πτύξεως, βασικὸν καθῆκον τοῦ ὁποίου εἶναι «νὰ συμβάλῃ εἰς τὴν ὑγιᾶ οἰκονομικὴν ἀνάπτυξιν τῶν εὑρισκομένων εἰς τὴν διαδικασίαν τῆς ἀναπτύξεως περιοχῶν, τόσον εἰς τὰς χώρας — μέλη ὅσον καὶ ἀλλαχοῦ, διὰ καταλλήλων μέτρων, περιλαμβανόντων τὴν ἐνθάρρυνσιν τῆς εἰσροῆς κεφαλαίων ἀναπτύξεως πρὸς τὰς ἐν λόγῳ περιοχάς». Τὸ πνεῦμα στενῆς συνεργασίας τὸ ὁποῖον ἐπικρατεῖ μεταξὺ τῶν μελῶν τοῦ Ο.Ο.Σ.Α. καὶ ἡ ὑπ' αὐτῶν ἀναγνώρισις τῆς σημασίας τῆς παροχῆς βοηθείας πρὸς τὰς ὑπαναπτύκτους χώρας κατὰ τὰς προσπαθείας των πρὸς ἐπίτευξιν ὑψηλοτέρων βιοτικῶν ἐπιπέδων, μεγαλυτέρας ἐλευθερίας καὶ καλυτέρου τρόπου ζωῆς διὰ τοὺς λαούς των, προσφέρουν τὸ κατάλληλον κλῖμα διὰ τὴν ἐπεξεργασίαν τῶν λεπτομερειῶν καὶ τῆς διαδικασίας πρὸς ἀποτελεσματικὴν ἐφαρμογὴν τοῦ προτεινομένου συστήματος.

Εἶμαι πεπεισμένος ὅτι ὑπὸ τὴν αἰγίδα τοῦ Ὀργανισμοῦ Οἰκονομικῆς Συνεργασίας καὶ Ἀναπτύξεως, ὁ μηχανισμὸς τῶν Ὀργανισμῶν Βιομηχανικῆς Ἀναπτύξεως εἶναι δυνατὸν νὰ καταστῇ ταχέως πραγματικότης καὶ νὰ συμβάλῃ ἀποφασιστικῶς εἰς τὴν πρόοδον τῶν ἐλευθέρων ὑπὸ ἀνάπτυξιν οἰκονομιῶν.

ΑΝΑΚΟΙΝΩΣΕΙΣ ΜΗ ΜΕΛΩΝ

ΓΕΩΜΕΤΡΙΑ.—On pseudoanalytic mappings, *by Nic. Petridis**. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Κωνστ. Παπαϊωάννου.

INTRODUCTION.

In the present note we extend the notion of quasiconformal functions to pseudoanalytic mappings of a Riemann surface R into a complex analytic manifold M_N of (complex) dimension $N > 1$. The main problem of this extension is the intrinsicness of the definition, that is, the invariance of the definition under acceptable coordinate transformations. This problem is answered by the fundamental theorem 2. which serves also as an existence theorem.

Immediate consequences of theorem 2 are 1) A pseudoanalytic mapping accepts uniformization, fundamental property of quasiconformal functions [9], 2) The dilatation of a pseudoanalytic mapping which in some sense measures its deviation from the analytic ones is bounded. In fact properties 1) and 2) are characteristic and are consequences of the sole requirement of the intrinsicness of the definition. As a first application of

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the notion of pseudoonalytic mappings we consider the case in which M_N is the complex projective space P_N. This extends the notion of meromorphic curves [1], [11] to the pseudomeromorphic ones. We apply a purely geometric technique initiated by L. Ahlfors [1] and H. Weyl [11] and developed by S. S Chern [3] in his lectures on the theory of meromorphic curves at the University of Chicago (1959).

The introduction of a uniformization parameter plays an instrumental role for the definition of osculating spaces and associated curves which however turn out to be intrinsic notions independent of any particular parameter. We subsequently prove the first and second main theorems for pseudomeromorphic curves.

A method of procedure, in the case of meromorphic curves, is to exhaust R by a sequence of compact polyhedra D_ε with boundaries and consider the mappings $f: D_\varepsilon \rightarrow P_N$. We determine the polyhedra D_ε through a uniformization parameter; in other words, we let the mapping f determine the «form» of D. This innovation, without substracting anything essential from the results, makes the results much more clear and simple. In the case $N=1$ our pseudomeromorphic curve coincides with the notion of pseudomeromorphic functions as defined and examined by G. Hälstrom [1]. However, the convenient choice of D, as said above, makes the results of the present paper much simpler and more directly connected with the corresponding results of the theory of meromorphic curves. The technique applied in this note makes possible an extension of Picard's theorem for meromorphic functions, to the pseudomeromorphic curves, as it will be shown in a future note.

I. PSEUDOANALYTIC MAPPINGS

Definition 1: A mapping $f=f(z)$ of the complex plane $D(z=x+iy)$ into the complex plane $f=u+iv$ is said to be quasiconformal if i) it is C^1 mapping ii) the Jacobian $J(z)$ of the mapping is positive everywhere with the exception of a countable (at most) set of points which, however, are isolated in D iii) if $df=pdz+qd\bar{z}$, where $\bar{z}$ denotes the conjugate of z, the ratio

$$k(z)=\frac{|p|+|q|}{|p|-|q|}$$

called dilatation of f(z) at z, is bouned over D.

If we put $f_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$, $f_{\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$ we see immediately that $p = f_z$, $q = f_{\bar{z}}$.

A quasiconformal function is an interior mapping in the sense of Stoilow [9]. Using Stoilow's uniformization of an interior mapping [10], we construct the «graph» of the quasiconformal function. This is a complex manifold which is topologically equivalent to the domain of definition of the mapping and carries on a complex structure equivalent to the complex structure of the image of the mapping. In particular, this graph leads to the following uniformization theorem, implied by works of Kakutani [8] and Teichmueller.

Theorem 1: If $f(z)$ is a quasiconformal function defined on the once punctured disc $D = \{z/0 < |z| < 1\}$, there is a homeomorphism $\zeta = \varphi(z)$ of the unit disc $\{z/|z| < 1\}$ such that i) $\varphi(0) = 0$, ii) $\varphi(z)$ is quasiconformal at least in D iii) the function $F(\zeta)$, defined by $F(\zeta) = f\varphi^{-1}(\zeta)$ is holomorphic in $D' = \{\zeta/0 < |\zeta| < 1\}$; the point $\zeta = 0$ is a pole, an isolated essential singularity or a regular point of $F(\zeta)$ if and only if $z = 0$ is a pole, an isolated essential singularity of $f(z)$ or $\lim f(z)$, $z \to 0$, exists and is finite, respectively.

By this theorem we are suggested the following definition.

Definion 2: Let $f(z)$ be a quasiconformal function defined in a neighbourhood of the point $z = 0$, with this point deleted; let, moreover, $\zeta = \varphi(z)$ be a homeomorphism of the disc $d = \{\zeta/|\zeta| < 1\}$ such that it is quasiconformal in $\{\zeta/o < |\zeta| < 1\}$ and $\varphi(0) = 0$. If the function $F(\zeta)$ defined by $F(\zeta) = f\varphi^{-1}(\zeta)$, is analytic in $\{\zeta/o < |\zeta| < 1\}$ the disc d will be called uniformi zation disc of $f(z)$ and ζ will be called uniformization parameter of $f(z)$ in a neighbourhood of $z = 0$.

Of course, a uniformization parameter as defined above is not uniquely determined. However, it is a matter of some computations to show that if w and v are two uniformization parameters of the quasiconformal function $f(z)$, defined by $w = \varphi(z)$, $v = g(z)$, where $\varphi(w)$, $g(v)$ are quasiconformal homeomorphisms, then $w = \varphi^{-1}g(v)$ is a one-one conformal correspondence between w and v. Now we continue to the definition of pseudo-analytic mappings.

Definition 3: A preudoanalytic mapping $f: R \to M_N$, of a Riemann surface R into a complex (analytic) manifold M_N of dimension N, is a mapping which is locally determined by expresisng the coordinates of the image point as quasiconformal functions of the coordinate of the original point.

Remark: Clearly, the analytic mappings of a Riemann surface into a complex manifold, as well as the quasiconformal functions, are particular cases of pseudoanalytic mappings.

The crucial part of Definition 3 is that the quasiconformal functions which define locally a pseudoanalytic mapping must keep their quasiconformal character under any admissible change of coordinates, either in R or in M_N. It is however a fact that a holomorphic function of quasiconformal functions, taken as a composite function may not be quasiconformal. For example, the functions $f_1(z)=k\bar{z}+z$, $f_2(z)=(1-k)\bar{z}+z$, $k<1$, are quasiconformal, but $F(z)=f_1(z)+f_2(z)$ is not.

In other words, there is a question of whether definition 3 defines anything else beyond the analytic mappings for $N>1$. The answer is given by theorem 2 which gives the interconnection of the coordinate functions of the mapping, so that a pseudoanalytic mapping to be defined. As a necessary and sufficient condition it is the best possible.

Theorem 2: Necessary and sufficient conditions for the mapping $f: R \rightarrow M_N$, expressed locally by the functions $f_1(z), f_2(z), \dots, f_N(z)$, to be pseudoanalytic is that these coordinate functions be C^1 functions with isolated critical points and satisfy the same Beltrami equation

$$\frac{\partial g}{\partial \bar{z}} = \mu \frac{\partial g}{\partial z} \tag{1}$$

where $\mu(z)$ is a measurable function satisfying $|\mu(z)| \leq k < 1$

Proof: The proof is based on theorem 4B [2], characterizing the solutions of a Beltrami equation. Suppose that the coordinate functions $f_1(z)$, $f_2(z), \dots, f_N(z)$ satisfy equation (1). Let w be a uniformization parameter of $f_i(z)$ defined $w=\varphi(z)$, around z_0. It is easy to see that $\varphi(z)$ satisfies (1) and, moreover, it is a homeomorphism. So, according to theorem 4B [2], w is a uniformization parameter of $f_j(z)$ for $j=1,2,\dots,N$. The fact that all $f_i(z)$ $(i=1,2,\dots,N)$ accept a common uniformization parameter is sufsicient to keep the quasiconformal character of the coordinate functions invariant under holomorphic transformations. For the necessity of the condition, assume that $\left(\frac{\partial f_1}{\partial \bar{z}}\right)_0 \left(\frac{\partial f_2}{\partial z}\right)_0 - \left(\frac{\partial f_1}{\partial z}\right)_0 \left(\frac{\partial f_2}{\partial \bar{z}}\right)_0 \neq 0$ where $(\ )_0$ denotes evaluation at z_0.

Let a, b constants such that $b\left(\frac{\partial f_2}{\partial z}\right)_0 = a\left(\frac{\partial f_1}{\partial z}\right)_0$

Then the function $F_1(z) = af_1(z) - bf_2(z)$ is clearly not quasiconformal.

II. PSEUDOMEROMORPHIC CURVES

The theory of meromorphic curves was presented by H. and J. Weyl [11] as a direct attempt to generalize Nevanlinna's theory of a meromorphic function. In this chapter we define the notion of pseudomeromorphic curves and we extend the first two main theorems of Weyl-Ahlfors' theory [1], [11] of meromorphic curves to the pseudomeromorphic ones.

Defininion 4: A pseudomeromorphic curve (f, R) is defined to be a pseudoanalytic mapping $f: R \rightarrow P_N$, where R is a Riemann surface (=one dimensional complex analytic manifold) and P_N is the N-dimensional complex projective space.

Remark: It is easy to show that a pseudomeromorphic function in the sense of G. Halstrom [7], is a pseudomeromorphic curve for $N=1$, in the sense of the definition above. So, the results obtained here will concern pseudomeromorphic functions as a particular case.

Associated Curves: Let g(p) be a quasiconformal function and w a uniformization parameter defined by $w=\varphi(p)$ in some neighbourhood of p. If $G(w) = g\,\varphi^{-1}(w)$ we define the operator D_w on g(p) as follows:

$$D_w\, g(p) = G'\varphi(p)$$

where G' is the complex derivative of G(w). It is easy to see that $D_w g(p)$ is a quasiconformal function accepting common uniformization parameters with g(p) and, consequently, having the same dilatation as g(p). Successive applications of D_w define D_w^j for $j=1,2,\ldots$ Consider now the pseudomeromorphic curve (f, R); we put $D_w^j f(p) = [D_w^j f_0(p), D_w^j f_1(p), \ldots, D_w^j f_N(p)]$ and we consider the k-dimensional space

$$f(p) \wedge D_w f(p) \wedge \ldots \wedge D_w^k f(p)$$

which will be called the k-osculating space of the curve at p. The locus of the k-osculating spaces is a curve in the Grassman manifold G(N, k) of the k-dimensional linear subspaces of P_N, and will be called the k-associated curve. The k-associated curve does not depend on the particular uniformization parameter used for its definition, as it is easy to show; it is intrinsically determined by the curve. The homogeneous coordinates of a point of the k associated curve are of the form

$$\pm D_w^{j_1} f_{i_1}(p)\, D_w^{j_2} f_{i_2}(p) \dots D_w^{j_{k+1}} f_{i_{k+1}}(p)$$

where $(i_1, i_2, \dots, i_{k+1})$ is a subset of $(0, 1, 2, \dots, N)$ and $j_1, j_2, \dots, j_{k+1}$ is a permulation of $0, 1, 2, \dots, k$. It becomes clear that an associated curve is itself a pseudomeromorphic curve accepting common uniformization parameters as the curve (f, R) and having the same dilatation with (f, R). A pseudomeromorphic curve (f, R) will be called non-degenerated if it is not contained in a P_{N-1}, i.e. if

$$f \wedge D_w f \wedge \dots \wedge D_w^N f \not\equiv 0$$

In the sequel we shall consider only non-degenerated pseudomeromorphic curves.

Some preliminary notions: Let V_{N+1} be a complex vector space of dimension $N+1$, equipped with a scalar product defined by $\langle Z, W \rangle = \sum_{i=0}^{N} z_i \bar{w}_i$, where $Z = (z_0, z_1, \dots, z_N)$, $W = (w_0, w_1, \dots, w_N)$ are any two vectors of V_{N+1}. The complex projective space P_N, is the orbit space of $V_{N+1} - 0$ under the group $Z \to tZ$, where t is any complex number $\neq 0$. The differential form

(1) $$ds^2 = \frac{\langle Z, Z\rangle \langle dZ, dZ\rangle - \langle Z, dZ\rangle \langle dZ, Z\rangle}{\langle Z, Z\rangle^2}$$

is invariant under this group and defines an Hermitian metric in P_N (5). If $(Z_0, Z_1, \dots, Z_N)$ is a frame of V_{N+1} and define the differential forms $\omega_{\alpha\beta}$ [4] by

(2) $$dZ_\alpha = \sum_{\beta=0}^{N} \omega_{\alpha\beta} Z_\beta$$

the metric (1) takes the form

(3) $$ds^2 = \sum_{i=1}^{N} \omega_{0i} \bar{\omega}_{0i}$$

This Hermitian metric determines an exterior quadratic form, the associated 2-form [5], given by

(4) $$\Omega = \frac{i}{2} \sum_{j=1}^{N} \omega_{0j} \wedge \bar{\omega}_{0j}$$

from which we derive the volume element of P_N [5].

For the Grassman manifold $G(N, p)$, form (1) is written

(5) $$ds_p^2 = \langle X_1, X_1\rangle \langle dX_1, dX_1\rangle - \langle X_1, dX_1\rangle \langle dX_1, X_1\rangle$$

where X_1 is a decomposable element of degree p+1 and length 1 of the Grassman algebra of V_{N+1}. Finally (5) is written (taking in consideration (2))

$$(6) \qquad ds_p^2 = \sum \omega_{ah} \bar{\omega}_{ah}, \quad (0 \leq a \leq p, \; p+1 \leq h \leq N)$$

and the 2-associated form of G (N, p) is given by

$$(7) \qquad \Omega_p = \frac{i}{2} \sum \omega_{ah} \wedge \omega_{ah}, \quad (0 \leq a \leq p, \; p+1 \leq h \leq N)$$

where the $\omega_{\alpha\beta}$'s are the ones defined by (2).

The first Main Theorem: If P^0_{N-p-1} is a fixed linear subspace of P_N of dimension N-p-1, the first main theorem of rank p gives an estimate of how many times the p-associated curve $A_p(z)$, of the pseudomeromorphic curve (f(z), R) meets this linear space. Let H be the subspace of G(N, p) consisting of all linear subspaces P_p of P_N, of dimension p, having a point in common with the fixed subspace P^0_{N-p-1}. If P^0_p is a linear subspace completely orthogonal to P^0_{N-p-1}, X_0 a decomposable (p+1)-vector representing P^0_p (i.e., X_0 perpendicular to P^0_p) and X a decomposable (p+1)-vector representing an element of H, the differential form

$$(8) \qquad \langle dX, X \rangle - \bar{h}(X) dh(X)$$

where $h(X) = \frac{\langle X, X_0 \rangle}{|\langle X, X_0 \rangle|}$, is defined in G (Np) − H.

If $(Z_0, Z_1, \ldots, Z_N)$ is a frame of V_{N+1} and $X_1 = Z_0 \wedge Z_1 \wedge \ldots \wedge Z_p$ exterior differentiation of X_1 and substitution from (2) gives

$$(9) \qquad \langle dX_1, X_1 \rangle = \sum \omega_{\alpha\alpha}, \quad (0 \leq \alpha \leq p)$$

Finally, after exterior differentiation of (8) and considering the fact that $h(X_1) dh(X_1)$ is a closed form, we obtain

$$(10) \qquad d[\langle dX_1, X_1 \rangle - \overline{h(X_1)} dh(X_1)] = 2i\Omega_p$$

If X is any vector ≠ 0 and if we put $X_1 = \frac{X}{|X}$ we have

$$\langle dX_1, X_1 \rangle = \frac{1}{2|X|^2} [\langle dX, X \rangle - \langle X, dX \rangle] = (d' - d'') \log X$$

where d′ and d″ denote differentiation with respect to X and X respectively. Similarly, we have

$$\overline{h(X_1)} dh(X_1) = (d' - d'') \log |\langle X, X_0 \rangle$$

So (10) becomes

$$(11) \qquad d(d' - d'') \log \frac{|X|}{|\langle X, X_0 \rangle|} = 2i\Omega_p$$

Consider now the pseudomeromorphic curve $f: R \to P_N$ where R is a compact Riemann surface bounded by a sectionally smooth curve C. The p associated curve $A_p: R \to G(N, p)$ induces a mapping A_p^* of the differential forms of $G(N, p)$ to differential forms of R. So, applying A_p^* on (11) we get

$$(12) \qquad A_p^* \, d(d' - d'') \log \frac{|X|}{|\langle X, X_o \rangle|} = 2i \, A_p^* \, \Omega_p$$

Here as well as throughout this paper by an ε-neighbourhood of a point $m \in R$ (or an isolated boundary point of R) we shall mean a neighbourhood of m bounded by the simple closed smooth curve C_ε defined by $|\varphi(z)| = \varepsilon$ where $\varphi(z)$ is a quasiconformal homeomorphism defining a uniformization parameter $w = \varphi(z)$ around m. Clearly all ε-neighbourhoods of a point are conformally equivalent. If by $X_o^{\perp}$ we denote a point of R such that $\langle A_p(X_o^{\perp}), X_o \rangle = 0$, assuming that all such points are interior in R, we isolate them by disjoint ε-neighbourhoods so that all C_ε's and C are also disjoint. If R_ε is the complement in R of the union of all ε-neighbourhoods of the points $X_o^{\perp}$, applying Stokes theorem to (12) and then taking limits as $\varepsilon \to 0$, we obtain.

$$(13) \qquad \frac{1}{2\pi i} \int_C A_p^* \, (d' - d'') \log \frac{|X|}{|\langle X, X_o \rangle|} - \frac{1}{2\pi i} \sum \lim_{\varepsilon \to o} \int_{C_\varepsilon} A_p^* \, (d' - d'' \log \frac{|X|}{|\langle X, X_o \rangle|} = \frac{1}{\pi} V_p(R)$$

where the summation is taken over all C_ε's, oriented negatively with respect to the interior of R_ε, and $V_p(R)$=area of $A_p(R)$. It remains to see the value of the quantity under the summation sign in (13). We have

$$(14) \qquad \int_{C_\varepsilon} A_p^* \, (d' - d'') \log \frac{|X|}{|\langle X, X_o \rangle|} = \int_{\varphi C_\varepsilon} A_p \, \varphi^{-1})^* (d' - d'') \log \frac{|X|}{|\langle X, X_o \rangle|}$$

Since $A_p \varphi^{-1}$ is an holomorphic function, it commutes with $d' - d''$ and the second member of (14) becomes

$$(15) \qquad \int_{\varphi C_3} (d' - d'') \log \frac{|A_p \varphi^{-1}(w)|}{|\langle A_p \varphi^{-1}(w), X_o \rangle|} = \int_{\varphi C_\varepsilon} (d' - d'') \log |A_p \varphi^{-1}(w)| - \int_{\varphi C_\varepsilon} (d' - d'') \log |\langle A_p \varphi^{-1}(w), X_o \rangle|$$

Now we apply the following lemma [3].

Lemma 1: If $F(z)$ is a C^1 function then $(d' - d'') F(z) = ir \, F_r \, d\theta$ along the circle r=constant, where r, θ, are the polar coordinates of z.

If in (15) we put $\langle A_p \varphi^{-1}(w), X_o \rangle = w^k g(w)$, where $g(0) \neq 0$, and considering the lemma above we obtain

$$\int_{\varphi C_\varepsilon} (d' - d'') \log |\langle A_p \varphi^{-1}(w), X_o \rangle| = 2k\pi i + i\varepsilon \int_0^{2\pi} \frac{\partial \log |g(w)|}{\partial \varepsilon} d\theta$$

and consequently

$$\lim_{\varepsilon \to 0} \int_{\varphi C_\varepsilon} (d' - d'') \log |\langle A_p \varphi^{-1}(w), X_o \rangle| = 2k\pi i$$

Since $|A_p \varphi^{-1}(w)| \neq 0$, the same argument gives

$$\lim_{\varepsilon \to 0} \int_{\varphi C_\varepsilon} (d' - d'') \log |A_p \varphi^{-1}(w)| = 0$$

So, if we put $\Sigma k = n_p(R, X_o^{\perp})$, where the summation is taken over all $X_o^{\perp}$ points of R, formula (13) becomes

$$(16) \qquad \frac{1}{2\pi i} \int_C A_p^* (d' - d'') \log \frac{|X|}{|\langle X, X_o \rangle|} + n_p(R, X_o^{\perp}) = \frac{1}{\pi} V_p(R)$$

This formula gives the unintegrated form of the first main theorem of order p of the pseudomeromorphic curve $f: R \to P_N$. In (16) $n_p(R, X_o^{\perp})$ indicates the number of times $A_p(z)$ meets the linear space $\overset{o}{P}_{N-p-1}$.

The integrated first main theorem will be derived assuming that the Riemann surface R is obtained from a compact one R_1 by deleting a finite number of points which will be called points at infinity [3]. We isolate the points at infinity by disjoint ε-neighbourhoods; let C_ε be ths boundary of such a a neighbourhood and R_ε the complement of the union of the ε-neihhbourhoods of all points at infinity. The unintegrated form of the first main theorem appled on R_ε becomes:

$$(17) \quad n_p(R_\varepsilon, X_o^{\perp}) - \Sigma \frac{1}{2\pi i} \int_{C_\varepsilon} A_p^* (d' - d'') \log \frac{|X|}{|\langle X, X_o \rangle|} = \frac{1}{\pi} V_p(R_\varepsilon)$$

where C_ε's are taken positively oriented and the summation is taken over all C_ε's. If in (17) we put $\varepsilon = \frac{1}{\rho}$, then divide both members by ρ and finally integrate with respect to ρ we obtain

$$(18) \qquad N_p(\rho, X_o^{\perp}) + M_p(\rho) = T_p(\rho)$$

where $T_p(\rho) = \frac{1}{\pi} \int_{\rho_0}^{\rho} V_p(R_\rho) \frac{d\rho}{\rho}$ is the order function in the Ahlfors-Shimizu definition.

$$N_p(\rho, X_o^{\perp}) = \int_{\rho_o}^{\rho} n_p(R_\rho, X_o^{\perp}) \frac{d\rho}{\rho}$$

and

$$(19) \quad M_p(\rho) = -\Sigma \frac{1}{2\pi i} \int_{\rho_o}^{\rho} \Big[\int_{C_\rho} A_p^* (d' - d'') \log \frac{|X|}{|\langle X, X_o \rangle|} \Big] \frac{d\rho}{\rho}$$

After some standard computations and application of lemma 1 we get:

$$\int_{C_\rho} A_p^* (d' - d'') \log \frac{|X|}{|\langle X, X_o \rangle|} = - i\rho \frac{\partial}{\partial \rho} \int_0^{2\pi} \log \frac{|A_p \varphi^{-1}(w)|}{|\langle A_p \varphi^{-1}(w), X_o \rangle|} d\theta$$

Finally, dividing by ρ and integrating with respect to ρ we obtain:

$$(20) \int_{\rho_o}^{\rho} \Big[\int_{C_\rho} A_p^* (d' - d'') \log \frac{|X|}{|\langle X, X_o \rangle|} \Big] \frac{d\rho}{\rho} = - i \int_0^{2\pi} \log \frac{|A_p \varphi^{-1}(w)|}{|\langle A_p \varphi^{-1}(w), X_o \rangle|} d\theta + iK_o$$

where the integral of the second member is taken for $|w| = \rho$ around a point at infinity and K_o is the value of the same integral for $|w| = \rho_o$. Taking now summation of (20) over all points at infinity (19) becomes

$$M_p(\rho) = \Sigma \frac{1}{2\pi} \int_0^{2\pi} \log \frac{|A_p(z)|}{|\langle A_p(z), X_o \rangle|} d\theta \Big|_{|w| = \rho} - K$$

From Schwartz inequality we have $\log \frac{|A_p(z)|}{|\langle A_p(z), X_o \rangle|} \geqq 0$

So, (18) becomes

$$(21) \quad N_p(\rho, X_o^{\perp}) - T_p(\rho) + \Sigma \frac{1}{2\pi} \int_0^{2\pi} \log \frac{|A_p(z)|}{|\langle A_p(z), X_o \rangle|} d\theta \Big|_{|w| = \rho} = \text{Constant} \geqq 0$$

This formula expresses the integrated form of the first main theorem. Sometimes the first main theoreme is useful in the following weaker form

$$(22) \qquad N_p(\rho, X_o^{\perp}) \leqq T_p(\rho) + \text{Constant}$$

A first consequence of (21) is that the measure of all A_{N-p-1} non intersected by the p-associated curve of the pseudomeromorphic curve (f, R) is of measure zero. This is obtained by integrating (21) over the volume element $dG(N, N\ p\text{-}1)$ restricted to those A_{N-p-1} for which $n_p(\rho, X_o^{\perp}) = 0$. This result takes a much stronger form in the Ahlfors defect relations [1], which, as we shall show in a future note, hold for the pseudomeromorphic curves as well.

The Second Main Theorem: The mapping $A_p: R \rightarrow G(N, p)$ maps the invariant Hermitian metric of $G(N, p)$ to an Hermitian metric in R with certain singularities. The second main theorem expresses a relation between the Euler characteristic of G and the curvature of this induced metric. The metric (6) of $G(N, p)$ is mapped by A_p^* to the differential form

$$(1) \qquad A_p^* \, ds_p^2 = \frac{\langle A_{p-1}, A_{p-1}\rangle \langle A_{p+1}, A_{p+1}\rangle}{\langle A_p \; A_p\rangle^2} \, d\varphi(z) d\overline{\varphi(z)}$$

where, as everywhere in this paper, $\varphi(z)$ is a quasiconformal homeomorphism defining locally a uniformization parameter w of the mapping $(w=\varphi(z))$. The singularities of (1) appear at the zeroes of

$$(2) \qquad h_p = \frac{|A_{p-1}| \; |A_{p+1}|}{|A_p|^2}$$

If, at some point, the zeroes of $|A_{p-1}|$, $|A_p|$, $|A_{p+1}|$ are of order d_{p-1}, d_p, d_{p+1}, respectively, the point is singular if $m_p = d_{p-1} - 2d_p + d_{p+1} > 0$. Such a point will be called p stationary point of stationary index m_p. Assuming that no stationary point lies on the boundary C of R, we isolate the stationary points by disjoint ε-neighbourhoods and let R_ε the complement in R of the union of these neighbourhoods. Clearly, (1) defines a positive definite Hermitian metric in R_ε. The connection form of this metric [3] is

$$(3) \qquad \varphi_{11} = -idy + (d' - d'') \log h_p$$

where $d' \log h_p = \frac{\partial \log h_p}{\partial w} d\varphi(z)$, $d'' \log hp = \frac{\partial \log h_p}{\partial \bar{w}} d\overline{\varphi(z)}$

Applying Gauss-Bonnet formulla on (1) and taking limits as $\varepsilon \to 0$ we obtain

$$(4) \qquad -2\pi i\,[X(R)-k] + \int_C \varphi_{11} + \Sigma \lim_{\varepsilon \to 0} \int_{C\varepsilon} \varphi_{11} = \lim_{\varepsilon \to 0} \iint_{R\varepsilon} \Phi_{11}$$

where k is the number of singular points, Σ is taken over all singular points and $\Phi_{11} = d\varphi_{11} = -2d'd'' \log h_p$.

Considering the local expression (3) of φ_{11} and applying lemma 1 we find $\lim_{\varepsilon \to 0} \int_{C\varepsilon} \varphi_{11} = -2\pi i\,(m_p + 1)$. We also have

$$\iint_{R\varepsilon} \Phi_{11} = 2i\left[\iint_{R\varepsilon} A_{p-1}^* \Omega_{p-1} - 2\iint_{R\varepsilon} A_p^* \Omega_p + \iint_{R\varepsilon} A_{p+1}^* \Omega_{p+1}\right] =$$
$$= 2i\,[V_{p-1}(R_\varepsilon) - 2\,V_p(R_\varepsilon) + V_{p+1}(R_\varepsilon)]$$

Since for $\varepsilon \to 0$ we have $R_\varepsilon \to R$, formula (4) is finally written

$$(5) \qquad -X(R) + \frac{1}{2\pi i} \int_C \varphi_{11} = \frac{1}{\pi}\,[V_{p-1}(R) - 2\,V_p(R) + V_{p+1}(R)] + w_p$$

where $w_p = \Sigma m_p$, Σ being considered over all singular points. Formula (5) gives the unintegrated form of the second main theorem. The integrated form of the second main theorem is derived from the unintegrated one, by applying the same method as in the first main theorem. That is, we restrict ourselves to a Riemann surface R obtained from a compact one R_1 by deleting a finite number of points at infinity. So, the notation being as in the case of the first main theorem, formula (5) is written for R_ε.

$$(6) \quad -X(R_\varepsilon) - \frac{1}{2\pi i} \Sigma \int_{C\varepsilon} \varphi_{11} = \frac{1}{\pi} \left[V_{p-1}(R_\varepsilon) - 2 V_p(R_\varepsilon) + V_{p+1}(R_\varepsilon) \right] + w_p.$$

If in (6) we put $\rho = \frac{1}{\varepsilon}$, divide by ρ and then integrate over ρ, we obtain

$$(7) \quad -X(R_\rho) \log \frac{\rho}{\rho_0} - \frac{1}{2\pi i} \Sigma \int_{\rho_0}^{\rho} \left[\int_{C\rho} \varphi_{11} \right] \frac{d\rho}{\rho} = T_{p-1}(\rho) - 2 T_p(\rho) + + T_{p+1}(\rho) + \int_{\rho_0}^{\rho} w_p(\rho) \frac{d\rho}{\rho}$$

From the local form of $\varphi_{11} = id\theta - i\rho \frac{\partial}{\partial\rho} \log h_p d\theta$ we obtain:

$$\int_{\rho_0}^{\rho} \left[\int_{C\rho} \varphi_{11} \right] \frac{d\rho}{\rho} = 2\pi i \log \frac{\rho}{\rho_0} - i \int_{C\rho} \log h_p$$

If finally we notice that R_ρ is obtained from R_1 by deleting r small discs, we have $X(R_\rho) = X(R_1) - r$. So (7) becomes

$$(8) \quad -X(R_1) \log \frac{\rho}{\rho_0} + \Sigma S_p(\rho) = T_{p-1}(\rho) - 2 T_p(\rho) + T_{p+1}(\rho) + W_p(\rho)$$

where $S_p(\rho) = \frac{1}{2\pi} \int_{C\rho} \log h_p d\theta, \; W_p(\rho) = \int_{\rho_0}^{\rho} w_p(p) \frac{d\rho}{\rho}$

and the summation is taken over all points at infinity. Formula (8) gives the integrated form of the second main theorem.

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Ὁ κ. **Κωνστ. Π. Παπαϊωάννου** ἀνακοινῶν τὴν ἀνωτέρω μελέτην εἶπε τὰ ἑξῆς.

Ὁ κ. Νικόλαος Πετρίδης ἐσπούδασε τὰ μαθηματικὰ εἰς τὸ Πανεπιστήμιον Ἀθηνῶν κατὰ τὰ ἔτη 1948 - 1953. Τὴν πρώτην ἐργασίαν του παρουσίασεν ὡς φοιτητὴς ἐπὶ ἀλγεβρικοῦ θεωρήματος τοῦ ἀειμνήστου Παναγιώτου Ζερβοῦ. Ἀποφοιτήσας μὲ τὸν βαθμὸν ἄριστα ἀπὸ τὸ Πανεπιστήμιον Ἀθηνῶν, ἔτυχεν ὑποτροφίας τοῦ Ἱδρύματος Κρατικῶν ὑποτροφιῶν διὰ τὸ Πανεπιστήμιον τοῦ Σικάγου, εἰς τὸ ὁποῖον ἐφοίτησεν ἐπὶ πενταετίαν, παρακολουθήσας τὰ μαθήματα διαπρεπῶν καθηγητῶν ὡς οἱ A. Albert, S. S. Chern, S. Maclane, M. Stone. Τὴν ἐπὶ διδακτορίᾳ διατριβήν του ἐξεπόνησεν ὁ κ. Πετρίδης ὑπὸ τὴν ἐποπτείαν τοῦ Καθηγητοῦ S. S. Chern, ἑνὸς τῶν μεγαλυτέρων γεωμετρῶν τῆς ἐποχῆς μας. Ἡ διατριβὴ αὕτη εἶναι ἰδιαιτέρως ἀξιόλογος καὶ ἀποτελεῖ πρωτότυπον συμβολὴν εἰς τὰ δύσκολα θέματα ἐπὶ τῶν ἐπιφανειῶν τοῦ Riemann.

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Τὴν παροῦσαν ἐργασίαν του, τὴν ὁποίαν ἔχω τὴν τιμὴν νὰ ἀνακοινώσω εἰς τὴν Ἀκαδημίαν, παρουσιάζω μὲ ἰδιαιτέραν χαράν, διότι ὁ κ. Πετρίδης διετέλεσεν καὶ ἀγαπητὸς μαθητής μου.

Εἶναι γνωστὴ ἡ ἔννοια τῆς συμμόρφου ἀπεικονίσεως χωρίου τοῦ μιγαδικοῦ ἐπιπέδου $OX\Psi$ $(x+i\psi=Z)$ *ἐντὸς τοῦ μιγαδικοῦ ἐπιπέδου* $ouv(u+iv=w)$. *Εὐκόλως ἀποδεικνύεται ὅτι μία ἀναλυτικὴ συνάρτησις* $w=f(z)$ *ὁρίζει τοπικῶς μίαν σύμμορφον ἀκεικόνισιν. Καὶ ἀντιστρόφως ἐφ' ὅσον ἐννοοῦμεν ὡς συμμόρφους ἀπεικονίσεις, τὰς ἀπεικονίσεις, αἱ ὁποῖαι εἶναι ὄχι μόνον ἰσογώνιοι ἀλλὰ καὶ ὁμοιόστροφοι, αἱ ἀπεικονίσεις αὗται ὁρίζονται ἀπὸ ἀναλυτικὰς συναρτήσεις. Ἐπὶ τῶν παρατηρήσεων αὐτῶν στηριζόμενος ἀρχικῶς ὁ Riemann καὶ ἐν συνεχείᾳ πλειὰς διαπρε-*

πῶν ἐρευνητῶν ἐδημιούργησαν τὴν «γεωμετρικὴν» θεωρίαν τῶν συναρτήσεων μιᾶς μιγαδικῆς μεταβλητῆς. Εἰς τὴν ἀνάπτυξιν τῆς θεωρίας ταύτης συνέβαλεν οὐσιωδῶς καὶ ὁ ἀείμνηστος Κωνσταντῖνος Καραθεοδωρῆ. Εἰς τὰ κλασσικὰ αὐτὰ θέματα ὑφίστανται εἰσέτι σημαντικὰ ἄλυτα προβλήματα. Τοῦτο ὅμως δὲν ἠμπόδισε τὴν δημιουργίαν εὐρυτέρων θεωριῶν ἀποσκοπουσῶν εἰς τὴν γενίκευσιν τῆς ἐννοίας τῆς συμμόρφου ἀπεικονίσεως, τῆς ἀναλυτικῆς ἀπεικονίσεως, τῆς μερομόρφου ἀπεικονίσεως, τῆς ἀλγεβρικῆς ἀπεικονίσεως κτλ. Τὰς ἐργασίας ἐπὶ τῶν κατευθύνσεων αὐτῶν δυνάμεθα νὰ κατατάξωμεν εἰς δύο κατηγορίας, ἤτοι: 1ον. εἰς ἐκείνας, αἵ ὁποῖαι ἐνεπνεύσθησαν κυρίως ἀπὸ τὴν κλασσικὴν μαθηματικὴν ἀνάλυσιν· εἰς τὴν κατηγορίαν αὐτὴν ἀνήκουν καὶ αἱ ἐργασίαι τοῦ ἀειμνήστου Γεωργίου Ρεμούνδου, καὶ 2ον. εἰς τὰς ἐργασίας, αἵ ὁποῖαι, ἐν συνδυασμῷ ἐνίοτε καὶ μὲ τὰς πρώτας, τοποθετοῦνται εἰς τὰ πλαίσια τῶν λεγομένων νεωτέρων μαθηματικῶν.

Εἰς τὰς τελευταίας ἀνήκει καὶ ἡ παροῦσα ἐργασία τοῦ κ. Νικολάου Πετρίδη, εἰς τὴν ὁποίαν ἐπιτυγχάνεται κατὰ γονιμώτατον τρόπον ἡ σύνδεσις ζητημάτων τῆς κλασσικῆς ἀναλύσεως μὲ ζητήματα νεωτέρων μαθηματικῶν. Εἰς τὴν ἐργασίαν αὐτὴν ἐπεκτείνεται ἡ ἔννοια τῆς Quasiconformal ἀπεικονίσεως εἰς τὴν περίπτωσιν ψευδοαναλυτικῶν ἀπεικονίσεων μιᾶς ἐπιφανείας R τοῦ Riemann ἐντὸς μιᾶς μιγαδικῆς ἀναλυτικῆς πολλαπλότητος M_N μιγαδικῆς διαστάσεως $N > I$. Ὁ νέος ὁρισμὸς τῆς Quasiconformal ἀπεικονίσεως, τὸν ὁποῖον δίδει ὁ κ. Πετρίδης ἀφ' ἑνὸς μὲν παραμένει ἀναλλοίωτος εἰς τοὺς διαφόρους μετασχηματισμούς, ἀφ' ἑτέρου δὲ προσφέρεται διὰ τὴν γενίκευσιν γνωστῶν ἐννοιῶν τῆς θεωρίας τῶν Ἀναλυτικῶν συναρτήσεων.

ΜΑΘΗΜΑΤΙΚΑ.—Une méthode nouvelle pour la majoration et pour la minoration des valeurs absolues des zéros des polynômes; extension au cas des zéros des Series de Taylor, *par* ***Jeanne Ferentinou - Nicolacopoulou*****. Ἀνεκοινώθη ὑπὸ τοῦ Ἀκαδημαϊκοῦ κ. Κωνστ. Παπαϊωάννου.

I. INTRODUCTION

Soit

$$a_o z^n + a_1 z^{n-1} + \cdot \cdot \cdot + a_n \qquad (1)$$

où $a_o \neq o$, un polynôme à coefficients dans le corps complexe C. D'après un

* ΙΩΑΝΝΑΣ ΦΕΡΕΝΤΙΝΟΥ - ΝΙΚΟΛΑΚΟΠΟΥΛΟΥ, Νέα μέθοδος διὰ τὴν ἀνεύρεσιν ἀνωτέρων καὶ κατωτέρων φραγμάτων τῶν ἀπολύτων τιμῶν τῶν ριζῶν τῶν πολυωνύμων. Ἐπέκτασις εἰς τὴν περίπτωσιν τῶν σειρῶν τοῦ Taylor.