easy to determine the coordinates (a, b) of the origin of the coordinate system (x′,y′), which are $a = \frac{D}{2A}$, $b = \frac{E}{2C}$.

Example 3.5. Let the equation $x^2 + 2y^2 + x + y - \frac{5}{8} = 0$ be given. It is easy to see that the coefficients $A = 1$, $C = 2$, $D = 1$, $E = 1$, and $Z = -\frac{5}{8}$ satisfy the condition established above. Therefore the corresponding conic (ellipse) has equation $x^2 + 2y^2 = 1$ and the center O′ has coordinates $a = \frac{1}{2}$, $b = \frac{1}{4}$. It is not difficult to verify that the above set of transformations applied to $x^2 + 2y^2 = 1$ produces the given equation $x^2 + 2y^2 + x + y - \frac{5}{8} = 0$. Of course this set of transformations is not of the same type with that of section 1.1 (Example 1.1).

An analogous condition could be established for the two other conics (hyperbola and parabola). Also the above problem can be generalized in case the given conic's equation is $f_2(x,y) \equiv Ax^2 + Bxy + Cy^2 + Dx + Ey + Z = 0$. In that case we must apply the transformation $x = x'\cos\omega - y'\sin\omega + a$, $y = x'\sin\omega + y'\cos\omega + b$. The new condition for the coefficients of $f_2(x,y) = 0$ will have as special case the one established above for $\omega = 0$.

The problem 3.5 may be put in another way. Given a central conic C_{oo} one applies the transformation $x = x' + a$, $y = y' + b$ in the equation of C_{oo}. The new equation represents geometrically the same conic C_{oo} but in the system $O'x'y'$ where O' has coordinates a, b with respect to the basic system Oxy.

The latter equation is transformed geometrically to the system Oxy. The new conic $\overline{C_{oo}}$ will be an image of C_{oo} with its center not coinciding with the point O'. But $\overline{C_{oo}}$ does not coincide necessarily with the arbitrarily taken conic $f_1(x, y) = 0$. In order that the latter happens we must have the established condition satisfied by the coefficients of $f_1(x, y) = 0$. All the conics satisfying that condition are elements of one of the sets denoted in section 3.4. by $C_o, \overset{(1)}{C_o}, \overset{(2)}{C_o}, \ldots$

3.6. The condition of section 3.5 may be interpreted geometrically as follows. Let two rectangular systems Oxy and $O'x'y'$ be given. Let a conic (ellipse) $f_1(x,y) \equiv Ax^2 + Cy^2 + Dx + Ey + Z = 0$ be given. The condition that this conic has as its image under the set of transformations of section 3.5 a central conic is that its center O'' is symmetric to O' with center of symmetry the point O (Fig. 9).

Really, let A'' be a point of $f_1(x,y) = 0$. Its image under the above mentioned set of transformations is the point A. From the triangles formed in Figure 9 it is easy to see that $OA = O''A''$ and that $NM'' = N'M$. Hence it is clear the curve OAM is the image of $O''A''M''$ and is a central curve.

From the above considerations we arrive at certain conclusions. Any curve (ellipse) $f_1(x,y) \equiv Ax^2 + Cy^2 + Dx + Ey + Z = 0$ which satisfies the relation $\frac{D^2}{4A} + \frac{E^2}{2C} - 1 = Z$ has as its image under the set of transformations of section 3.5 the curve $Ax^2 + Cy^2 = 1$. In this case the origin O′ of the auxiliary rectangular system O′x′y′ has coordinates $a = \frac{D}{2A}$, $b = \frac{E}{2C}$.

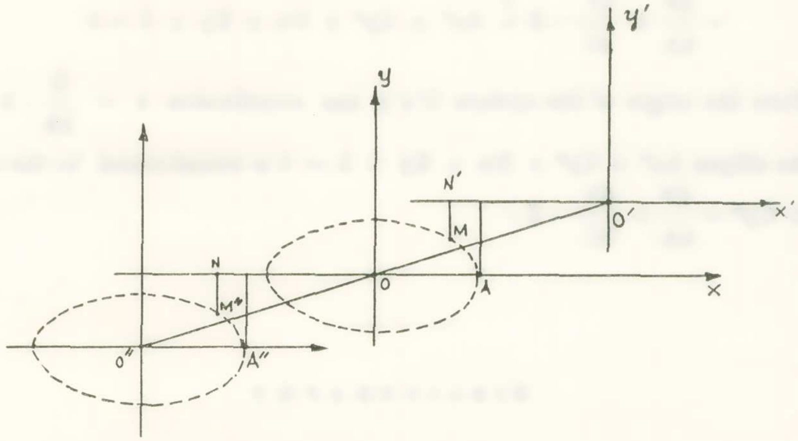

Fig. 9

Really if we apply in $Ax^2 + Cy^2 = 1$ the transformation $x = x' + \frac{D}{2A}$, $y = y' + \frac{E}{2C}$, then we get $A\left(x' + \frac{D}{2A}\right)^2 + C\left(y' + \frac{E}{2C}\right)^2 = 1$ and after the necessary manipulations:

$$Ax'^2 + Cy'^2 + Dx' + Ey' + \left(\frac{D^2}{4A} + \frac{E^2}{4C} - 1\right) = 0$$

But because the coefficients of $f_1(x,y) = 0$ satisfy the well known relation we finally arrive at the curve $Ax^2 + Cy^2 + Dx + Ey + Z = 0$. And this proves our assertion.

All curves (ellipses) $f_1(x,y) = 0$ which have the same A and C and differ in the other coefficients but satisfy our condition have as image the same central, namely $Ax^2 + Cy^2 = 1$. In order to construct any one of them we determine tbe point O″ with coordinates $a'' = -a = -\frac{D}{2A}$, $b'' = -b = -\frac{E}{2C}$. With center O″ and axes parallel to Ox, Oy we construct the ellipse

$Ax^2 + Cy^2 = 1$ directly without using the rest of the constructions of Figure 9.

The above discussion may be easily extended to the other conics and generalized for any conic $Ax^2 + Bxy + Cy^2 + Dx + Ey + Z = 0$.

Finally a topic for further investigation is the transformation:

$$Ax^2 + Cy^2 = \frac{D^2}{4A} + \frac{E^2}{4C} - Z \overset{a}{\sim} A\left(x' + \frac{D}{2A}\right)^2 + C\left(y' + \frac{E}{2C}\right)^2 =$$

$$= \frac{D^2}{4A} + \frac{E^2}{4C} - Z \overset{g}{\sim} Ax^2 + Cy^2 + Dx + Ey + Z = 0$$

Here the origin of the system $O'x'y'$ has coordinates $a = \frac{D}{2A}$, $b = \frac{E}{2C}$ and the ellipse $Ax^2 + Cy^2 + Dx + Ey + Z = 0$ is transformed to the central $Ax^2 + Cy^2 = \frac{D^2}{4A} + \frac{E^2}{4C} - Z$.

BIBLIOGRAPHY

GLAVAS, C.B.: «Contributions to Curve Tracing», *Proceedings of the Academy of Athens*, **39** (1964), p. 27-43.

GLAVAS, C.B.: «The Principle of Geometrical Equivalence and some of its Consequences to the Theory of Curves», *Proceedings of the Academy of Athens*, **32** (1957), p. 122-131.

OAKLEY, C.O.: *Analytic Geometry*. New York, Barnes and Noble Inc., 1960 (1949).

SMITH, P.F. and OTHERS: *New Analytic Geometry*. Revised Edition. Boston, Ginn and Co., 1928.

ΠΕΡΙΛΗΨΙΣ

Ἡ σπουδὴ τῶν κωνικῶν τομῶν ἐκ τῆς ἐξισώσεώς των εἰς ὀρθογωνίους Καρτεσιανὰς συντεταγμένας πραγματοποιεῖται συνήθως διὰ τῆς ἐφαρμογῆς δύο διαδοχικῶν ἀναλυτικῶν μετασχηματισμῶν. Οὕτω ἐν ἀρχῇ γίνεται «παράλληλος μεταφορὰ» τῶν ἀξόνων εἰς νέαν ἀρχὴν καὶ ἐν συνεχείᾳ «στροφὴ» τούτων καθ' ὡρισμένην γωνίαν. Διὰ τῆς μεθόδου ταύτης ἐπιτυγχάνεται τελικῶς ἡ εὕρεσις τῆς καλουμένης προτύπου ἐξισώσεως ἑκάστης κωνικῆς. Ἐπειδὴ πρόκειται περὶ σειρᾶς ἀναλυτικῶν μετασχηματισμῶν, ἡ πρότυπος ἐξίσωσις, ἥτις ἀναφέρεται εἰς ἄλλο, διάφορον τοῦ ἀρχικοῦ,

σύστημα ὀρθογωνίων ἀξόνων, παριστᾷ γεωμετρικῶς μίαν καὶ τὴν αὐτὴν καμπύλην μὲ τὴν ἀρχικὴν ἐξίσωσιν. Ἡ δοθεῖσα ὅθεν κωνικὴ κατασκευάζεται ἐκ τῆς προτύπου ἐξισώσεώς της.

Εἰς τὴν ἀνακοίνωσιν ταύτην γίνεται ἡ σπουδὴ τῶν κωνικῶν διὰ τῆς ἐφαρμογῆς τῶν ἀρχῶν τῆς ἀναλυτικῆς καὶ γεωμετρικῆς ἰσοδυναμίας, ὡς αὗται καθορίζονται εἰς προηγουμένας ἀνακοινώσεις [*Πρακτικὰ τῆς Ἀκαδημίας Ἀθηνῶν*, 32 (1957) καὶ 39 (1964)]. Πρὸς τοῦτο δίδεται ἔμφασις εἰς τὰ χρησιμοποιούμενα ὀρθογώνια ἢ πλαγιογώνια συστήματα συντεταγμένων, τὰ ὁποῖα ἀποδεικνύονται ἀναλυτικῶς καὶ γεωμετρικῶς ἰσοδύναμα καὶ ὡς τοιαῦτα θεωροῦνται ἀνεξάρτητα καὶ ὅταν ἀκόμη πρόκειται περὶ τοῦ ἰδίου συστήματος εἰς διαφόρους θέσεις τοῦ ἐπιπέδου. Ἡ διαφορὰ τῆς μεθόδου ταύτης ἀπὸ τὴν προηγουμένην ἔγκειται κυρίως εἰς τὸ γεγονός, ὅτι ἡ ἀρχικὴ ἐξίσωσις τῆς κωνικῆς μετατρέπεται εἰς ἄλλην προτύπου μορφῆς, ἡ ὁποία ὅμως ἀναφέρεται εἰς τὸ ἴδιον σύστημα συντεταγμένων μὲ τὴν δοθεῖσαν. Ἐκ τῆς κωνικῆς, ποὺ παριστᾷ ἡ πρότυπος ἐξίσωσις, κατασκευάζεται βάσει κανόνος ἡ ἀρχικὴ καμπύλη.

Ἐκ τῆς ὡς ἄνω σπουδῆς τῶν κωνικῶν προκύπτουν ὡρισμέναι διαπιστώσεις. Δύο διάφοροι κωνικαὶ μὲ κέντρον τὴν ἀρχὴν καὶ ἄξονας ἐπὶ τῶν ἀξόνων τῶν συντεταγμένων (κεντρικαὶ) δυνατὸν νὰ ἔχουν, ὑπὸ διαφόρους μορφὰς μετασχηματισμῶν, ὡς εἰκόνα τὴν αὐτὴν μὴ κεντρικὴν κωνικὴν καὶ ἀντιστρόφως. Τὸ σύνολον τῶν κωνικῶν τοῦ ἐπιπέδου βάσει καταλλήλου ὁρισμοῦ ἰσότητος μεταξὺ τούτων κατανέμεται εἰς κλάσεις ἰσοδυναμίας. Ἀκόμη δεικνύεται, ὅτι τὸ ἐν λόγῳ σύνολον ἀποτελεῖ ὁμάδα ὑπὸ ὡρισμένον νόμον συνθέσεως. Ἐν συνεχείᾳ τίθεται τὸ πρόβλημα, ἄν, δοθείσης μιᾶς κωνικῆς τοῦ ἐπιπέδου διὰ τῆς ἐξισώσεώς της καὶ ὡρισμένου τύπου ἀναλυτικῶν καὶ γεωμετρικῶν μετασχηματισμῶν, εἶναι δυνατὸν νὰ προσδιορισθῇ μία κεντρικὴ κωνικὴ ὡς εἰκών της. Ἡ σχετικὴ ἔρευνα καταλήγει εἰς τὴν διατύπωσιν μιᾶς σχέσεως μεταξὺ τῶν συντελεστῶν τῆς δοθείσης ἐξισώσεως ὡς συνθήκης ἐπιλύσεως τοῦ προβλήματος. Τῆς σχέσεως ταύτης δίδεται καὶ γεωμετρικὴ ἑρμηνεία.

Περαιτέρω δεικνύεται, ὅτι αἱ κωνικαί, τῶν ὁποίων αἱ ἐξισώσεις ἔχουν τοὺς αὐτοὺς συντελεστὰς τῶν ὅρων x^2 καὶ y^2 καὶ ἱκανοποιοῦν τὴν ἐν λόγῳ σχέσιν, ἔχουν τὴν αὐτὴν κεντρικὴν κωνικὴν ὡς εἰκόνα, ἥτις εὐχερῶς δύναται νὰ προσδιορισθῇ. Ἡ κατασκευὴ ἑκάστης τῶν κωνικῶν τούτων ἐπιτυγχάνεται βάσει ἁπλοῦ κανόνος.

Τὰ συμπεράσματα τῆς ὡς ἄνω ἐργασίας δύνανται νὰ γενικευθοῦν εἰς ἄλλας περιπτώσεις. Ἐν τέλει διατυποῦται ἓν γενικώτερον πρόβλημα πρὸς περαιτέρω ἔρευναν.