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ΜΑΘΗΜΑΤΙΚΑ. — **Nonlinear Quasi-Contractions on Metric Spaces**, by *Lj. B. Ćirić* and *J. S. Ume**, διὰ τοῦ Ἀκαδημαϊκοῦ κ. *Νικολάου Ἀρτεμιάδη*.

ABSTRACT

Let (X, d) be a complete metric space, $T : X \to X$ be an orbitally continuous mapping and let $\varphi : [0, +\infty) \to [0, +\infty)$ be a nondecreasing and right continuous function such that $\varphi(t) < t$ for $t > 0$ and $\lim_{t\to\infty} [t - \varphi(t)] = \infty$. Furthermore, suppose that there exist fixed positive integers p and q such that:

$$d(T^p x,\ T^q y) \leqslant max\{\varphi[d(T^r x, T^s y)],\ \varphi[d(T^r x, T^{r'} x)],\ \varphi[d(T^s y, T^{s'} y)] : r, r' \in P;\ s, s' \in Q\}$$

holds for all $x, y \in X$, where $P = \{0, 1, ..., p\}$, $Q = \{0, 1, ..., q\}$. Under these assumptions our main result states that T has a unique fixed point. Furthermore, it is shown that the condition for T to be continuous is unnecessary if $P = \{0, p\}$ or $Q = \{0, q\}$. Our work generalizes corresponding results of the first author, Fisher, Ivanov, Kaminski, Pal and Maiti and several other authors.

Let (X, d) be a complete metric space and T be a selfmapping on X such that

$$d(Tx,\ Ty) \leq \varphi[d(x,\ y)] \tag{1}$$

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for all $x, y \in X$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a function satisfying

$$\varphi(t) < t \text{ for all } t > 0. \tag{2}$$

Banach fixed point pririnciple states that if $\varphi(t)$ is linear, that is, if $\varphi(t) = \lambda \cdot t$ for some $0 \le \lambda < 1$, then T has a unique fixed point. Many authors tried to replace the condition of linearity for φ by more general conditions ([1], [2], [6], [7], [10], [11], [12], [13], [15], [16]). Boyd and Wong proved the following result.

THEOREM 1. (Boyd and Wong [1]). *Let $T : X \to X$ be a mapping satisfying (1), where φ satisfies (2) and $\varphi(t)$ is upper semi-continuous from the right. Then T has a unique fixed point in X and for each $x \in X$, $\{T^n x\}$ converges to a unique fixed point in X.*

Similarly as Banach fixed point theorem, the theorem of Boyd-Wong also have applications (e.f. [2]). So the study of mappings satisfying a contractive condition more general than (1) plays an important role in fixed point theory.

The purpose of this paper is to consider mappings which satisfy a significantly weakened contractive condition in the Boyd-Wong's theorem, adding the assumptions for $\varphi(t)$ to be nondecreasing and to satisfy the condition

$$\lim_{t \to \infty} [t - \varphi(t)] = +\infty.$$

Our main result is the following.

THEOREM 2. *Let (X, d) be a complete metric space, $T : X - X$ be an orbitally continuous mapping and let $\varphi : [0, +\infty) \to [0, +\infty)$ be a right continuous function which satisfies (2). Suppose that there exist fixed poritive integers p and q such that:*

$$d(T^p x, T^q y) \le \max \{\varphi[d(T^r x, T^s y)], \varphi[d(T^r x, T^{r'} x)], \varphi[d(T^s y, T^{s'} y)] : \\ r, r' \in P; s, s' \in Q\} \tag{3}$$

hold for all $x, y \in X$, where $P = \{0, 1, \dots, p\}$, $Q = \{0, 1, \dots, q\}$. If $\varphi(t)$ in addition satisfies the following conditions:

$$\varphi(\mathrm{t}) \text{ is a nondecreasing function,} \tag{4}$$

$$\lim_{t \to \infty} [\mathrm{t} - \varphi(\mathrm{t})] = +\infty, \tag{5}$$

then T has a unique fixed point, say $u \in X$ and $\lim_{n\to\infty} T^n x = u$ for each $x \in X$. Further, if $P = \{0, p\}$ or $Q = \{0, q\}$, then the assumption that T is orbitally continuous is unnecessary. Furthermore, if $P = \{0\}$ and $Q = \{0\}$ then the conditions (4) and (5) are also unnecessary.

PROOF. First we shall show that for any given $x \in X$, the orbit $\{T^n x\}_{n=0}^{\infty}$ is bounded.

Let n be any fixed positive integer. Then there are positive integers $i = i(n)$ and $j = j(n)$ such that

$$\delta_n(x) = \text{diam}\ \{x, Tx, T^2x, \ldots, T^n x\} = d(T^i x, T^j x). \qquad (6)$$

Observe that by monotonicity of φ we can write the inequality (3) in the following form:

$$d(T^p x, T^q y) \leq \varphi[\max\{d(T^r x, T^s y), d(T^r x, T^{r'} x), d(T^s y, T^{s'} y) : r, r' \in P \text{ and } s, s' \in Q\}]. \qquad (7)$$

Without loss of generality we may suppose that $p \leq q$, $i < j$ and that $n > q$ and $j > q$, since the case $j \leq q$ is trivial.

First assume that $i \geq p$. Then from (7) and by the monotonicity of φ we have $\delta_n(x) = d(T^p T^{i-p} x, T^q T^{j-q} x) \leq \varphi$ [diam $\{T^{i-p}x, \ldots, T^i x, T^{j-q} x, \ldots, T^j x\}$] $\leq \varphi[\delta^n(x)]$ which is in contradiction with (2) for $\delta_n(x) > 0$. Therefore, $i < p$. Since by the triangle inequality,

$$d(T^i x, T^j x) \leq d(T^i x, T^p x) + d(T^p x, T^j x)$$

and as $j > q$, using (6) we have

$$\delta_n(x) \leq d(T^i x, T^p x) + d(T^p x, T^q T^{j-q} x). \qquad (8)$$

Since from (7)

$$d(T^p x, T^q T^{j-q} x) \leq \varphi[\text{diam}\{x, Tx, \ldots, T^p x, T^{j-q} x, \ldots, T^j x\}],$$

and as φ is monotonous, we have

$$d(T^p x, T^q T^{j-q} x) \leq \varphi[\delta^n(x)].$$

Now, by (8) we obtain

$$\delta_n(x) - \varphi[\delta_n(x)] \leq d(T^i x,\ T^p x) \leq K, \tag{9}$$

where $K = \max\{d(T^m x,\ T^p x):\ m \in P\}$. Since the sequence $\{\delta_n(x)\}$ is nondecreasing, the lim $\delta_n(x)$ exists. Suppose that lim $\delta_n(x) = +\infty$. Then (5) implies that the lefthand side of (9) becomes unbounded when n tends to infinity, and that the right-hand side is bounded, a contradiction. Therefore, $\lim_{n\to\infty} \delta_n(x) = \delta(x) < +\infty$, that is

$$\delta(x) = \text{diam}\ \{x,\ Tx,\ T^2x, \ldots,\ T^n x, \ldots\} < +\infty. \tag{10}$$

Now we show that $\{T^n x\}$ is a Cauchy sequence .Set

$$a_n = \delta(T^n x) = \text{diam}[\{T^n x,\ T^{n+1}x, \ldots,\}] \tag{11}$$

($n = 0,\ 1,\ 2,\ldots$). By (10), $\{a_n\}$ is a sequence of finite nonnegative numbers. Since $a_n \geq a_{n+1}$, it follows that $\{a_n\}$ converges to some $a \geq 0$. We shall prove that $a = 0$. Let n be arbitrary and let r, s be any positive integers such that $r,\ s \geq n + p + q$. Then from (7),

$$\begin{aligned} d(T^r x,\ T^s y) &= d(T^p T^{r-p} x,\ T^q T^{s-q} x) \\ &\leq \varphi[\text{diam}\{T^{r-p}x,\ T^{r-p+1}x, \ldots,\ T^s y\}] \leq \varphi[\delta(T^{r-p}x)] \leq \varphi[\delta(T^n x)] \end{aligned}$$

and hence

$$\sup\{d(T^r x,\ T^s y) : r,\ s \geq n + q\} \leq \varphi[\delta(T^n x)],$$

that is, $a_{n+q} \leq \varphi(a_n)$. Hence, as $a \leq a_{n+q}$, we have $a \leq \varphi(a_n)$. Suppose that $a > 0$. Then by the right continuity of φ we have

$$a \leq \lim_{a_n \to a+} \varphi(a_n) = \varphi(a) < a,$$

a contradiction. Therefore, $a = 0$. Thus, we have proved that

$$\lim_{n\to\infty} \text{diam}\ [\{T^n x,\ T^{n+1}x, \ldots,\}] = 0$$

and consequently the sequence $\{T^n x\}$ is a Cauchy sequence. By the completeness of X there is some $u \in X$ such that

$$u = \lim_{n \to \infty} x_n \tag{12}$$

where $x_n = T^n x$. If T is orbitally continuous (c.f. [5]), then

$$Tu = \lim Tx_n = \lim x_{n+1} = u,$$

which means that u is a fixed point of T.

The uniqueness of a fixed point of T follows from (3) and (2).

Suppose now that $P = \{0, p\}$ or $Q = \{0, q\}$ and that T may be discontinuous. Without loss of generality we may consider only the case $P = \{0, p\}$ and $Q = \{0, 1, 2, ..., q\}$. Then from (7) with $x = u$, where u is defined by (12), and $y = T^{n-q}x$ we have

$$d(T^p u, T^n x) = d(T^p u, T^q T^{n-q} x) \le \varphi[\mathrm{diam}\{u, T^p u, T^{n-q}x, \ldots, T^n x\}]. \tag{13}$$

Suppose that $d(T^p u, Tu) > 0$. Since $\lim_{n\to\infty} T^n x = u$, for n large enough we have:

$$\mathrm{diam}\{u, T^p_u, T^{n-q}_x, ..., T^n_x\} \le d(u, T^p u) + \max\{d(u, T^{n-i}x) : i = 0, 1, \ldots, q].$$

By monotonicity of φ, from (11) we get

$$d(T^p u, T^n x) \le \varphi[d(u, T^p u) + \max\{d(u, T^{n-i}x)\ i = 0,1, \ldots, q\}], \tag{14}$$

Set $c_n = d(u, T^p u) + \max\{d(u, T^{n-i}x) : i = 0, 1, \ldots, q]$. Then $c_n \to d(u, T^p u)$ when $n \to \infty$. Now from (14), we have

$$d(T^p u, u) = \lim_{u \to \infty} d(T^p u, T^n x) \le \lim_{n \to \infty} \varphi(c_n) = \lim_{c_n \to d(u, T^p u)+} \varphi(c_n) = \varphi[d(u, T^p u)] < d(u, T^p u),$$

a contradiction. Therefore, $T^p u = u$. Since $\{T^n u\}$ must be a Cauchy sequence, it follows that $Tu = u$.

Let now $P = \{0\}$ and $Q = \{0\}$. Then (3) becomes the Boyd-Wong's contractive condition, and so Theorem 2 becomes Theorem 1.

REMARK 1. If a space X is bounded, then the condition (5) for φ is unnecessary.

REMARK 2. The following simple example shows that if in Theorem 2, $p \geq 2$ and T is not orbitally continuous, then T may have no fixed point. Let $X = [0, 1]$ and let $T : X \to X$ be defined as follows: $T(0) = 1$; $T(x) = \frac{x}{2}$, if $x \neq 0$. Then it is easy to see that T satisfies (3) with $p = q = 2$ and $\varphi(t) = \frac{t}{2}$, but T has not a fixed point.

REMARK 3. If $T : X \to X$ in Theorem 1 satisfies (3) with $p \neq q$, then it also satisfies (3) with $p = q$.

PROOF. Suppose that $p < q$. Then we have

$$d(T^q x,\ T^q y) = d(T^p\ T^{q-p} x,\ T^q y) = d(T^p z,\ T^q y),$$

where for notational simplicity we set $z = T^{q-p}x$. Since

$$\begin{aligned} &\max\ \{\varphi[d(T^r z,\ T^s y)],\ \varphi[d(T^r z,\ T^{r'} z)] : r, r' \in P \text{ and } s \in Q\} \\ = &\max\ \{\varphi[d(T^r x,\ T^s y)],\ \varphi[d(T^r x,\ T^{r'} x)] : q - p \leq r, r' \leq q \text{ and } s \in Q\} \\ \leq &\max\ \{\varphi[d(T^r x,\ T^s y)],\ \varphi[d(T^r x,\ T^{r'} x)] : r, r' \in Q \text{ and } s \in Q\}, \end{aligned}$$

from (3) we get

$$d(T^q x,\ T^q y) \leq \max\ \{\varphi[d(T^r x,\ T^s y)],\ \varphi[d(T^r x,\ T^{r'} x)],\ \varphi[d(T^s y,\ T^{s'} y)] : \\ r,\ r' \in Q \text{ and } s,\ s' \in Q\}.$$

REMARK 4. Independently, Matkowski ([10], [11]) and Ivanov [7] introduced the condition (5): $\lim_{n \to \infty} [t - \varphi(t)] = +\infty$. In [6] the first author showed that each of the hypothesis (4) and (5) in Theorem 2 is essential, even for $p = q = 1$.

As a consequence of Theorem 2 it is easy to obtain the following result:

THEOREM 3. *Let (X, d) be a complete metric space and let $\varphi : [0, +\infty) \to [0, +\infty)$ be as in Theorem 2. If $T : X \to X$ is continuous and satisfies the following condition:*

$$d(T^2 x,\ T^2 y) \leq a\varphi[d(x,\ y)] + b\varphi[d(Tx,\ Ty)] + c\varphi[d(x,\ Tx)] + h\varphi[d(y,\ Tx)]$$

where a, b, c and h are nonnegative reals such that $a + b + c + h \leq 1$, then T has a unique fixed point.

REMARK 5. Taking in Theorem 2: $\varphi(t) = \lambda \cdot t$ with $0 \leq \lambda < 1$ we obtain the fixed point theorem of *B.* Fisher [8] and A. Kaminski [9]. For $p = 1$ (and q arbitrary) we obtain the theorem of Pal and Maiti [14].

REMARK 6. Taking in Theorem 2: $\varphi(t) = \lambda \cdot t$ with $0 \leq \lambda < 1$ and $q = p$, we obtain the fixed point therem, given in [4]. For $p = q = 1$ we obtain the theorem in [3].

REMARK 7. The following example shows that Theorem 2 is not only a formal generalization of the corresponding theorems of the first author [3], [4], Ivanov [7], Fisher [8], Kaminski [9] and Pal and Maiti [14].

EXAMPLE. Let $X = [0, 1]$ be equipped with the Euclidean metric d and let $T : X \rightarrow X$ be a mapping defined as follows:

$$T(x) = x - \frac{1}{2} x^2, \text{ for } x \in X \text{ and } x \neq \frac{2}{3},$$
$$T\left(\frac{2}{3}\right) = 1.$$

Furthermore, let $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ be defined as follows:

$$\begin{aligned} \varphi(t) &= t - \frac{1}{2} t^2, \text{ for } 0 \leq t \leq 1 \\ \varphi(t) &= 1, \quad \text{for } t > 1. \end{aligned}$$

Then it is clear that φ satisfies the conditions (2), (7) and (8).

Now we shall show that T satisfies (3) with $p = q = 3$. If $x, y \in X$ and $x \neq \frac{2}{3}$, $y \neq \frac{2}{3}$ then we have

$$\begin{aligned} d(Tx, Ty) &= |x - y - \frac{1}{2}(x^2 - y^2)| = |x - y||1 - \frac{1}{2}(x + y)| \\ &\leq |x - y|\left(1 - \frac{1}{2}|x - y|\right) = d(x, y) - \frac{1}{2} d^2(x, y), \end{aligned}$$

that is,

$$d(Tx, Ty) \leq \varphi[(d(x, y)].$$

Using (2) and this inequality we get

$$d(T^3x,\ T^3y) \leq \varphi[d(T^2x,\ T^2y)] \leq \varphi^2[d(Tx.\ Ty)] \leq \varphi^3[d(x,\ y)].$$

Since from (4), $\varphi(t) \leq t$ implies $\varphi^2(t) \leq \varphi(t)$ and $\varphi^3(t) \leq \varphi(t)$, we have

$$d(T^3x,\ T^3y) \leq \varphi[d(x,\ y)] \text{ for all } x \neq \frac{2}{3} \text{ and } y \neq \frac{2}{3}.$$

Since $T^3\left(\frac{2}{3}\right) = \frac{3}{8}$, $d[T\left(\frac{2}{3}\right),\ T^2\left(\frac{2}{3}\right)] = \frac{1}{2}$, for any $x \in X$ we have

$$d[T^3\left(\frac{2}{3}\right),\ T^3(x)] = \left|\frac{3}{8} - T^3(x)\right| \leq \frac{3}{8} = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2 = \varphi\left(\frac{1}{2}\right) =$$

$$= \varphi[d[T\left(\frac{2}{3}\right),\ T^2\left(\frac{2}{3}\right)]].$$

Therefore, for all $x,\ y \in X$ we have that T satisfies the following condition:

$$d(T^3x,\ T^3y) \leq \max\ \{\varphi[d(x,\ y)],\ \varphi[d(Tx,\ T^2x)],\ \varphi[d(Ty,\ T^2y)]\}.$$

Since T is orbitally continuous, we can apply our Theorem 2. On the other hand, since $d[T\left(\frac{2}{3}\right),\ T^2(0)] = 1 = \operatorname{diam}(X)$, T does not satisfy the condition (3) with $p = 1$. Also for any $p,\ q \geq 1$, T does not satisfy the condition (3) with $\varphi(t) = \lambda \cdot t$, for any $0 < \lambda < 1$.

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ΠΕΡΙΛΗΨΗ

Μὴ γραμμικὲς ἡμισυσταλτικὲς ἀπεικονίσεις ἐπὶ μετρικῶν χώρων

Ἔστω $(X,\ d)$ ἕνας πλήρης μετρικὸς χῶρος,

$T : X \to X$ μία τροχιακὰ συνεχὴς ἀπεικόνιση, καὶ

$\varphi : [0,\ +\infty) \to [0,\ +\infty)$ μία μὴ φθίνουσα καὶ συνεχὴς ἐκ δεξιῶν συνάρτηση τέτοια ὥστε

$$\varphi(t) < t \text{ γιὰ } t > 0 \text{ καὶ } \lim\, [t - \varphi(t)] = +\infty.$$

Ἐπιπλέον, ὑποθέτομε ὅτι ὑπάρχουν θετικοὶ ἀκέραιοι ἀριθμοὶ p καὶ q, τέτοιοι ὥστε ἡ σχέση

$$d\left(T^p_x,\ T^q_y\right) \le \max\left\{\varphi\left[d\left(T^r_x,\ T^s_y\right)\right],\ \varphi\left[d\left(T^r_x,\ T^{r'}_x\right)\right],\ \varphi\left[d\left(T^s_y,\ T^{s'}_y\right)\right] : r,\ r' \in P\ ;\ s,\ s' \in Q\right\}$$

νὰ ἰσχύει γιὰ ὅλα τὰ $x,\ y \in X$ ὅπου $P = \{0,\ 1,\ \ldots,\ p\}$, $Q = \{0,\ 1,\ \ldots,\ \mathrm{q}\}$

Ὑπὸ τὶς ὡς ἄνω προϋποθέσεις, τὸ κύριο ἀποτέλεσμα τῆς ἐργασίας εἶναι ὅτι ἡ ἀπεικόνιση T ἔχει ἕνα μοναδικὸ σταθερὸ σημεῖο. Ἐπὶ πλέον οἱ συγγραφεῖς ἀποδεικνύουν ὅτι ἡ συνθήκη ὅτι ἡ T πρέπει νὰ εἶναι συνεχὴς δὲν εἶναι ἀναγκαία ἂν

$$P = \{0,\ q\} \text{ ἢ } Q = \{0,\ q\}$$

Ἡ ἐργασία τῶν κ.κ. Ćirić καὶ Ume γενικεύει ἀνάλογα ἀποτελέσματα τῶν Ćirić, Fisher, Ivanov, Kaminski, Pal καὶ Maiti, καθὼς καὶ ἄλλων συγγραφέων.